Geodesics in large planar quadrangulations

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Outline

Statistics of geodesics

Geodesic points

Geodesic loops

Confluence of geodesics
Reminder: geodesic = shortest path between two points
Outline

Statistics of geodesics

Geodesic points

Geodesic loops

Confluence of geodesics
Quadrangulations with geodesic boundary
Quadrangulations with geodesic boundary

Simply pointed
Quadrangulations with geodesic boundary

\[
\min \ell(v) = 1
\]

Schaeffer well-labeled tree
Quadrangulations with geodesic boundary

\[ \prod_{m=1}^{i} R_m \]

\[ \min \ell(v) = 1 \]
Quadrangulations with geodesic boundary

The generating function for quadrangulations with geodesic boundary is therefore:

\[ Z_i(g) = \prod_{j=1}^{i} R_j = R^i \frac{(1 - x)(1 - x^i + 3)}{(1 - x^3)(1 - x^{i+1})} \]

Reminder: \( g \) weight per square, \( R(g) = \frac{1 - \sqrt{1 - 12g}}{6g} \)

\[ x(g) + \frac{1}{x(g)} + 1 = \frac{1}{gR(g)^2} \]
Quadrangulations with a marked geodesic
Quadrangulations with a marked geodesic

Almost the same as quadrangulations with geodesic boundary...
Quadrangulations with a marked geodesic

Arbitrary geodesic boundaries may have “pinch points”. Marked geodesics correspond to irreducible boundaries.
Quadrangulations with a marked geodesic

An arbitrary geodesic boundary may be decomposed into irreducible components.

\[ Z_i = i, U_i = i = Z_i - \sum_{j=1}^{i-1} Z_{i-j} \]
Quadrangulations with a marked geodesic

\[ U_i(g) = Z_i(g) - \sum_{j=1}^{i-1} U_j(g)Z_{i-j}(g) \quad \text{i.e.} \quad \hat{U}(g; t) = \frac{\hat{Z}(g; t)}{1 + \hat{Z}(g; t)} \]

From the exact formula for \( Z_i \) we can perform asymptotic analysis:

\[ U_i(g)|_{g^n} \sim \frac{\sqrt{12^n}}{2 \pi n^{5/2}} \delta_1 \quad \text{as} \quad n \to \infty \]

where:

\[ \delta(t) = \frac{3t}{70(1-2t)^4(t-(1-2t)\log(1-2t))^2} \left(2t (3+177t^2-412t^3+708t^4-624t^5+224t^6)+3(1-2t)^6\log(1-2t)\right) \]
Quadrangulations with a marked geodesic

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where:

\[ \delta_i \sim \frac{9}{7} 2^i i^3 \quad \text{as} \quad i \to \infty \]
Quadrangulations with a marked geodesic

In the local limit:

\[ U_i(g) \mid_{g^n} \sim \frac{\sqrt{12^n}}{2 \pi n^{5/2}} \times \frac{3}{7} \cdot i^3 \times 3 \cdot 2^i \]
Quadrangulations with a marked geodesic

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! \[ \frac{12^n}{2 \sqrt{\pi n^{5/2}}} : \text{asymptotic number of pointed quadrangulations} \]
Quadrangulations with a marked geodesic

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\( \frac{12^n}{2 \sqrt{\pi n^{5/2}}} \): asymptotic number of pointed quadrangulations

\( \frac{3}{7} \cdot i^3 \): average number of vertices at distance \( i \) from the origin
Quadrangulations with a marked geodesic

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- $\frac{12^n}{2\sqrt{\pi n^{5/2}}}$: asymptotic number of pointed quadrangulations
- $\frac{3}{7} \cdot i^3$: average number of vertices at distance $i \geq 1$ from the origin
- $3 \cdot 2^i$: mean number of geodesics between two given points at distance $i \geq 1$
Quadrangulations with a marked geodesic

In the local limit:

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- \( \frac{12^n}{2 \sqrt{\pi n^{5/2}}} \): asymptotic number of pointed quadrangulations
- \( \frac{3}{7} \cdot i^3 \): average number of vertices at distance \( i \)' from the origin
- \( 3 \cdot 2^i \): mean number of geodesics between two given points at distance \( i \)' from the origin

A similar result holds in the scaling limit \( i = r \cdot n^{1/4} \):

\[ U_i(g)|_{g^n} \sim \frac{12^n}{2 \sqrt{\pi n^{7/4}}} \times \rho(r) \times 3 \cdot 2^i \]

\( \rho(r) \): canonical two-point function
Geodesic watermelons

Our method does not easily give access to higher moments for the number of geodesics. We shall consider quadrangulations with several marked geodesics, which might have complicated crossings.
Geodesic watermelons

