Random sampling of plane partitions

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Young tableaux: natural generalization of integer partitions in 3D,
huge literature, e.g. the Alternating Sign Matrix Conjecture (Zeilberger 1995),
MacMahon: beautiful (and simple) generating function (∼ 1912)
for long, no bijective proof,
Krattenthaler, 1999, proof based on interpretation the hook-length formula,
sampling of plane partitions in a box $a \times b \times c$
  $\rightarrow$ hexagon tilings by rhombi,
2002: Pak’s bijection for general planes partitions,
2004: Boltzmann sampling
today: efficient samplers for some classes of plane partitions.
Motivations

- mathematics,
- statistical physics,
- random sampling according to a natural parameter (volume),
- very large object $\rightarrow$ observation of limit properties,
- in particular : limit shape
  - Cerf and Kenyon,
  - Okounkov and Reshetikhin
- phenomena such as frozen boundaries,
- ...

\[ \begin{array}{c}
-4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
5 & 4 & 3 & 2 & 1 & 0 & -1 & -2 & -3
\end{array} \]
Boltzmann sampling technics

+ Explicit bijection with a constructible class

⇓

Polynomial-time sampler for plane partitions
Plan of the talk

1 Pak’s bijection
2 Boltzmann sampler
3 Analysis of Complexity
Planes partitions

- \( \lambda \): Integer partition \( \simeq \text{Shape} \) of plane partition
e.g.: \( \lambda = \{4,3,1,1\} \).
- \( h(i,j) \): hook length of the cell \((i,j)\)
- **Plane partitions** of shape \( \lambda \) \((\mathcal{P})\)
  - \( \lambda \) filled with integers \( > 0 \), decreasing in both dimensions
  - matrix filled with integers \( \geq 0 \), decreasing in both dimensions
- **Reverse plane partition** of shape \( \lambda \) \((\mathcal{RP})\)
  - \( \lambda \) filled with integers \( \geq 0 \) that are increasing in both dimensions
- **Size** of a plane partition: sum of the entries

![Diagram](image)

**Random sampling of plane partitions**
Boxed and skew planes partitions

- **Bounding rectangle** of a plane partition
  the smallest rectangle containing all the non-zero cells
- **$(a \times b)$-boxed plane partitions** ($\mathcal{P}_{a,b}$)
  the size of the bounding rectangle is at most $(a \times b)$
- **Skew plane partitions** ($\mathcal{S}$)
  plane partition of shape $\lambda/\mu$, where $\lambda, \mu$ are integer partitions and $\lambda \supset \mu$
- **Corner** of a skew plane partition

$$\mathcal{S} \equiv \mathcal{RP}$$
Specialization of reverse plane partitions

reverse plane partition

plane partition

boxed

skew

shape = infinite rectangle

shape = box

any shape

Random sampling of plane partitions
Counting plane partitions

Hook content formula:
\[ \sum_{A \in \mathcal{RP}(\lambda)} z^{|A|} = \prod_{(i,j) \in [\lambda]} \frac{1}{1 - z^{h(i,j)}} \]

Set \( \lambda \) to be an infinite rectangle:
\[ \prod_{i,j \geq 0} \frac{1}{1 - z^{i+j+1}} \]

Generating function of plane partitions (MacMahon, 1912):
\[ P(z) = \prod_{r \geq 1} (1 - z^r)^{-r} \]

- combinatorial isomorphisms with constructible classes (symbolic methods)
  \[ \mathcal{P} \simeq \mathcal{M} , \quad \mathcal{P}_{a,b} \simeq \mathcal{M}_{a,b} \quad \text{and} \quad S_D \simeq \mathcal{M}_D \]

- non-trivial bijection, for long, non constructive proof...
\[
\prod_{i,j \geq 0} \frac{1}{1 - z^{i+j+1}} = \prod_{i,j \geq 0} \text{SEQ}(\mathbb{Z} \times \mathbb{Z}^i \times \mathbb{Z}^j) = \text{MSET}(\mathbb{Z} \times \text{SEQ}(\mathbb{Z})^2)
\]

- \( \mathcal{M} = \text{MSET}(\mathbb{N}^2) \sim \text{multiset of pairs of integers} \)
  
  → example: \( \{(0, 0), (1, 0), (2, 0), (2, 0), (0, 1), (1, 2)\} \), size = 15

  → size of \((i, j) : (i + j + 1)\)

- **Diagram** of an element \( \in \mathcal{M} \)

  \[
  \begin{array}{ccc}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  1 & 1 & 2 \\
  \end{array}
  \]

  size = 15

  \[|D| = \sum_{i,j} m_{i,j} (i + j + 1)\]

  → sum of the hook lengths weighted by the values of the cells.
Isomorphic classes – 2

