Deformation techniques for triangular arithmetic

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Triangular sets

Def.

- A triangular set is a family of polynomial $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ in $R[x_1, \ldots, x_n]$, where
  - R a ring;
  - T_i is in $R[x_1, \ldots, x_i]$;
  - T_i is monic in X_i;
  - T_i is reduced modulo $T_1, \ldots, T_{i-1}$.

Example

$$
\begin{aligned}
T_1(x_1) &= x_1^2 + 3x_1 \\
T_2(x_1, x_2) &= x_2^2 + x_2 x_1
\end{aligned}
$$

Why?

All-purpose polynomial system solving

- 1980's, 1990's: Wu, Kalkbrenner, Lazard, Wang, Aubry, Moreno Maza
- still a lot of hard questions, especially in positive dimension.

The appropriate data-structure for a lot of more specialized questions

- often situations involving some form of symmetry;
- Galois theory, crypto, factoring in algebraic extensions, ...
- often in dimension zero.

Goal of this work

Our question.

- Given $\mathbf{T}$ and A and B both reduced modulo $\langle \mathbf{T} \rangle$, how much does it cost to compute $C = AB \bmod \langle \mathbf{T} \rangle$?

Example (continued)

$$
\begin{aligned}
A &= x_1x_2 + x_2 + x_1 + 1 \\
B &= x_1x_2 + x_2 + x_1 + 1 \\
AB &= x_1^2x_2^2 + 2x_2^2x_1 + 2x_2x_1^2 + 4x_1x_2 + x_2^2 + 2x_2 + x_1^2 + 2x_1 + 1 \\
C &= -6x_2x_1 + 2x_2 - x_1 + 1
\end{aligned}
$$

Why?

1. General philosophy: as with univariate polynomials, multiplication controls all other operations (especially when no splitting happens).

- inversion;
- resultant;
- Hensel lifting;
- change of order;
- $\ldots$

2. Some concrete examples:

- see later.

Previous work

The complexity measure is

$$\delta = d_1 \cdots d_n, \quad \text{with} \quad d_i = \deg(T_i, x_i)$$

(this is essentially the input and output size).

Theo. [Li, Moreno Maza, Schost]

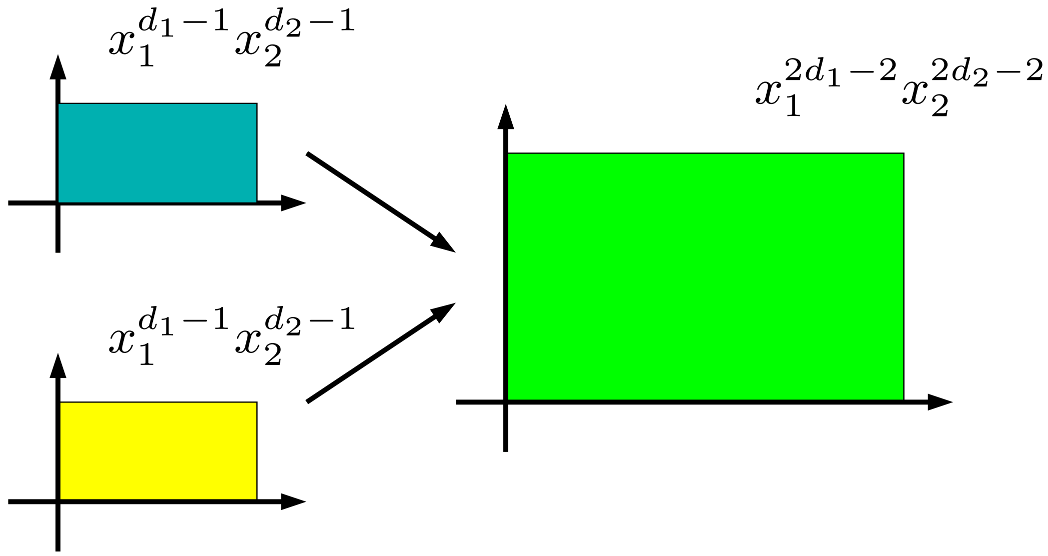
- The product $AB \bmod \langle \mathbf{T} \rangle$ can be computed in time $O\tilde{}(4^n \delta)$.

The notation $O\tilde{}$ hides logarithmic factors.

Theo. [Li, Moreno Maza, Schost]

- Let $r = \max d_i$.
- Suppose that for all i, T_i is in $R[x_i]$, and that R has r interpolation points. Then the product $AB \bmod \langle \mathbf{T} \rangle$ can be computed in time $O\tilde{}(\delta r)$.