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However one can consider “geodesic watermelons”: sets of $k$ non-crossing geodesics with common endpoints. These correspond to $k$ quadrangulations with geodesic boundary placed side-by-side.
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Weakly avoiding case: the whole must be irreducible

$$U_i^{(k)} = (Z_i)^k - \sum_{j=1}^{i-1} U_j^{(k)} (Z_{i-j})^k$$
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Weakly avoiding case: the whole must be irreducible

$$U_i^{(k)} = (Z_i)^k - \sum_{j=1}^{i-1} U_j^{(k)} (Z_{i-j})^k$$

Strongly avoiding case: each part must be irreducible

$$\tilde{U_i}^{(k)} = (U_i)^k$$
Geodesic watermelons

In the weakly avoiding case, in the local limit:

\[ U_i^{(k)}(g) \big|_{g^n} \sim \frac{12^n}{2 \pi n^{5/2}} \times \frac{3}{7} \cdot i^3 \times k \cdot (3 \cdot 2^i)^k \]

\( k \cdot (3 \cdot 2^i)^k \): mean number of \( k \)-watermelons
Geodesic watermelons

In the weakly avoiding case, in the local limit:

$$U_i^{(k)}(g) \bigg|_{g^n} \sim \frac{\sqrt{12^n}}{2 \pi n^{5/2}} \times \frac{3}{7} \cdot i^3 \times k \cdot (3 \cdot 2^i)^k$$

$k \cdot (3 \cdot 2^i)^k$: mean number of $k$-watermelons

The $k$ factor corresponds to symmetry breaking: among the $k$ delimited regions, only one has macroscopic ($\propto n$) size.
Geodesic watermelons

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\( k \ (3 - 2^i)^k \): mean number of \( k \)-watermelons

The \( k \) factor corresponds to symmetry breaking: among the \( k \) delimited regions, only one has macroscopic (\( \propto n \)) size.

Further computations (\( k = 2 \)):

- Two weakly avoiding geodesics of length \( i \) have in average \( i/3 \) common vertices
- They delimit two regions with respective areas \( n \) vs \( O(i^3) \)
Geodesic watermelons

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\]

\(k \cdot (3 \cdot 2^i)^k\): mean number of \(k\)-watermelons

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Further computations \((k = 2)\):

\[\begin{align*}
\text{! two weakly avoiding geodesics of length } i & \text{ have in average } i/3 \text{ common vertices} \\
\text{! they delimit two regions with respective areas } n & \text{ vs } O(i^3)
\end{align*}\]

Similar results hold in the scaling limit:

\[
U_{i}^{(k)}(g) \bigg|_{g^n} \sim \frac{\sqrt{12^n}}{2\pi n^{7/4}} \times \rho(r) \times k \cdot (3 \cdot 2^i)^k
\]
In the strongly avoiding case, in the local limit:

\[ \tilde{U}_i^{(k)}(g) \big|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} \times \frac{3 \cdot 4^{k-1}}{7} i^{6-3k} \times k \cdot (3 \cdot 2^i)^k \]
In the strongly avoiding case, in the local limit:

\[ \tilde{U}_i^{(k)}(g) \bigg|_{g^n} \sim \frac{12^n}{2} \frac{1}{\pi n^{5/2}} \times \frac{3 \cdot 4^{k-1}}{7} i^{6-3k} \times k \cdot (3 \cdot 2^i)^k \]

The constraint of strong avoidance is relevant. In the scaling limit:

\[ \tilde{U}_i^{(k)}(g) \bigg|_{g^n} \sim \frac{12^n}{2} \frac{1}{\pi n^{3k/4+1}} \times \sigma^{(k)}(r) \times k \cdot (3 \cdot 2^i)^k \]

\( \sigma^{(k)}(r) \): new scaling functions
Outline

Statistics of geodesics

Geodesic points

Geodesic loops

Confluence of geodesics
Geodesic points

Consider a quadrangulation with two marked points (1,2) at distance $i$. Consider a third point (3) lying on a geodesic between them, say at distance $s$ from 1 (hence $t = i - s$ from 2).
Geodesic points

Consider a quadrangulation with two marked points (1,2) at distance \( i \). Consider a third point (3) lying on a geodesic between them, say at distance \( s \) from 1 (hence \( t = i - s \) from 2). Apply the Miermont bijection with sources 1,2 and delays \( \tau_1 = -s \), \( \tau_2 = -t \), and obtain a well-labeled map with two faces:
Geodesic points