- \( \mathcal{M}_{a,b} = \text{MSET}(\mathcal{Z} \times \text{SEQ}_{<a}(\mathcal{Z}) \times \text{SEQ}_{<b}(\mathcal{Z})) \)
  \[= \prod_{0 \leq i < a \atop 0 \leq j < b} \text{SEQ}(\mathcal{Z} \times \mathcal{Z}^i \times \mathcal{Z}^j) \]
  \[\sim \text{MSET}(\mathbb{N}_{<a} \times \mathbb{N}_{<b}) \]

- \( \mathcal{M}_D = \prod_{(i,j) \in D} \text{SEQ}(\mathcal{Z} \times \mathcal{Z}^{i-\ell(i)} \times \mathcal{Z}^{j-d(j)}) = \prod_{(i,j) \in D} \text{SEQ}(\mathcal{Z}^{h(i,j)}) \)

Diagrams

- Hook length of \((i, j) \in D : h(i, j) = (i - \ell(i)) + (j - d(j)) + 1 \)
  
  \[\ell(i) \leftarrow \text{min. abscissa such that } (\ell(i), j) \in D\]
  
  \[d(j) \leftarrow \text{min. ordinate such that } (i, d(j)) \in D\]
\[ \mathcal{M} \]

\[ \mathcal{M}_D \]

\[ \mathcal{M}_{a,b} \]

\[ \mathcal{R}\mathcal{P} \]

\[ \mathcal{P}_{a,b} \]

plane partition

boxed

skew

reverse partition
Pak’s bijection
Pak’s bijection – principles

- sequential update of the corners of the multiset $M$
- at each step, the current plane partition (of shape $\lambda$) correspond to the restriction of $M$ to $\lambda$
- prop. 1: for any corner, the value of the cell, in the plane partition = the maximum value of a monotone path, in the multiset.
- prop. 2: for any extreme cell, diagonal sum, in the plane partition = rectangular sum, in the multiset.
- order constraint, size constraint
- dynamic programming

simple algorithm, but difficult proof!
Application of Pak’s algorithm on an example.
**Pak’s algorithm**

**Input**: a diagram $D$ of a multiset in $\mathcal{M}$.

**Output**: a plane partition.

Let $\ell$ be the length and $w$ be the width of $D$.

**for** $i := \ell - 1$ **downto** 0 **do**
  **for** $j := w - 1$ **downto** 0 **do**
    **for** $c := 1$ **to** $\min(w - 1 - j, \ell - 1 - i)$ **do**
      $x \leftarrow i + c$; $y \leftarrow j + c$;
      $D[x, y] \leftarrow \max(D[x + 1, y], D[x, y + 1])$;
      $+ \min(D[x + 1, y], D[x, y + 1])$;
      $- D[x, y]$;

  **Return** $D$;
Boltzmann sampler
Random sampling under Boltzmann model

- for any constructible class
- approximate size sampling,
- size distribution spread over the whole combinatorial class, but uniform for a sub-class of objects of the same size,
- control parameter,
- automatized sampling: the sampler is compiled from specification automatically,
- very large objects can be sampled.
In the unlabelled case, Boltzmann model assigns to any object \( c \in \mathcal{C} \) the following probability:

\[
\mathbb{P}_x(c) = \frac{x^{|c|}}{C'(x)}
\]

A Boltzmann sampler \( \Gamma C(x) \) for the class \( \mathcal{C} \) is a process that produces objects from \( \mathcal{C} \) according to this model.

→ 2 object of the same size will be drawn with the same probability. The probability of drawing an object of size \( N \) is then:

\[
\mathbb{P}_x(N = n) = \sum_{|c| = n} \mathbb{P}_x(c) = \frac{C'_n x^n}{C'(x)}
\]

Then, the expected size of an object drawn by a generator with parameter \( x \) is:

\[
\mathbb{E}_x(N) = x \frac{C'(x)}{C'(x)}
\]
• Free samplers: produce objects with randomly varying sizes!
• Tuned samplers: choose \( x \) so that expected size is \( n \).
• Run the targeted sampler until the output size is in the desired range (rejection).
• Size distribution of free sampler determines complexity.
Disjoint unions
Boltzmann sampler \( \Gamma C \) for \( \mathcal{C} = \mathcal{A} \cup \mathcal{B} \):
With probability \( \frac{A(x)}{C(x)} \) do \( \Gamma A(x) \) else do \( \Gamma B(x) \) \( \rightarrow \) Bernoulli.

Products
Boltzmann sampler \( \Gamma C \) for \( \mathcal{C} = \mathcal{A} \times \mathcal{B} \):
Generate a pair \( \langle \Gamma A(x), \Gamma B(x) \rangle \) \( \rightarrow \) independent calls.

Sequences
Boltzmann sampler \( \Gamma C \) for \( \mathcal{C} = \text{SEQ}(\mathcal{A}) \):
Generate \( k \) according to a geometric law of parameter \( A(x) \)
Generate a \( k \)-tuple \( \langle \Gamma A(x), \ldots, \Gamma A(x) \rangle \) \( \rightarrow \) independent calls.