The problem with multivariate polynomials



In n variables,

- polynomial multiplication induces a 2^n overhead;
- so triangular multiplication should avoid expansion.
- Evaluation / interpolation is the key.

This work

We extend the second approach of Li et al. to a few more general situations.

- The support of the polynomials determines the complexity;
- for nice cases, we get results of the same form

$$O\tilde{}(\delta r) \quad \text{with} \quad r = \sum d_i.$$

Remark

- For $d_i = 2$ and $\delta = 2^n$, taking the best of both approaches (general/specialized) gives

$$\delta \left(e^{\sqrt{\log \delta}}\right)^{O(1)}.$$

Examples

Power series

The monomial ideal

$$\mathbf{T} = \left| \begin{array}{l} x_n^{d_n} \\ \vdots \\ x_2^{d_2} \\ x_1^{d_1} \end{array} \right.$$

is triangular.

No quasi-linear algorithm was known until Schost, 2005.

Polynomial multiplication

Reductions:

- to multiply polynomials of degree d, enough to multiply series mod x^d;
- we can take $d = 2^k$.

From uni- to multivariate (with $d = 8 = 2^3$)

- write $x = x_0$, and introduce x_1, x_2;
- use the equality between ideals

$$\left|\begin{array}{l} x_2 - x_0^4 \\ x_1 - x_0^2 \\ x_0^8 \end{array}\right. \quad = \quad \left|\begin{array}{l} x_0^2 - x_1 \\ x_1^2 - x_2 \\ x_2^2 \end{array}\right.$$

- change of basis is free via base-2 decomposition of indices.

Exponential generating series multiplication

Let $(a_i)_{i\le d}$, $(b_i)_{i\le d}$ and $(c_i)_{i\le d}$ be sequences that satisfy

$$c_i = \sum_j \binom{i}{j} a_j b_{i-j} \qquad i \le d.$$

If we were over $\mathbb{Q}$, this would mean

$$\sum_{i\le d} \frac{c_i}{i!} x^i = \sum_{i\le d} \frac{a_i}{i!} x^i \sum_{i\le d} \frac{b_i}{i!} x^i \bmod x^d.$$

Here, we suppose that

- all odd integers can be inverted in the base ring;
  examples: $\mathbb{Q}$, any $\mathrm{GF}(2^m)$, any $\mathbb{Z}/2^\ell\mathbb{Z}$
- and also that $d = 2^k$.

Reduction to multivariate multiplication

For $i \geq 0$, let

$$i_\star = \frac{i!}{\text{largest power of 2 that divides } i!}$$

Prop.

- The c_i are given by

$$\sum_i \frac{c_i}{i_\star} x_0^{i_0} \cdots x_{k-1}^{i_{k-1}} = \sum_i \frac{a_i}{i_\star} x_0^{i_0} \cdots x_{k-1}^{i_{k-1}} \quad \sum_i \frac{b_i}{i_\star} x_0^{i_0} \cdots x_{k-1}^{i_{k-1}} \quad \bmod \langle \mathbf{T} \rangle,$$

  with

$$\mathbf{T} = \left| \begin{array}{l} x_0^2 - 2x_1 \\ \qquad \vdots \\ x_{k-2}^2 - 2x_{k-1} \\ x_{k-1}^2 \end{array} \right. \qquad \text{and} \quad i = i_0 + 2i_1 + \cdots + 2^{k-1} i_{k-1}$$

The canonical example

Addition of algebraic numbers:

$$f = \prod_i (x - f_i), \qquad g = \prod_j (x - g_j), \qquad h = \prod_{i,j} (x - (f_i + g_j)).$$

To compute h:

- compute the Newton sums of f and g;
- deduce those of h by exponential generating series product;
- get h by exponentiation.

In small characteristic, say over $\mathbb{F}_2$:

- we compute the Newton sums modulo a higher power of 2;
- the exponential series product is done multivariate-ly.

Artin-Schreier

To handle degree-p extensions

- of the form $\mathbb{F}[x]/\langle x^p - x - \alpha \rangle$,
- where char($\mathbb{F}$)$= p$.

Algorithms such as Couveignes' generate towers

$$\mathbf{T} = \left| \begin{array}{l} x_n^p - x_n - \alpha_n(x_1, \ldots, x_{n-1}) \\ \quad\vdots \\ x_2^p - x_2 - \alpha_2(x_1) \\ x_1^p - x_1 - \alpha_1 \end{array} \right.$$

and we have to compute modulo such $\mathbf{T}$'s.

