The complete counting function of Gessel walks is algebraic

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Context: nearest-neighbour walks in $\mathbb{N}^2$

▷ *Set of admissible steps* $\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$.

▷ $\mathfrak{S}$-*walks* = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in $\mathfrak{S}$.

▷ $f(n;i,j)$ = number of $\mathfrak{S}$-walks ending at (i,j) and consisting of exactly n steps. Complete generating function

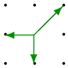
$$F(t;x,y) = \sum_{n=0}^{\infty}\Big(\sum_{i,j=0}^{\infty} f(n;i,j)x^i y^j\Big)t^n \in \mathbb{Q}[x,y][[t]].$$

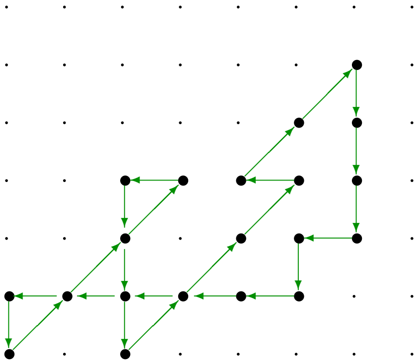
Questions: Starting from $\mathfrak{S}$, what can be said about $F(t;x,y)$? Is it *algebraic*, or *holonomic transcendental*, or *non-holonomic*?

$F(t;1,1) \rightsquigarrow$ number of walks with prescribed number of steps;
$F(t;0,0) \rightsquigarrow$ number of walks returning to the origin (excursions);
$F(t;1,0) \rightsquigarrow$ number of walks ending on the horizontal axis.

Examples: Kreweras and Gessel walks

Kreweras walks: $\mathfrak{S}$-walks with $\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$
Gessel walks: $\mathfrak{S}$-walks with $\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$





Example: A Kreweras excursion of length 24.

Main results

Theorem (Kreweras 1965; 100 pages combinatorial proof!)

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)}\,t^{3n}.$$

Theorem (Gessel's conjecture; Kauers, Koutschan, Zeilberger 2008)

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

Question: What about $K(t;x,y)$ and $G(t;x,y)$?

Theorem (Bousquet-Mélou 2005) $K(t;x,y)$ is algebraic.

Theorem (B. & Kauers 2008) $G(t;x,y)$ is algebraic.
In particular, $g(n;i,j)$ is holonomic for any pair $(i,j) \in \mathbb{N}^2$.

$\longrightarrow$ Effective, computer-driven, discovery and proof.

Methodology

Experimental mathematics approach:

(S1) high order expansions of generating power series;

(S2) guessing differential and/or algebraic equations they satisfy;

(S3) empirical certification of the guessed equations (sieving by inspection of their analytic, algebraic, arithmetic properties);

(S4) rigorous proof, based on (exact) polynomial computations.

Step (S1): high order series expansions

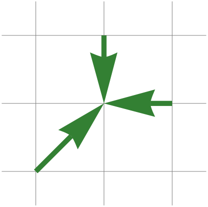
$f(n;i,j)$ satisfies the recurrence with constant coefficients

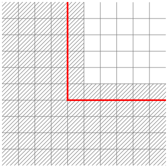
$$f(n+1;i,j) = \sum_{(u,v)\in\mathfrak{S}} f(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

$+$ init. cond. $f(0;i,j) = \delta_{0,i,j}$ and $f(n;-1,j) = f(n;i,-1) = 0$.

Example: for the Kreweras walks,

$$\begin{aligned} k(n;i,j) &= k(n-1;i+1,j) \\ &\quad + k(n-1;i,j+1) \\ &\quad + k(n-1;i-1,j-1) \end{aligned}$$





The recurrence is used to compute $F(t;x,y) \bmod t^N$ for large N.

$$\begin{aligned} K(t;x,y) &= 1 + xyt + (x^2y^2+y+x)t^2 + (x^3y^3+2xy^2+2x^2y+2)t^3 \\ &\quad + (x^4y^4+3x^2y^3+3x^3y^2+2y^2+6xy+2x^2)t^4 \\ &\quad + (x^5y^5+4x^3y^4+4x^4y^3+5xy^3+12x^2y^2+5x^3y+8y+8x)t^5+\cdots \end{aligned}$$

Step (S2): guessing equations for $F(t;x,y)$, a first idea

In terms of generating series, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}&\big(xy - t(x+y+x^2y^2)\big)\,K(t;x,y)\\ &\qquad = xy - tx\,K(t;x,0) - ty\,K(t;0,y)\end{aligned}} \qquad \text{(KerEq)}$$

▷ This *kernel equation* is simply a multivariate analogue of $(1-t-t^2)\cdot\sum_{n\geq 0}\mathfrak{F}_n t^n = 1,$ where $\mathfrak{F}_n$ are the Fibonacci numbers.

