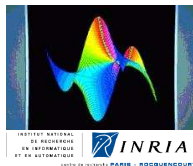


A Non-Holonomic Systems Approach to Special Function Identities

Frédéric Chyzak
(joint work with M. Kauers & B. Salvy)

Project-Team Algorithms, INRIA



June 22, 2009

I Introduction

Computer Algebra & Special-Function Identities

Goal

Symbolic manipulation of expressions and **automatic proof of identities** involving **special functions**, combinatorial sequences, etc.

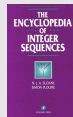
Linear differential/recurrence equations are a good data structure!

- Zeilberger's *Holonomic systems approach* (1990).
- Equations defeat closed-form expressions: BS & Zimmermann (1994 + **gfun**), FC & BS (1998), FC (2000), etc.
- Meunier's ESF & the DDMF by ABe, FC, SG & MM.

An ubiquitous class



About **25%** of Sloane's encyclopedia,
60% of Abramowitz & Stegun.



eqn + initial cond. = data structure

Examples: Sums & Integrals


$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad [\text{Strehl92}]$$

$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2 \quad [\text{Blodgett90}]$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2} \quad [\text{GIMo94}]$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{[n/2]!} \quad [\text{Doetsch30}]$$

$$\int_0^\infty \int_0^\infty J_1(x) J_1(y) J_2(c\sqrt{xy}) \frac{dx dy}{e^{x+y}} \quad \text{satisfy a 2nd-order ODE}$$

+ many, many more in, e.g., 

Examples: q -Sums, Integral Transforms & Symmetric Functions

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}} \quad [\text{Andrews74}]$$

$$\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^n \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k} (q; q)_{n-k}} \quad [\text{Paule85}]$$

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}} dx = (-1)^n \pi I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx) dx = \frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$

$$\langle \exp((p_1^2 - p_2)/2 - p_2^2/4) | \exp(t(p_1^2 + p_2)/2) \rangle = \frac{e^{-\frac{1}{4}t(t+2)}}{\sqrt{1-t}}$$

Examples: Non-“Holonomic” Sums & Integrals

$$\sum_{k=0}^n \binom{n}{k} i(k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n \quad [\text{Abel1826}]$$

$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle \quad [\text{Frobenius1910}]$$

$$\sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^n \binom{n}{k} B_{m+k} \quad [\text{Gessel03}]$$

$$\int_0^{\infty} x^{k-1} \zeta(n, \alpha + \beta x) dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha)$$

$$\int_0^{\infty} x^{\alpha-1} \text{Li}_n(-xy) dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\int_0^{\infty} x^{s-1} \exp(xy) \Gamma(a, xy) dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

Two Ideas — Several Generations of Algorithms

Ideas

- **Confinement** in **finite-dimensional** vector-spaces
→ closures under algebraic operations
- **Creative telescoping** (\approx differentiation under the integral sign)
→ closures under \int , \sum , $\langle \dots | \dots \rangle$

Classes of inputs

- (q -)hypergeometric/hyperexponential: Zeilberger (1990), Paule–Schorn (1995); Riese (2003); Almkvist–Z. (1990)
- higher-order eqns: Zeilberger (1990), Takayama (1989–90), FC–BS (1998), FC (2000)
- Abel-type/Stirling/Euler and Bernoulli: Majewicz (1996); Kauers (2007); Chen & Sun (2009)
- previous and more: the present work

II Algebraic Closures

Before the examples, recall . . .

Gauss' **hypergeometric** series,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} u_n x^n \quad \text{for} \quad u_n := \frac{(a)_n (b)_n}{(c)_n n!}, \quad \text{solves}$$

$$(\theta + c - 1)\theta(F) = x(\theta + a)(\theta + b)(F) \quad \text{for} \quad \theta := x \frac{d}{dx},$$

$$\text{since} \quad \frac{u_{n+1}}{u_n} = \frac{(n+a)(n+b)}{(n+c)(n+1)}.$$

$$\theta^2(F) \in \langle F, \theta(F) \rangle \quad (\text{over } \mathbb{Q}(a, b, c)(x))$$