Summary

Multivariate power series

- $T_i = x_i^{d_i}$

Polynomial multiplication

- $T_i = x_i^2 - x_{i-1}$

Exponential generating series

- $T_i = x_i^2 - 2x_{i-1}$

Artin-Schreier (over $\mathbb{F}_2$)

- $T_i = x_i^2 + x_i + \alpha_i(x_1, \ldots, x_{i-1})$

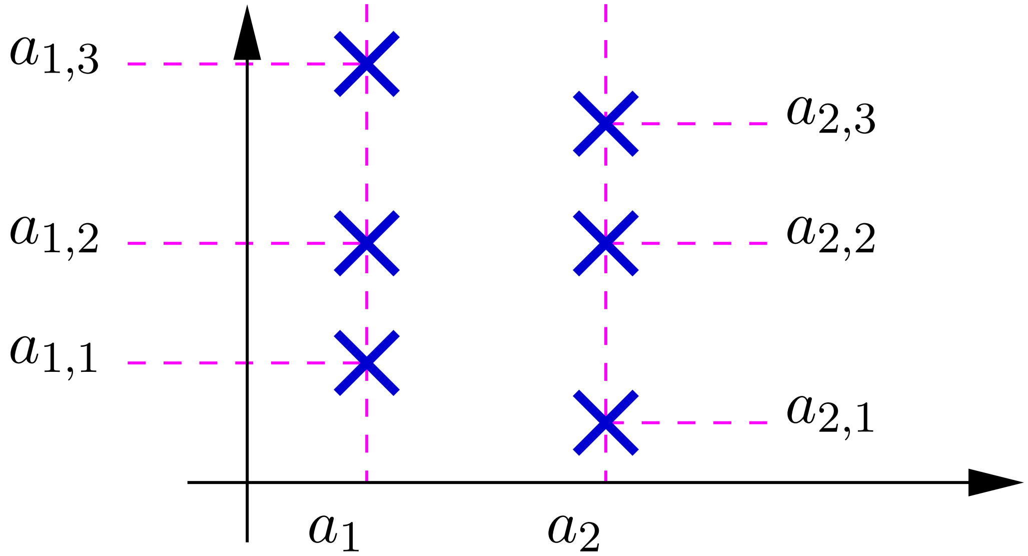
The split case

The nice case

When all polynomials split into linear factors, fast algorithms are available. Precisely, suppose that there exists:

- $a_1, \ldots, a_{d_1}$ such that
$$T_1 = (x_1 - a_1) \cdots (x_1 - a_{d_1})$$
- for all $i \le d_1$, $a_{i,1}, \ldots, a_{i,d_2}$ such that
$$T_2(a_i, x_2) = (x_2 - a_{i,1}) \cdots (x_2 - a_{i,d_2})$$
- for all $i \le d_1, j \le d_2$, $a_{i,j,1}, \ldots, a_{i,j,d_3}$ such that
$$T_3(a_i, a_j, x_3) = (x_3 - a_{i,j,1}) \cdots (x_2 - a_{i,j,d_3})$$
- $\ldots$ up to n.

What this means



The points $P_{i,j,k,\ldots} = (a_i, a_{i,j}, a_{i,j,k}, \ldots)$ are the coordinates of the points

$$V = V(T_1, \ldots, T_n).$$

Modular multiplication

Remember that in one variable,

$$\text{FFT multiplication} = \text{multiplication mod } x^d - 1$$

Hence, to multiply A, B modulo $\langle \mathbf{T} \rangle$:

- evaluate A, B at V;
- multiply the values;
- interpolate the result.

Extra condition

- the differences $a_i - a_{i'}, \ldots$ should all be units.

Evaluation and interpolation

Similar to multidimensional FFT:

- Evaluate A at $x_1 = a_1, \ldots, a_{d_1}$. We get $A_1, \ldots, A_{d_1}$.

  Cost: $\mathsf{M}(d_1)\log(d_1) \times (d_2 \cdots d_n) \;=\; \delta\,\frac{\mathsf{M}(d_1)\log(d_1)}{d_1}$.

- For $i \le d_1$, evaluate A_i at $a_{i,1}, \ldots, a_{i,d_2}$. We get $A_{i,1}, \ldots, A_{i,d_2}$.

  Cost: $d_1\mathsf{M}(d_2)\log(d_2) \times (d_3 \cdots d_n) \;=\; \delta\,\frac{\mathsf{M}(d_2)\log(d_2)}{d_2}$.

- $\ldots$ up to n.

Total cost: $\delta \sum_i \frac{\mathsf{M}(d_i)\log(d_i)}{d_i} \le \delta \log^2(\delta)\log\log(\delta)$.