▷ A similar kernel equation holds for $F(t;x,y)$, for any $\mathfrak{S}$-walk.

Corollary. $F(t;x,y)$ is holonomic (resp. algebraic) if and only if $F(t;x,0)$ and $F(t;0,y)$ are both holonomic (resp. algebraic).

▷ This simplification is crucial: equations for $G(t;x,y)$ are huge.

Step (S2): guessing equations for $F(t;x,0)$ and $F(t;0,y)$

Task 1: Given the first N terms of $S = F(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a *differential equation* satisfied by S at precision N:

$$\mathcal{L}_{x,0}(S) = c_r(x,t)\cdot\frac{d^r S}{dt^r} + \cdots + c_1(x,t)\cdot\frac{dS}{dt} + c_0(x,t)\cdot S = 0 \bmod t^N.$$

Task 2: Search for an *algebraic equation* $\mathcal{P}_{x,0}(S) = 0 \bmod t^N$.

- Both tasks amount to linear algebra in size N over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- (Right) gcds of several candidates provide minimal equations.

Step (S2): guessing equations for $K(t;x,0)$

The guessed *operator* of order 4 in $D_t = \frac{d}{dt}$, degree $(14,11)$ in (t,x)

$$\begin{aligned}\mathcal{L}_{x,0} = {} & t^3 \cdot (3t-1) \cdot (9t^2+3t+1) \cdot (3t^2+24t^2x^3-3xt-2x^2) \cdot \\ & \cdot (16t^2x^5+4x^4-72t^4x^3-18x^3t+5t^2x^2+18xt^3-9t^4) \cdot \\ & \cdot (4t^2x^3-t^2+2xt-x^2) \cdot D_t^4 + \cdots\end{aligned}$$

is such that $\mathcal{L}_{x,0}(K(t;x,0)) = 0 \bmod t^{100}$.

The guessed *polynomial* of tridegree $(6,10,6)$ in (T,t,x)

$$\begin{aligned}\mathcal{P}_{x,0} = {} & x^6t^{10}T^6 - 3x^4t^8(x-2t)T^5 + \\ & + x^2t^6\left(12t^2+3t^2x^3-12xt+\frac{7}{2}x^2\right)T^4 + \cdots\end{aligned}$$

is such that $\mathcal{P}_{x,0}(K(t;x,0),t,x) = 0 \bmod t^{100}$.

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

For Gessel walks, using $N = 1000$ terms of $G(t;x,y)$, we guessed

- $\mathcal{L}_{x,0}$: order 11 in D_t, bidegree $(96,78)$ in (t,x), 61 digits coeffs
- $\mathcal{L}_{0,y}$: order 11 in D_t, bidegree $(68,28)$ in (t,y), 51 digits coeffs

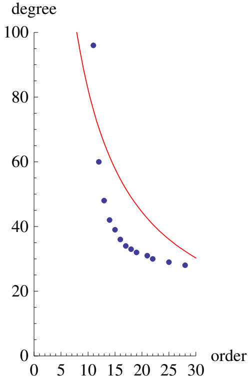
such that $\quad \mathcal{L}_{x,0}(G(t;x,0)) = \mathcal{L}_{0,y}(G(t;0,y)) = 0 \bmod t^{1000}$.

• For a *fixed value* x_0, and *modulo a prime* p, many (non-minimal) operators in $\mathbb{Z}_p[t]\langle D_t\rangle$ for $G(t;x_0,0)$ can be guessed by Hermite-Padé.

• *Still:* reconstructing from one of them an operator in $\mathbb{Q}[t,x]\langle D_t\rangle$ for $G(t;x,0)$ is *too costly*.

• *However,* the reconstruction (wrt x) is feasible if applied to the *minimal-order* operator = *gcrd*.

▷ Guessing $\mathcal{L}_{x,0}$ by *undetermined coefficients* would have required solving a dense linear system 91956×91956 with ≈ 1000 digits entries!