This generalizes to

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n n!} x^n.$$

Algebraic Closures by Vector-Space Confinement (1)

Example of a product of generating series: Clausen's identity,

$${}_2F_1\left(\begin{matrix} a, b \\ a + b + 1/2 \end{matrix} \middle| x\right)^2 = {}_3F_2\left(\begin{matrix} 2a, 2b, a + b \\ 2(a + b), a + b + 1/2 \end{matrix} \middle| x\right).$$

Proof: After setting $f = g^2$,

- ① $f \in \langle g^2 \rangle$, $\theta(f) = 2g\theta(g) \in \langle g\theta(g) \rangle$,
 $\theta^2(f) \in \langle \theta(g)^2, g\theta^2(g) \rangle \subset \langle g^2, g\theta(g), \theta(g)^2 \rangle =: V$,
 $\theta^3(f) \in \langle g^2, g\theta(g), \theta(g)^2, g\theta^2(g), \theta(g)\theta^2(g) \rangle \subset V$.
- ② 4 vectors in dim. 3 \rightarrow 3rd-order ODE for f (confinement).
- ③ Same ODE for the ${}_3F_2$ + match up to $O(x^4)$ + Cauchy thm.

Algebraic Closures by Vector-Space Confinement (2)

Example of the generating series of a product: Mehler's identity for Hermite polynomials,

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}.$$

Proof:

- ① Define Hermite polynomials $H_n(t)$ (D-finite over $\mathbb{Q}(t)(n)$):
recurrence of order 2;
- ② For the product, introduce the vector basis over $\mathbb{Q}(x, y)(n)$

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

→ recurrence of order at most 4; (confinement)

- ③ Translate into differential equation and solve.

Multivariate Confinement Implies Several Relations

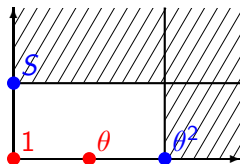
Example: Contiguity relation (Gauss, 1812) for hypergeometric series,

$$F(a, x) := {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) := \sum_{n=0}^{\infty} u_{a,n} x^n, \quad u_{a,n} := \frac{(a)_n (b)_n}{(c)_n n!}.$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow (1-x)\theta^2(F) = ((a+b)x + 1 - c)\theta(F) + abx F,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S(F) := F(a+1, x) = \frac{1}{a}\theta(F) + F.$$

$\dim. = 2 \Rightarrow S^2(F), S(F), F$ dependent
over $\mathbb{Q}(b, c)(a, z)$:



$$(a+1)(x-1)S^2(F) + ((b-a-1)x + 2 - c + 2a)S(F) + (c-a-1)F = 0$$

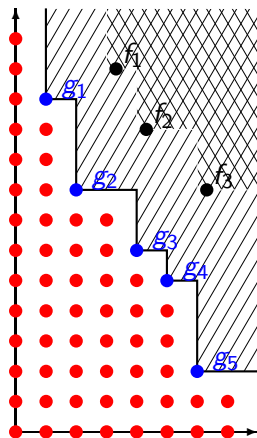
Crash Course on Gröbner Bases

Generalize Euclidean division and the Euclidean (gcd) algorithm

- 1 **Monomial ordering**: total order \prec on the monomials, compatible with the product, with minimal element 1
- 2 **Gröbner basis** of a (left) ideal I w.r.t. \prec : generators **matching the stairs** of I

$$\text{GB}(f_1, f_2, f_3) = (g_1, \dots, g_5)$$

- 3 **Quotient**: vector basis **below the stairs**
- 4 **Reduction** of P modulo I : unique remainder written on this basis



Dimension of Ideals and ∂ -Finite Functions

$M_k(I) := \langle m : m \text{ is below the stairs and of total degree } \leq s \rangle$

Theorem (Hilbert)

For any I , there is an integer $\delta(I)$ such that $\#M_k(I) = O(s^{\delta(I)})$.