The general case

Deformation

Given $\mathbf{T} = (T_1, \ldots, T_n)$ in $R[x_1, \ldots, x_n]$.

- Build $\mathbf{U} = (U_1, \ldots, U_n)$, which completely splits, with pairwise distinct roots.
  - constraint: $\deg(U_i, X_i) = \deg(T_i, X_i) = d_i$;
  - other than that, we are free to pick $\mathbf{U}$ as we wish;
  - this requires that the base field has enough elements.
- Build $\mathbf{V} = (V_1, \ldots, V_n)$, with

$$V_i = \eta T_i + (1 - \eta) U_i.$$

Algorithm

- compute the product modulo $\langle \mathbf{V} \rangle$;

  by evaluation / interpolation
- let $\eta = 1$ in the result.

The roots of $\mathbf{V}$

Hensel's lemma

- Because $\mathrm{subs}(\eta = 0, \mathbf{V}) = \mathbf{U}$, and because the roots of $\mathbf{U}$ are simple in R^n, $\mathbf{V}$ admits pairwise distincts roots in $R[[\eta]]^n$.

Cost to reach some precision r.

- Lift the x_1-coordinates
  $\mathsf{M}(d_1)\log(d_1)\,\mathsf{M}(r)$
- Substitute them in $V_2, \ldots, V_n$
  $\mathsf{M}(d_1)\log(d_1)\,(d_2 + d_2 d_3 + \cdots + d_2 \cdots d_n)\,\mathsf{M}(r)$
- Lift the x_2-coordinates
  $d_1 \mathsf{M}(d_2)\log(d_2)\,\mathsf{M}(r)$
- Substitute them in $V_3, \ldots, V_n$
  $d_1 \mathsf{M}(d_2)\log(d_2)\,(d_2 d_3 + \cdots + d_2 \cdots d_n)\,\mathsf{M}(r)$
- $\ldots$

$$\Bigg\} \quad \mathsf{M}(r)\delta\log^2(\delta)\log\log(\delta).$$

Overall complexity

Precision

- Let $r = \deg(AB \bmod \langle \mathbf{V} \rangle, \eta)$.
- All computations done mod η^{r+1}.

  No loss of precision in the algorithm

Total cost

- Given $\mathbf{T}$ and the roots of $\mathbf{U}$,

$$\mathsf{M}(r)\, \delta \sum_i \frac{\mathsf{M}(d_i) \log(d_i)}{d_i} \quad i.e. \quad \mathsf{M}(r)\, \delta \log^2(\delta) \log\log(\delta)$$

  (constructing $\mathbf{U}$ is negligible)

Bounding the precision

Basic remarks

Remarks.

- The needed precision r is

$$\deg(x_1^{2d_1-2} \cdots x_n^{2d_n-2} \bmod \langle \mathbf{V} \rangle, \eta).$$

- In general, $r = \delta$, so this approach is useless.
- r should be smaller for cases with structure.

Example

- Suppose the T_i and U_i are univariate.
- Then $V_i = x_i^{d_i} + v_{i,d_i-1} x_i^{d_i-1} + \cdots + v_{i,0}$, linear in η.
- Each reduction decrements one partial degree and increments the degree in η.
- So $r = \sum_i (d_i - 1)$.

Recurrence relations

Let $\mathbf{e} = (e_1, \ldots, e_n)$ and

$$W(\mathbf{e}) = \deg(x_1^{e_1} \cdots x_n^{e_n} \bmod \langle \mathbf{V} \rangle, \eta).$$

The support of $\mathbf{V}$ determines recurrence relations on W. Write

$$V_n = x_n^{d_n} + \sum_{i \leq S} v_{\boldsymbol{\alpha}_i} \mathbf{x}^{\boldsymbol{\alpha}_i},$$

for some exponents $\boldsymbol{\alpha}_i$ in $\mathbb{N}^n$ and coefficients $v_{\boldsymbol{\alpha}_i}$ linear in η.

Then,

- $W(\mathbf{e}) = W(e_1, \ldots, e_{n-1})$ for $e_n < d_n$;
- else,

$$W(\mathbf{e}) = 1 + \max_i \; W(\mathbf{e} + \boldsymbol{\beta}_i), \qquad \boldsymbol{\beta}_i = \boldsymbol{\alpha}_i - (0, \ldots, 0, d_n).$$