The generating function for such objects is

\[ \Delta_s \Delta_t X_{s,t} = X_{s,t} - X_{s-1,t} - X_{s,t-1} + X_{s-1,t-1} \]

where

\[ X_{s,t} = \frac{[3][s + 1][t + 1][s + t + 3]}{[1][s + 3][t + 3][s + t + 1]} \]

with \([m] \equiv \frac{1 - x^m}{1 - x}\)
**Geodesic points**

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Upon evaluating \(X_{s,t}\big|_{g_n}\) for \(n \to \infty\) and normalizing by the number of quadrangulations with two marked points at distance \(i = s + t\), we obtain the mean number of geodesic points:
Geodesic points

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$$\Delta_s \Delta_t X_{s,t} = X_{s,t} - X_{s-1,t} - X_{s,t-1} + X_{s-1,t-1}$$

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with $[m] \equiv \frac{1 - x^m}{1 - x}$

Upon evaluating $X_{s,t}|_{g^n}$ for $n \to \infty$ and normalizing by the number of quadrangulations with two marked points at distance $i = s + t$, we obtain the mean number of geodesic points:

$$\xi(s),_{s+t} = \frac{1}{N_{s+t}} \Delta_s \Delta_t \xi(s, t)$$

$$\xi(s,t) = \frac{9}{140} \frac{(1+s)(1+t)(3+s+t)}{(3+s)(3+t)(1+s+t)} st (29+20(s+t)+5(s^2+t^2+st))(4+s+t)$$

$$N_i = \frac{3}{35} (i+1)(5i^2+10i+2)$$
Geodesic points

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Upon evaluating $X_{s,t}|_{g^n}$ for $n \to \infty$ and normalizing by the number of quadrangulations with two marked points at distance $i = s + t$, we obtain the mean number of geodesic points:

$$\mathbb{E}(s),_{s+t} \to \frac{3s(5 + s)}{(3 + s)(2 + s)}$$

for $t \to \infty$
The generating function for such objects is

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Upon evaluating $X_{s,t}|_{g^n}$ for $n \to \infty$ and normalizing by the number of quadrangulations with two marked points at distance $i = s + t$, we obtain the mean number of geodesic points:

$$\langle \epsilon(s), s+t \rangle \to 3 \quad \text{for} \ s, t \to \infty$$
Geodesic points

\[ \langle c(s) \rangle_d \] \hspace{2cm} \langle c(s) \rangle_\infty \]
We actually have access to the full probability law for the number of geodesic points at fixed distances. The g.f. for doubly-pointed quadrangulations with exactly $c$ geodesic points at distances $s, t$ is:

$$
\Delta_s \Delta_t X^{(c)}_{s,t} \quad \text{with} \quad X^{(c)}_{s,t} = \frac{1}{c} \left( \frac{X_{s,t} - 1}{X_{s,t}} \right)^c
$$
Geodesic points

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For $s, t \to \infty$ we find the probability:

$$p_{\infty}(c) = \frac{1}{2} \left( \frac{2}{3} \right)^c$$
**Geodesic points**

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In the scaling limit, we expect all geodesic points to be at distance $o(n^{1/4})$. By this argument, Miermont was able to prove that the unicity of the geodesic between two generic points in the scaling limit of quadrangulations.
Outline

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Geodesic points

Geodesic loops

Confluence of geodesics
Geodesic loops

Consider a triply pointed quadrangulation (1,2,3) and study the length of the shortest cycle going through sho
Geodesic loops

Consider a triply pointed quadrangulation $(1,2,3)$ and study the length of the shortest cycle going through $3$ separating $1$ from $2$. $u \leq \min(d_{13}, d_{23})$
Geodesic loops

Apply the Miermont bijection with sources 1,2,3 and delays $\tau_1 = -s = u - d_{13}, \tau_2 = -t = u - d_{23}, \tau_3 = -u$. 
Geodesic loops

Apply the Miermont bijection with sources 1, 2, 3 and delays $\tau_1 = -s = u - d_{13}$, $\tau_2 = -t = u - d_{23}$, $\tau_3 = -u$. 