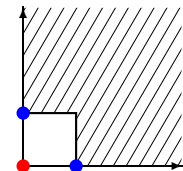
- $\delta(I)$ is the (Hilbert) dimension of I .
- Finite measure of infinite-dimensional vector-spaces.
- Can be obtained by a Gröbner-basis calculation.
- $I \subset J \Rightarrow \delta(I) \geq \delta(J)$

Definition (annihilator and ∂ -finiteness)

- $\text{ann } f \subset S = \{ P \in S : P(f) = 0 \}$
- f is ∂ -finite \Leftrightarrow linear dim. of quotient is finite $\Leftrightarrow \delta(\text{ann } f) = 0$

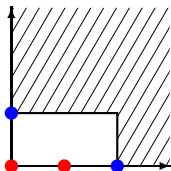
Examples

Binomial coeffs $\binom{n}{k}$ w.r.t. S_n, S_k ;
Hypergeometric sequences:



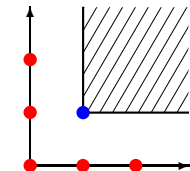
$$\delta(I) = 0, \dim S/I = 1$$

Bessel $J_\nu(x)$ w.r.t. S_ν, D_x ;
Orthogonal polys w.r.t. S_n, D_x :



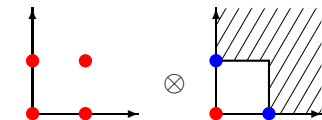
$$\delta(I) = 0, \dim S/I = 2$$

Stirling nbs w.r.t. S_n, S_k :



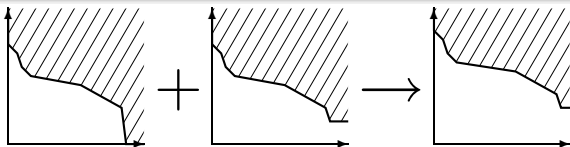
$$\delta(I) = 1, \dim S/I = \infty$$

Abel-type w.r.t. S_m, S_k, S_r, S_s ,
 $h(m, k)(k+r)^k(m-k+s)^{m-k} \frac{r}{k+r}$:



$$\delta(I) = 2 \text{ in space of dimension } 4$$

Closure Properties



Proposition

$$\delta(\text{ann}(f + g)) \leq \max(\delta(\text{ann } f), \delta(\text{ann } g)),$$

$$\delta(\text{ann } fg) \leq \delta(\text{ann } f) + \delta(\text{ann } g), \quad \delta(\text{ann } \partial f) \leq \delta(\text{ann } f).$$

Algorithm (for a product fg): for a *graded* ordering,

for $s = 0, 1, 2, \dots$, **until** $\delta(I) \leq \text{bound}$:

for each $|\alpha| \leq s$, **reduce** $\partial^\alpha(fg)$ to a sum $\sum u_{\alpha;\beta,\gamma}(x)\partial^\beta(f)\partial^\gamma(g)$
over $\beta \in M_s(\text{ann } f), \gamma \in M_s(\text{ann } g)$

search for $Q(x)$ -linear relations, set I to the ideal they generate

return I , a subideal of $\text{ann } fg$

Product of Stirling Numbers of the Second Kind

$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ and $\left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$, given by $\delta(l_1) = \delta(l_2) = 1$:

$$\left\{ \begin{smallmatrix} n+1 \\ k+1 \end{smallmatrix} \right\} = (k+1) \left\{ \begin{smallmatrix} n \\ k+1 \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}, \quad \Delta_m \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = 0,$$

$$\Delta_n \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\} = 0, \quad \left\{ \begin{smallmatrix} m+1 \\ k+1 \end{smallmatrix} \right\} = (k+1) \left\{ \begin{smallmatrix} m \\ k+1 \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}.$$

Computations up to $s = 3$: $\binom{6}{3} = 20 > 19 = \dim M_3(I)$.

$u_{n,m,k} := \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$, given by $\delta(I) = 2$:

$$u_{n+1,m+1,k+1} - (k+1)u_{n+1,m,k+1} - (k+1)u_{n,m+1,k+1} + (k+1)^2 u_{n,m,k+1} - u_{n,m,k} = 0.$$

III Closures under \sum and \int

Summation by Creative Telescoping

Goal: evaluate $S_n := \sum_{k=0}^n \binom{n}{k}$ to 2^n .