Simplest case

With $T_i = x_i^2 + x_{i-1}$, we take in characteristic $\neq 2$

$$U_i = x_i^2 - 1 \implies V_i = x_i^2 + \eta x_{i-1} - (1 - \eta).$$

Initial conditions

- $W(e_1) = \lfloor e_1/2 \rfloor \leq e_1/2$

Recurrence

- for $e_n = 0, 1$

$$W(e_1, \ldots, e_n) = W(e_1, \ldots, e_{n-1}).$$

- for $e_n \geq 2$

$$W(e_1, \ldots, e_n) \leq 1 + \max \; W(e_1, \ldots, e_n - 2), W(e_1, \ldots, e_{n-1} + 1, e_n - 2)$$

or

$$W(e_1, \ldots, e_n) \leq 1 + W(e_1, \ldots, e_{n-1} + 1, e_n - 2).$$

Simplest case

Unrolling until $e_n = 0, 1$

- $W(e_1, \ldots, e_n) = \lfloor e_n/2 \rfloor + W(e_1, \ldots, e_{n-1} + \lfloor e_n/2 \rfloor)$

Ansatz: assume

$$W(e_1, \ldots, e_{n-1}) \leq w_1 e_1 + \cdots + w_{n-1} e_{n-1}, \quad w_i \geq 0.$$

Then

$$W(e_1, \ldots, e_n) \leq \frac{e_n}{2} + w_1 e_1 + \cdots + w_{n-1} e_{n-1} + w_{n-1} \frac{e_n}{2}$$

so we also have sub-linearity in n variables with $w_n = (w_{n-1} + 1)/2$.

Finally,

$$r \ \leq \ \sum_{i \leq n} 2 w_i \quad \text{with} \quad w_1 = \frac{1}{2}, \quad w_i = \frac{w_{i-1} + 1}{2},$$

so $r \in O(n)$.

Simple case

With $T_i = x_i^2 + x_{i-1}$, we take in characteristic 2

$$U_i = x_i^2 - x_i \implies V_i = x_i^2 + \eta x_{i-1} - (1-\eta)x_i.$$

Initial conditions

- $W(e_1) = e_1 - 1 \leq e_1$.

Recurrence

- for $e_n = 0, 1$

$$W(e_1, \ldots, e_n) = W(e_1, \ldots, e_{n-1}).$$

- for $e_n \geq 2$

$$W(e_1, \ldots, e_n) \leq 1 + \max\ W(e_1, \ldots, e_n - 1), W(e_1, \ldots, e_{n-1} + 1, e_n - 2)$$

Simple case

Unrolling until $e_n = 0, 1$

- We can follow the directions

$$\boldsymbol{\beta}_1 = (0, \ldots, 0, -1), \qquad \boldsymbol{\beta}_2 = (0, \ldots, 1, -2)$$

- We do a_1 steps in the first direction, a_2 steps in the second one.

When we stop, the n-coordinate is either 0 or 1:

$$e_n - a_1 - 2a_2 \in \{0, 1\}$$

So,

$$W(\mathbf{e}) \leq \max_{\substack{a_1, a_2 \in \mathbb{N} \\ e_n - a_1 - 2a_2 \in \{0,1\}}} a_1 + a_2 + W(\mathbf{e} + a_1\boldsymbol{\beta}_1 + a_2\boldsymbol{\beta}_2)$$

Simple case

Induction: assume

$$W(e_1, \ldots, e_{n-1}) \leq w_1 e_1 + \cdots + w_{n-1} e_{n-1}, \quad w_i \geq 0$$

Then,

$$\begin{aligned} W(\mathbf{e}) = & \max_{\substack{a_1, a_2 \in \mathbb{N} \\ e_n - a_1 - 2a_2 \in \{0,1\}}} a_1 + a_2 + w_1 e_1 + \cdots + w_{n-1} e_{n-1} + w_{n-1} a_2 \\ \leq & \max_{\substack{a_1, a_2 \in \mathbb{R}_{\geq 0} \\ e_n - a_1 - 2a_2 = 0}} a_1 + a_2 + w_1 e_1 + \cdots + w_{n-1} e_{n-1} + w_{n-1} a_2 \end{aligned}$$

- The max is either at $(a_1 = 0, a_2 = e_n/2)$ or $(a_1 = e_n, a_2 = 0)$.
- $w_n = \max 1, (w_{n-1} + 1)/2$
- as before, $r = O(n)$.

In general

The w_i's satisfy the recurrence

$$w_n = \max_i \frac{w_1\beta_{i,1} + \cdots + w_{n-1}\beta_{i,n-1} + 1}{-\beta_{i,n}}$$

Artin-Schreier, with $p = 2$

$$V_n = x_n^2 + x_n + \eta x_1 \cdots x_{n-1} + \cdots$$

- $w_1 = 1$
- $w_n = (w_1 + \cdots + w_{n-1} + 1)/2$
- $r = O(\frac{3}{2}^n)$.