Step (S2): guessing equations for $G(t; x, y)$?

Feasible in principle: kernel equation + closure by differential lclm.

• Obstacle: this lclm has order 20 in D_t, tridegree $(359, 717, 279)$ in $(t, x, y) \to$ size of 1.5 billion integer coefficients (!)

• Thus: at this point, we had *guesses* for differential equations for $G(t; x, 0)$ and $G(t; 0, y)$, but *no proof* that they are correct and *no hope* to compute a candidate differential equation for $G(t; x, y)$.

▷ Remember: it was believed (e.g. by Gessel and Zeilberger, who popularized the problem) that $G(t; x, y)$ is not algebraic.

▷ This explains why no one (including us) tried – at this stage – to search for algebraic equations. Worse: no one even remarked that Gessel's expression ${}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix} \middle| \, 16t^2\right)$ for excursions *is algebraic*.

Step (S3): empirical certification of guesses

Provide convincing evidence that the candidate $\mathcal{L}_{x,0}$ is correct:

1. Size sieve: $\mathcal{L}_{x,0}$ has reasonable bit size compared to an artefact solution of the Hermite-Padé approximation problem.
2. Algebraic sieve: High order series matching.
   $\mathcal{L}_{x,0}$ verifies $\mathcal{L}_{x,0}(F(t;x,0)) = 0 \bmod t^{2 \cdot N}$.
3. Analytic sieve: singularity analysis.
   $\mathcal{L}_{x,0}$ is Fuchsian (all of its singular points are regular singular).
4. Arithmetic sieve: $\mathcal{L}_{x,0}$ is globally nilpotent (see below).

Step (S3): G-series and global nilpotence

Def. A power series $\sum_{n\geq 0} \frac{a_n}{b_n} t^n$ in $\mathbb{Q}[[t]]$ is called a *G-series* if it is (a) holonomic; (b) analytic at $t=0$; (c) $\exists C>0$, $\operatorname{lcm}(b_0, \dots, b_n) \leq C^n$.

Examples: ${}_2F_1\left(\begin{matrix}\alpha\ \beta\\ \gamma\end{matrix}\middle|\, t\right)$, $\alpha, \beta, \gamma \in \mathbb{Q}$; algebraic functions (Eisenstein).

Thm. (Chudnovsky 1985) The minimal-order differential operator annihilating a G-series is *globally nilpotent*: for almost all prime numbers p, it right-divides $D_t^{p\mu}$ modulo p, for some $\mu \in \mathbb{N}$.

Examples: $t(1-t)D_t^2 + (\gamma - (\alpha+\beta+1)t)D_t - \alpha\beta t$; algebraic resolvents.

Thm. (B. & Kauers) If $F(t; x, 0)$ is holonomic, then it's a G-series.

▷ The guessed operators for $K(t; x, 0)$, $G(t; x, 0)$, $G(t; 0, y)$ pass this arithmetic test: they are all globally nilpotent.

▷ And, unexpectedly, even more. . .

Step (S3): Grothendieck's conjecture and the big surprise

Conjecture (Grothendieck) $\mathcal{L}(S) = 0$ possesses a basis of *algebraic solutions* if and only if $\mathcal{L}$ globally nilpotent with $\mu = 1$.

▷ Big surprise: the guessed operators for $G(t; x, 0)$ and $G(t; 0, y)$ share this property for $5 \leq p < 100 \Rightarrow$ this strongly indicates that $G(t; x, 0)$ and $G(t; 0, y)$, and thus $G(t; x, y)$, *should be algebraic!*

Once we suspect algebraicity of $G(t; x, 0)$ and $G(t; 0, y)$, we guess candidates for annihilating polynomials

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[x, t, T]$ of tridegree $(32, 43, 24)$ in (x, t, T), 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[y, t, T]$ of tridegree $(40, 44, 24)$ in (y, t, T), 23 digits

such that

$$\mathcal{P}_{x,0}(x, t, G(t; x, 0)) = \mathcal{P}_{0,y}(x, t, G(t; 0, y)) = 0 \bmod t^{1200}.$$

Step (S4): warm-up – Gessel excursions

Theorem $G(t;0,0) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}$ *is algebraic*.