Given Pascal's triangle rule,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} = 2\binom{n}{k} + \binom{n}{k+1} - \binom{n}{k},$$

summation over k yields $S_{n+1} = 2S_n$.

The initial condition $S_0 = 1$ concludes the proof.

Creative Telescoping for Sums (Zeilberger, 1990)

$$U_n = \sum_{k=a}^b u_{n,k} = ?$$

Given $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k))(u)(n, k) = 0,$$

summation leads by “telescoping” to

$$A(n, S_n)(U)(n) = [B(n, k, S_n, S_k)(u)(n, k)]_{k=a}^{k=b+1} \stackrel{\text{often}}{=} 0.$$

Adapts easily to $U(x) = \sum_{k=a}^b u_k(x)$.

Creative Telescoping for Integrals (Zeilberger, 1990)

$$U(x) = \int_a^b u(x, y) dy = ?$$

Given $A(x, D_x)$ and $B(x, y, D_x, D_y)$ such that

$$(A(x, D_x) + D_y B(x, y, D_x, D_y))(u)(x, y) = 0,$$

integrating leads by “telescoping” to

$$A(x, D_x)(U)(x) = [B(x, y, D_x, D_y)(u)(x, y)]_{y=a}^{y=b+1} \stackrel{\text{often}}{=} 0.$$

Adapts easily to $U_n = \int_a^b u_n(y) dy$.

Towards Algorithms for Creative Telescoping

$$\text{Example: } U(x) := \int_0^1 \frac{\cos xy}{\sqrt{1-y^2}} dy = ?$$

$$U'(x) = \int_0^1 -y \frac{\sin xy}{\sqrt{1-y^2}} dy,$$

$$U''(x) = \int_0^1 -y^2 \frac{\cos xy}{\sqrt{1-y^2}} dy = -U(x) + \int_0^1 \sqrt{1-y^2} \cos xy dy,$$

$$U''(x) + U(x) = \left[\sqrt{1-y^2} \frac{\sin xy}{x} \right]_{y=0}^{y=1} + \int_0^1 \frac{y}{\sqrt{1-y^2}} \frac{\sin xy}{x} dy = -\frac{U'(x)}{x}.$$

$$U''(x) + \frac{U'(x)}{x} + U(x) = \left[-\frac{1-y^2}{xy} D_x(u)(x,y) \right]_{y=0}^{y=1} = 0.$$

$$D_x^2 + \frac{1}{x} D_x + 1 + D_y \frac{1-y^2}{xy} D_x \in \text{ann } u \quad \rightarrow \quad U(x) = J_0(x).$$

Creative telescoping = finite difference under the sum + summation by parts
 Creative telescoping = differentiation under the integral + integration by parts

Example: Rediscovering Pascal's Triangle Rule

Reduce all monomials of degree $\leq s = 2$:

$$\begin{aligned}
 1 &\rightarrow \mathbf{1}, & S_n &\rightarrow \frac{n+1}{n+1-k} \mathbf{1}, & S_k &\rightarrow \frac{n-k}{k+1} \mathbf{1} \\
 S_n^2 &\rightarrow \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)} \mathbf{1}, & S_k^2 &\rightarrow \frac{(n-k-1)(n-k)}{(k+2)(k+1)} \mathbf{1}, & S_n S_k &\rightarrow \frac{n+1}{k+1} \mathbf{1}.
 \end{aligned}$$



Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

$D_2, D_2 S_n, D_2 S_k, D_2 S_n^2, D_2 S_k^2, D_2 S_n S_k$ **confined** in

$\langle \mathbf{1}, k \mathbf{1}, k^2 \mathbf{1}, k^3 \mathbf{1}, k^4 \mathbf{1} \rangle$ over $K(n)$

$$\rightarrow D_2(S_n S_k - S_k - 1) \in \text{ann} \binom{n}{k}.$$

$\deg D_s = O(s) \Rightarrow$ this **had to happen** for some s .