$\min ! (v) = 1 - s$ 
$\min ! (v) = 0$ 
$\min ! (v) = 0$ 

$\min ! (v) = 1 - u$ 
$\min ! (v) = 1 - t$ 
$\min ! (v) = 0$
Geodesic loops

Apply the Miermont bijection with sources 1, 2, 3 and delays
\(\tau_1 = -s = u - d_{13}, \quad \tau_2 = -t = u - d_{23}, \quad \tau_3 = -u.\)
Geodesic loops

We arrive at a generating function:

\[ \tilde{G}(d_{13}, d_{23}, u) = \Delta_s \Delta_t \Delta_u \tilde{F}(s, t, u) \bigg|_{s=d_{13}-u}^{t=d_{23}-u} \]

where

\[ \tilde{F}(s, t, u) = X_{s,u} X_{t,u} X_{u,u} Y_{s,u,u} Y_{t,u,u} \]

\[ = \frac{[3][s+1][t+1][u+1]^4[s+2u+3][t+2u+3]}{[1]^3[s+u+1][s+u+3][t+u+1][t+u+3][2u+1][2u+3]} \]
Geodesic loops

We arrive at a generating function:

$$\tilde{G}(d_{13}, d_{23}, u) = \Delta_s \Delta_t \Delta_u \tilde{F}(s, t, u)\bigg|_{s=d_{13}-u \atop t=d_{23}-u}$$

where

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$$= \frac{[3][s+1][t+1][u+1]^4[s+2u+3][t+2u+3]}{[1]^3[s+u+1][s+u+3][t+u+1][t+u+3][2u+1][2u+3]}$$

We may sum over $d_{13}, d_{23}$ and find:

$$\tilde{G}(u) = \Delta_u \left( \frac{[3][u+1]^4}{[1]^3[2u+1][2u+3]} \right)$$
Geodesic loops

We arrive at a generating function:

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where

\[ \tilde{F}(s, t, u) = X_{s,u} X_{t,u} X_{u,u} Y_{s,u,u} Y_{t,u,u} \]

\[ = \frac{[3][s+1][t+1][u+1]^4[s+2u+3][t+2u+3]}{[1]^3[s+u+1][s+u+3][t+u+1][t+u+3][2u+1][2u+3]} \]

We may sum over \( d_{13}, d_{23} \) and find:

\[ \tilde{G}(u) = \Delta_u \left( \frac{[3][u+1]^4}{[1]^3[2u+1][2u+3]} \right) \]

We readily perform the scaling limit and find the law for \( U = u \cdot n^{-1/4} \):

\[ \rho(\tilde{U}) = -\frac{4}{i\pi} \int_{-\infty}^{\infty} d\xi e^{-\xi^2} \partial_U \left( \frac{\sinh^4(U\sqrt{-3i\xi/2})}{\sinh^2(2U\sqrt{-3i\xi/2})} \right) \]
Geodesic loops

$\bar{\rho}(U)$
Geodesic loops

\[ \bar{\rho}(U) \]

\[ \bar{\rho}(U) \sim 3U^3 \quad \text{for } U \to 0 \]

We can also plot:

\[ \bar{\rho}(D_{13}, D_{23}|U) = \frac{\bar{\rho}(D_{13}, D_{23}, U)}{\bar{\rho}(U)}. \]
Geodesic loops
Geodesic loops

Asymptotic regimes:

1. $U$: one distance is $\propto U$, the other finite.

$$
\bar{\rho}(D_{13}, D_{23}, U) \sim \frac{1}{2} \left( \rho(D_{13}) \frac{1}{U} \psi \left( \frac{D_{23}}{U} \right) + \rho(D_{23}) \frac{1}{U} \psi \left( \frac{D_{13}}{U} \right) \right)
$$

with

$$
\psi(z) = \frac{3}{2} \cdot \frac{2z - 1}{z^4} \quad z \in [1, \infty)
$$

Consistent with the absence of microscopic cycles separating two macroscopic components.

1. $U'$: both distances are $U + O(U^{-1/3})$

$$
\bar{\rho}(D_{13}, D_{23}, U) \sim (9U)^{2/3} \Phi \left( (D_{13} - U)(9U)^{1/3}, (D_{23} - U)(9U)^{1/3} \right)
$$

with

$$
\Phi(z, z') = e^{-(z+z')} \left( 2 - e^{-z} - e^{-z'} \right).
$$
Outline

Statistics of geodesics

Geodesic points

Geodesic loops

Confluence of geodesics
Le Gall has shown the surprising phenomenon of *confluence* of geodesics.
Confluence of geodesics

Consider the tree obtained by Schaeffer’s bijection with \( v_3 \) as origin:
Confluence of geodesics

Consider the tree obtained by Schaeffer’s bijection with $v_3$ as origin:

$$\min = 1 + \delta$$

$$\min = 1$$
Confluence of geodesics

Consider the tree obtained by Schaeffer’s bijection with $v_3$ as origin:

\[ \delta \]
Confluence of geodesics

Consider the tree obtained by Schaeffer’s bijection with $v_3$ as origin:
Confluence of geodesics