Proof 1: This ${}_3F_2$ series is a ${}_2F_1$ series in disguise:

$$ {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix} \middle| 16t^2\right) = \frac{1}{t^2}\left(\frac{1}{2}\, {}_2F_1\left(\begin{matrix} -1/6 & -1/2 \\ & 2/3 \end{matrix} \middle| 16t^2\right) - \frac{1}{2}\right). $$

Schwarz's classification of algebraic ${}_2F_1$'s allows to conclude.

Proof 2: *Guess* a polynomial $P(T,t)$ in $\mathbb{Q}[T,t]$ and then *prove* that P admits the power series $g(t) = G(\sqrt{t};0,0)$ as a root.

1. Such a P can be *guessed* from the first 100 terms of $g(t)$.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.
3. $r(t) = \sum_{n=0}^{\infty} g_n t^n$ being algebraic, it is holonomic, and so is (g_n):

$$ (n+2)(3n+5)g_{n+1} - 4(6n+5)(2n+1)g_n = 0, \qquad g_0 = 1. $$

$\longrightarrow$ solution $g_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}16^n$, thus $g(t) = r(t)$ is algebraic.

Step (S4): rigorous proof for Kreweras walks

1. Setting $y_0(t,x) = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2}$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\stackrel{!}{=}0} K(t;x,y) = -xy + xtK(t;x,0) + ytK(t;0,y)$$

shows that $U = K(t;x,0)$ satisfies the *reduced kernel equation*

$$\boxed{x \cdot y_0 - x \cdot t \cdot U(t,x) = y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. $U = K(t;x,0)$ is the *unique solution* in $\mathbb{Q}[[x,t]]$ of (RKerEq).

3. The guessed candidate $\mathcal{P}_{x,0}$ has one solution $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. Resultant computations + verification of initial terms
   $\Longrightarrow \quad U = H(t,x)$ also satisfies (RKerEq).

5. *Uniqueness*: $H(t,x) = K(t;x,0) \Rightarrow K(t;x,0)$ is algebraic!

Algebraicity of Kreweras walks: our Maple proof in a nutshell

```
[bostan@venus ~]$ maple
    |\^/|     Maple 11 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2007
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.

# high order expansion (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> prec:=80:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..prec),t,prec-1):
# guessing (S2)
> libname:=".",libname:gfun:-version();
                                         3.35
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T):
# rigorous proof (S4)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                           1
> nops(gfun:-algeqtoseries(p1,t,T,4,true));
                                           1
# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                        15, 657
```

Step (S4): rigorous proof for Gessel walks

Two difficulties: $G(t;x,y) \neq G(t;y,x)$ and $G(t;0,0)$ occurs in (KerEq)

$$\underbrace{\text{poly}(x,y,t)}_{\stackrel{!}{=}0} G(t;x,y) = xy + tG(t;0,0) - (1+y)tG(t;0,y) - tG(t;x,0)$$

$$\implies \quad y_0(t,x) = 0 + \frac{1}{x}t + \frac{x^2+1}{x^2}t^2 + \frac{x^4+3x^2+1}{x^3}t^3 + \cdots$$

$$\text{or} \quad x_0(t,y) = 0 + \frac{y+1}{y}t + \frac{(y+1)^3}{y^2}t^3 + \frac{2(y+1)^5}{y^3}t^5 + \cdots$$

This gives *two* equations connecting $G(t;x,0)$ and $G(t;0,y)$:

$$G(t;x,0) = xy_0/t + G(t;0,0) - (1+y_0)G(t;0,y_0)$$

$$(1+y)G(t;0,y) = yx_0/t + G(t;0,0) - G(t;x_0,0)$$

For fixed $G(t;0,0)$, they uniquely define $G(t;x,0)$ and $G(t;0,y)$.

▷ Resultant size: $\deg_T = 48, \deg_t = 90, \deg_y = 64$, digits $= 47$
$\longrightarrow$ fast algorithms needed (*B., Flajolet, Salvy & Schost 2006*).