More Examples

- $\frac{1}{n+k}$: essentially the same situation.
- $\frac{1}{n^2+k^2}$: confinement in a space of dimension $O(s^2)$,
no elimination of k succeeds.
- $f = \frac{a(x, y_1, \dots, y_r)}{b(x, y_1, \dots, y_r)}$: $D_s = b^s$,
confinement in a space of dimension $O(s^r)$ over $K(x)$,
elimination of y_1, \dots, y_r has to succeed.
Base case of proof that D-finite functions are “holonomic”.

Adding a Left Part and a Right Part

Specification of a Creative-Telescoping Algorithm

Input: generators of (a subideal of) $\text{ann } f$

Output: all (A, B) such that:

- $A + \partial_y B \in \text{ann } f$,
- A is free of y and ∂_y

Algorithms for specific classes. Often return truncated results.

Definition (telescoping ideal of I w.r.t. y)

$$T_y(I) := (I + \partial_y S_{x,y}) \cap S_x \quad \text{where}$$

$$S_{x,y} := K(x,y)\langle \partial_x, \partial_y \rangle \quad \text{and} \quad S_x := K(x)\langle \partial_x \rangle.$$

When $I = S_{x,y}G_1 + \cdots + S_{x,y}G_\ell$, this involves

$$(S_{x,y}G_1 + \cdots + S_{x,y}G_\ell) + \partial_y S_{x,y}.$$

Polynomial Growth and Creative Telescoping when $\delta > 0$

Definition (polynomial growth p)

There exists a sequence of polynomials $P_s(x, y)$, s.t. if $|a| + b \leq s$,
 $P_s \partial_{x_1}^{a_1} \dots \partial_{x_k}^{a_k} \partial_y^b$ reduces to polys of degree $O(s^p)$ in y .

Theorem (Chyzak, Kauers & Salvy, 2009)

$$\delta(T_y(I)) \leq \max(\delta(I) + p - 1, 0).$$

Proof: $P_s \partial_x^a \partial_y^b - \sum_{\beta \in M_s(I)} (\text{deg. in } y \leq O(s^p)) \partial^\beta \in I$
 \Rightarrow any choice $\partial_{x_1}, \dots, \partial_{x_{\delta(I)+p}}, \partial_y$ is algebraically dependent
 modulo $I \Rightarrow \delta(T_y(I)) \leq \delta(I) + p - 1$.

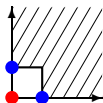
Corollary (sufficient condition for creative telescoping)

$$\delta(I) + p - 1 < k \Rightarrow \text{identities exist for the sum/int. w.r.t. } y.$$

Examples with Polynomial Growth $p = 1$

- Proper hypergeometric (Wilf & Zeilberger, 1992):

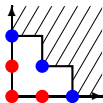
$$Q(n, k) \xi^k \frac{\prod_{i=1}^u (a_i n + b_i k + c_i)!}{\prod_{i=1}^v (u_i n + v_i k + w_i)!},$$



Q polynomial, a_i, b_i, u_i, v_i integers.

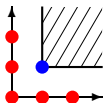
- Differential D-finite (special case of “holonomy”).
- Stirling: $\delta = 1 \rightarrow$ for ≥ 3 vars, e.g., Frobenius:

$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle.$$



- Abel type: $\delta = 2 \rightarrow$ for ≥ 4 vars, e.g., Abel:

$$\sum_{k=0}^n \binom{n}{k} i(k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n.$$



Algorithm I: Fasenmyer's Style

Polynomial growth + linear algebra $\rightarrow J := I \cap S_x[\partial_y]$.

Algorithm

For increasing values of s , until $\delta(J) \leq \text{bound}$:

- ① Reduce all $\partial_x^a \partial_y^b$ with $a + b \leq s$;
- ② Normalize to a common denominator;
- ③ Set up a linear system to cancel the positive powers of y ;
- ④ If a non-zero solution is found, it has the form $A(x, \partial_x) + \partial_y B(y, \partial_x, \partial_y)$; return it.