We were able to compute the continuous law for $\delta$ ($\delta \to \delta \cdot n^{-1/4}$):

$$
\tilde{\rho}(\delta) = \frac{\sqrt{3}}{\pi} \int_{-\infty}^{\infty} d\xi \ e^{-\xi^2} \sqrt{-3i\xi} \ e^{-2\delta} \sqrt{-3i\xi/2}
$$
Confluence of geodesics

The shape of a triangle will actually look like:
Confluence of geodesics

The shape of a triangle will actually look like:

Our computation of the three-point function can be refined into an expression involving six parameters: $d_{12}, d_{23}, d_{23}, \delta_1, \delta_2, \delta_3$. 
Confluence of geodesics

\[
\min = 1 - s'
\]

\[
\min = 1 - s''
\]

\[
\max(s', s'') = s = \frac{d_{12} + d_{31} - d_{23}}{2}
\]

\[
|s' - s''| = \delta_1
\]
Confluence of geodesics

Similarly we introduce the parameters $t', t'', u', u''$. We arrive at a generating function:

$$
\Delta_{s'}\Delta_{s''}\Delta_{t'}\Delta_{t''}\Delta_{u'}\Delta_{u''} \left( Y_{s',t',u'} Y_{s'',t'',u''} X_{s',t'} X_{s'',t''} \right)
$$
Confluence of geodesics

Similarly we introduce the parameters $t', t'', u', u''$. We arrive at a generating function:

$$\Delta_s'\Delta_s''\Delta_t'\Delta_t''\Delta_u'\Delta_u'' \left( Y_{s',t',u'} Y_{s'',t'',u''} X_{s',t',u'} X_{s'',t'',u''} \right)$$

Conventions for $X$ become irrelevant in the scaling limit:

$$\frac{3}{\alpha^2} Y(S', T', U'; \alpha) Y(S'', T'', U''; \alpha)$$

$$Y(S, T, U; \alpha) = \frac{\sinh(\alpha S) \sinh(\alpha T) \sinh(\alpha U) \sinh(\alpha(S + T + U))}{\sinh(\alpha(S + T)) \sinh(\alpha(T + U)) \sinh(\alpha(U + S))}$$
Confluence of geodesics

Similarly we introduce the parameters $t', t'', u', u''$. We arrive at a generating function:

$$
\Delta_s'\Delta_s''\Delta_t'\Delta_t''\Delta_u'\Delta_u'' \left( Y_{s'}, t', u' \ Y_{s''}, t'', u'' \ X_{s'}, t', u' \ X_{s''}, t'', u'' \right)
$$

Conventions for $X$ become irrelevant in the scaling limit:

$$
\frac{3}{\alpha^2} Y(S', T', U'; \alpha) Y(S'', T'', U''; \alpha)
$$

$$
Y(S, T, U; \alpha) = \frac{\sinh(\alpha S) \sinh(\alpha T) \sinh(\alpha U) \sinh(\alpha(S + T + U))}{\sinh(\alpha(S + T)) \sinh(\alpha(T + U)) \sinh(\alpha(U + S))}
$$

In the canonical ensemble we find a probability density function:

$$
\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} d\xi \ \frac{\xi}{i} e^{-\xi^2} (\cdots) \bigg|_{\alpha = \sqrt{\frac{3i\xi}{2}}}
$$
Confluence of geodesics

We can compute some marginal laws. $\delta_1 = \delta$ was seen before.
Confluence of geodesics

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\[ \tilde{G}(S, \delta_1; \alpha) = 6e^{-4\alpha S} e^{2\alpha \delta_1} \quad S > \delta_1 > 0 \]
Confluence of geodesics

Side of the “inner” triangle:

\[
\delta_{12} = D_{12} - \delta_1 - \delta_2
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We can also study the area of the inner triangle. We find it has an area \( \beta_n \) where \( \beta \in [0, 1] \) has density:

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\sqrt{\pi} \frac{1}{\Gamma(1/4)^2 (\beta(1 - \beta))^{3/4}}
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\[
\hat{\rho}(\delta_{12}) \propto \sqrt{\Gamma(1/4)^2 \pi} \frac{1}{\Gamma(1/4)^2 (\beta(1 - \beta))^{3/4}}
\]

(same as the area within a geodesic loop)
Conclusion

We have computed a number of properties of geodesics in planar quadrangulations, both in the local and scaling limit.

- The mean number of geodesics between two given points at distance $i$ is $3 \cdot 2^i$.
- The mean number of geodesic points at a given generic position is 3.
- Geodesic loops and confluence of geodesics can be quantitatively studied.

Still, the structure of a large random quadrangulation remains mysterious, inbetween tree and sphere.