Conclusion

1. Guess'n'Prove approach based on modern CA algorithms.
2. Brute-force approach and/or use of naive algorithms = hopeless.
   E.g. size of algebraic equations for $G(t; x, y) \approx$ 30Gb.
3. Going further: experimental classification of 2D and 3D walks:
   (*B. & Kauers FPSAC'09*) $\rightarrow$ 3500 cases treated; $\approx$ 4% holonomic.
   Matches the results of Bousquet-Mélou and Mishna (2D).
4. Remarkable properties discovered experimentally: explanation?
   4.1 algebraic cases: solvable Galois groups + genus 0, 1 and 5(!)

$$G(t;1,1) = -\frac{3}{6t} + \frac{\sqrt{3}}{6t}\sqrt{U(t) + \sqrt{\frac{16t(2t+3)+2}{(1-4t)^2U(t)} - U(t)^2 + 3}}$$

where $\quad U(t) = \sqrt{1 + 4t^{1/3}(4t+1)^{1/3}/(4t-1)^{4/3}}.$

   4.2 transcendental holonomic: operators factor as $L^{(2)} \cdot L_1^{(1)} \cdots L_t^{(1)}$
   $\longrightarrow$ iterated integrals of ${}_2F_1$'s (Dwork's conjecture)

F  $(t;0,0) = \frac{2}{t^2}\int_0^t \frac{\tau(1-12\tau^2)(4\tau^2+1)}{(1-4\tau^2)^{5/2}} \cdot {}_2F_1\left(\begin{matrix}5/4 & 7/4\\ & 2\end{matrix}\,\middle|\, \frac{64\tau^4}{(1-4\tau^2)^2}\right)d\tau.$

   4.3 all: $[t^n]\,F(t;1,1)$ grows like $\kappa n^\alpha \rho^n$ for $\kappa \in \mathbb{R}$, $\alpha \in \mathbb{Q}_-$, $\rho \in \overline{\mathbb{Q}}$.
   Moreover: $\rho \leq |\mathfrak{S}|$, with equality iff $\mathrm{drift}(\mathfrak{S}) = \sum_{s\in\mathfrak{S}} s \in \mathbb{N}^2$.

Experimental classification of 2D walks with holonomic $F(t;1,1)$

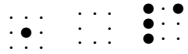
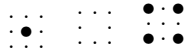
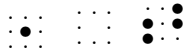
OEIS Tag	Sample step set	Equation sizes			OEIS Tag	Sample step set	Equation sizes		
A000012		1, 0	1, 1	1, 1	A000079		1, 0	1, 1	1, 1
A001405		2, 1	2, 3	2, 2	A000244		1, 0	1, 1	1, 1
A001006		2, 1	2, 3	2, 2	A005773		2, 1	2, 3	2, 2
A126087		3, 1	2, 5	2, 2	A151255		6, 8	4, 16	–
A151265		6, 4	4, 9	6, 8	A151266		7, 10	5, 16	–
A151278		7, 4	4, 12	6, 8	A151281		3, 1	2, 5	2, 2
A005558		2, 3	3, 5	–	A005566		2, 2	3, 4	–
A018224		2, 3	3, 5	–	A060899		2, 1	2, 3	2, 2
A060900		2, 3	3, 5	8, 9	A128386		3, 1	2, 5	2, 2
A129637		3, 1	2, 5	2, 2	A151261		5, 8	4, 15	–
A151282		3, 1	2, 5	2, 2	A151291		6, 10	5, 15	–
A151275		9, 18	5, 24	–	A151287		7, 11	5, 19	–
A151292		3, 1	2, 5	2, 2	A151302		9, 18	5, 24	–
A151307		8, 15	5, 20	–	A151318		2, 1	2, 3	2, 2
A129400		2, 1	2, 3	2, 2	A151297		7, 11	5, 18	–
A151312		4, 5	3, 8	–	A151323		2, 1	2, 3	4, 4
A151326		7, 14	5, 18	–	A151314		9, 18	5, 24	–
A151329		9, 18	5, 24	–	A151331		3, 4	3, 6	–

Equation sizes = {order, degree}(rec, diffeq, algeq).