This computes A in

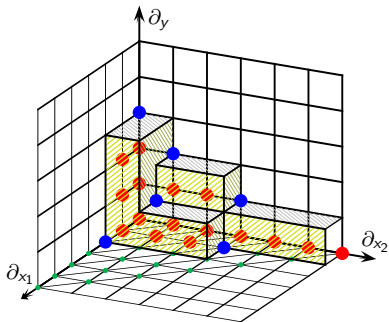
$$(J + \partial_y S_x[\partial_y]) \cap S_x, \quad \text{not in } T_t(I) := (I + \partial_y S_{x,y}) \cap S_x.$$

$$S_x = K(x)\langle \partial_x \rangle \subset S_x[\partial_y] \subset S_{x,y} := K(x,y)\langle \partial_x, \partial_y \rangle.$$

Algorithm II. Zeilberger's Style Extended to $\text{Dim} > 0$

Compute A in $T_t(I) := (I + \partial_y S_{x,y}) \cap S_x$. \leftrightarrow faster, more precise.

- ① $(q-)$ Hypergeometric case: Zeilberger 1990 (+ Schorn, Riese).
- ② ∂ -finite case ($\delta = 0$): Chyzak 2000 (+ Koutschan, Pech).
- ③ Non- ∂ -finite: **new**.



for $s = 0, 1, 2, \dots$, until $\delta(J) \leq \text{bound}$:

$$\text{set } A := \sum_{|\alpha| \leq s} \eta_\alpha \partial^\alpha$$

for undetermined coeffs $\eta_\alpha(y) \in K(x)$

$$\text{set } B := \sum_{\beta \in M_s(I)} \phi_\beta(y) \partial^\beta$$

for undetermined coeffs $\phi_\beta(y) \in K(x, y)$

reduce $A - \partial_y B$

onto the basis $M_{s+1}(I)$

extract coeffs to form a linear system

of first order w.r.t. ∂_y

solve and set J to the ideal of the A 's

return the pairs (A, B)

IV Conclusion

Conclusion

- Summary:
 - Linear differential/recurrence equations as a data structure;
 - Confinement in vector spaces + creative telescoping \rightarrow identities;
 - Input dimension + polynomial growth \rightarrow output dimension.
- Also:
 - Multiple summation/integration;
 - **Bounds** \rightarrow identities + their size + **complexity** of algorithms.
- Open questions:
 - Replace polynomial growth by something **intrinsic**;
 - Exploit symmetries;
 - Structured Padé-Hermite approximants;
 - Understand **non-minimality**.

Special-Functions Memento

$J_n(x)$, $Y_n(x)$, $I_n(x)$, $K_n(x)$ (Bessel fn), $H_n(x)$ (Hermite), $T_n(x)$ (Chebyshev) : EDO + rec.

$$\Gamma(s, x) = (\text{upper}) \text{ incomplete gamma fn} = \int_x^\infty t^{s-1} e^{-t} dt$$

$$B(s, t) = \text{beta fn} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad \zeta(s, x) = \text{Hurwitz's zeta fn} = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

$$(x)_n = \text{Pochhammer symbol} = x(x+1)\dots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$$

$$(x; q)_n = q\text{-Pochhammer symbol} = (1-x)(1-qx)\dots(1-q^{n-1}x)$$

$$\text{Li}_n(z) = \text{polylogarithms} = \sum_{k=1}^{\infty} \frac{z^k}{k^n} = \theta_z(\text{Li}_{n+1})(z)$$

$$B_n = \text{Bernoulli nb} = \frac{1}{2i\pi} \oint \frac{1}{e^z - 1} \frac{n!}{z^n} dz \quad E_n = \text{Euler nb} = \frac{1}{2i\pi} \oint \frac{2e^z}{e^{2z} + 1} \frac{n!}{z^n} dz$$

$$\langle n \rangle_m = \text{Eulerian nb} : \langle n+1 \rangle_{m+1} = (m+2)\langle n+1 \rangle_m + (n-m)\langle n \rangle_m$$

$$\left[n \right]_m = \text{Stirling nb of the 1st kind} = \left[n+1 \right]_{m+1} = \left[n \right]_m - (n+1) \left[n \right]_{m+1}$$

$$\left\{ n \right\}_m = \text{Stirling nb of the 2nd kind} = \left\{ n+1 \right\}_{m+1} = (m+1) \left\{ n \right\}_{m+1} + \left\{ n \right\}_m$$