Experimental classification of 2D walks with holonomic $F(t;1,1)$

OEIS Tag	Steps	Equation sizes			Asymptotics	OEIS Tag	Steps	Equation sizes			Asymptotics
A000012		1, 0	1, 1	1, 1	1	A000079		1, 0	1, 1	1, 1	2^n
A001405		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n}{\sqrt{n}}$	A000244		1, 0	1, 1	1, 1	3^n
A001006		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$	A005773		2, 1	2, 3	2, 2	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A126087		3, 1	2, 5	2, 2	$\frac{12\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^{3n/2}}{n^{3/2}}$	A151255		6, 8	4, 16	–	$\frac{24\sqrt{2}}{\pi}\frac{2^{3n/2}}{n^2}$
A151265		6, 4	4, 9	6, 8	$\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151266		7, 10	5, 16	–	$\frac{\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A151278		7, 4	4, 12	6, 8	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151281		3, 1	2, 5	2, 2	$\frac{1}{2}3^n$
A005558		2, 3	3, 5	–	$\frac{8}{\pi}\frac{4^n}{n^2}$	A005566		2, 2	3, 4	–	$\frac{4}{\pi}\frac{4^n}{n}$
A018224		2, 3	3, 5	–	$\frac{2}{\pi}\frac{4^n}{n}$	A060899		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A060900		2, 3	3, 5	8, 9	$\frac{4\sqrt{3}}{3\Gamma(\frac{1}{3})}\frac{4^n}{n^{2/3}}$	A128386		3, 1	2, 5	2, 2	$\frac{6\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n 3^{n/2}}{n^{3/2}}$
A129637		3, 1	2, 5	2, 2	$\frac{1}{2}4^n$	A151261		5, 8	4, 15	–	$\frac{12\sqrt{3}}{\pi}\frac{2^n 3^{n/2}}{n^2}$
A151282		3, 1	2, 5	2, 2	$\frac{A^2 B^{3/2}}{2^{3/4}\Gamma(\frac{1}{2})}\frac{B^n}{n^{3/2}}$	A151291		6, 10	5, 15	–	$\frac{4}{3\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A151275		9, 18	5, 24	–	$\frac{12\sqrt{30}}{\pi}\frac{(\sqrt{24})^n}{n^2}$	A151287		7, 11	5, 19	–	$\frac{\sqrt{8}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
A151292		3, 1	2, 5	2, 2	$\frac{\sqrt[4]{3}C^2 D^{3/2}}{8\Gamma(\frac{1}{2})}\frac{D^n}{n^{3/2}}$	A151302		9, 18	5, 24	–	$\frac{\sqrt{5}}{3\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
A151307		8, 15	5, 20	–	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$	A151318		2, 1	2, 3	2, 2	$\frac{\sqrt{5/2}}{\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
A129400		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{6^n}{n^{3/2}}$	A151297		7, 11	5, 18	–	$\frac{\sqrt{3}C^{7/2}}{2\pi}\frac{(2C)^n}{n^2}$
A151312		4, 5	3, 8	–	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	A151323		2, 1	2, 3	4, 4	$\frac{\sqrt{2}\,3^{3/4}}{\Gamma(\frac{1}{4})}\frac{6^n}{n^{3/4}}$
A151326		7, 14	5, 18	–	$\frac{2\sqrt{3}}{3\Gamma(\frac{1}{2})}\frac{6^n}{\sqrt{n}}$	A151314		9, 18	5, 24	–	$\frac{EF^{7/2}}{5\sqrt{95}\pi}\frac{(2F)^n}{n^2}$
A151329		9, 18	5, 24	–	$\frac{\sqrt{7/3}}{3\Gamma(\frac{1}{2})}\frac{7^n}{\sqrt{n}}$	A151331		3, 4	3, 6	–	$\frac{8}{3\pi}\frac{8^n}{n}$

Experimental classification of holonomic transcendental 3D walks

OEIS Tag	Step sets	Equation sizes		OEIS Tag	Step sets	Equation sizes	
A148060		9, 17	5, 28	A148438		7, 10	5, 17
A149090		9, 17	5, 28	A149589		10, 21	6, 29
A005817		2, 2	3, 4	A148005		5, 8	4, 15
A148052		7, 18	6, 27	A148068		7, 17	6, 25
A148072		12, 57	10, 69	A148162		4, 3	3, 6
A148284		14, 57	10, 71	A148331		11, 43	9, 53
A148507		4, 6	4, 11	A148525		7, 16	6, 25
A148548		7, 19	6, 28	A148689		8, 25	8, 31
A148703		4, 3	3, 6	A148790		6, 12	5, 18
A148934		5, 5	4, 11	A149279		14, 62	10, 75
A149290		11, 53	9, 61	A149363		7, 16	6, 24
A149632		7, 11	5, 16	A149713		8, 22	7, 29
A150054		12, 39	9, 52	A150370		14, 62	10, 75
A150410		4, 6	4, 11	A150471		12, 33	8, 42
A150499		14, 48	9, 61	A150764		7, 13	6, 19
A150950		8, 23	7, 29	A151053		14, 38	9, 48

Equation sizes = {order, degree}(rec, diffeq).