Remark: \( A(x), B(x), \) and \( C(x) \) is given by an oracle.
Generating multisets

\[ C = \text{MSET}(A) \cong \prod_{\gamma \in A} \text{SEQ}(\gamma) \Rightarrow C(z) = \prod_{\gamma \in A} (1 - z^{\mid \gamma \mid})^{-1} \]

\[ C(z) = \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} A(z^k) \right) = \prod_{k=1}^{\infty} \exp \left( \frac{1}{k} A(z^k) \right) \]
Sampling an object of $\mathcal{M}$

**Algorithm $\Gamma_M(x)$**

$M$ is the diagram of the multiset to be generated

- Draw $m$, the *max. index* of a subset, depending on $x$;
- For each index $k$ of a subset until $m - 1$
  - Draw the number $p$ of elements to sample, according to a Poisson law of parameter $\frac{x^k}{k(1-x^k)^2}$.
  - Perform $p$ calls to the sampler for $\mathcal{Z} \times \text{SEQ}((\mathcal{Z})^2$ with parameter $x^k$, and each time, add $k$ copies of the result to the multiset.
  
    Repeat $p$ times:
    
    $i \leftarrow \text{Geom}(x^k)$;
    $j \leftarrow \text{Geom}(x^k)$;
    $M[i, j] \leftarrow M[i, j] + k$

- for index $m$, draw the number $p$ of elements to generate, according to a *non zero* Poisson law.
Sampling $\mathcal{M}_{a,b}$ and $\mathcal{M}_D$

$\Gamma_{\mathcal{M}_{a,b}}(x)$ [Boltzmann sampler for $\mathcal{M}_{a,b}$]

$M$ is the diagram of the multiset to be generated

\begin{verbatim}
for i ← 0 to a - 1 do
    for j ← 0 to b - 1 do
        M[i, j] ← Geom($x^{i+j+1}$);
return M;
\end{verbatim}

$\Gamma_{\mathcal{S}_D}(x)$ [Boltzmann sampler for $\mathcal{M}_D$]

$M$ is the diagram of the multiset to be generated

\begin{verbatim}
for (i, j) ∈ D do
    M[i, j] ← Geom($x^{i+j+1}$);
return M;
\end{verbatim}

The free Boltzmann samplers operate in linear time in the size of the bounding rectangle of the diagram produced.
Targeted Boltzmann sampler for
- $\mathcal{M} \rightarrow$ plane partitions
- $\mathcal{M}_{a,b} \rightarrow$ boxed plane partitions
- $S_D \rightarrow$ skew planes partitions

Output: a *diagram* $D$.

Rejection

Pak’s algorithm transforms $D$ into a plane partition.

Size of the output plane partition = size of the original diagram.
Results

Theorem (Expected complexity)

- **Plane partitions:**
  - approximate-size: $O(n \ln(n)^3)$
  - exact-size: $O(n^{4/3})$

- $(p \times q)$-boxed plane partitions (for fixed $a, b$):
  - approximate-size: $O(1)$ as $n \to \infty$
  - exact-size: bounded by $C_{ab}n$

- **skew plane partitions ($S_D$):**
  - approximate-size: $O(1)$ as $n \to \infty$
  - exact-size: bounded by $C|D|.n$

where $C_1, C_1 > 0$ are constants.

<table>
<thead>
<tr>
<th>~size</th>
<th>10^4</th>
<th>10^5</th>
<th>10^6</th>
<th>10^7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma M$</td>
<td>~0.4s</td>
<td>~2-3s</td>
<td>~10s</td>
<td>~60s</td>
</tr>
<tr>
<td>rect. size</td>
<td>~50</td>
<td>~100</td>
<td>~200-300</td>
<td>~600-800</td>
</tr>
<tr>
<td>bijection</td>
<td>~0.05s</td>
<td>~10s</td>
<td>~20s</td>
<td>~250-300s</td>
</tr>
</tbody>
</table>
A plane partition of size $1005749$ drawn under Boltzmann distribution at $x = 0.9866$.

gen. time : $\sim 7s$

bij. time : $\sim 280s$
← A \((100 \times 100)\)-boxed plane partition of size 999400 drawn under Boltzmann distribution at \(x = 0.9931\).

\[
\text{gen. time} : \sim 5s \\
\text{bij. time} : \sim 0.7s
\]

→ A skew plane partition of size 1005532 on the index-domain:
\([0..99] \times [0..99] \setminus [0..49] \times [0..49]\), drawn under Boltzmann distribution at \(x = 0.9942\).

\[
\text{gen. time} : \sim 4s. \\
\text{bij. time} : \sim 0.35s.
\]
Analysis of Complexity
Theorem (Expected complexity)

- **Plane partitions**:
  - approximate-size: $O(n \ln(n)^3)$
  - exact-size: $O(n^{4/3})$

- $(p \times q)$-boxed plane partitions (for fixed $a, b$):
  - approximate-size: $O(1)$ as $n \to \infty$
  - exact-size: bounded by $C a b n$

- **skew plane partitions** ($S_D$):
  - approximate-size: $O(1)$ as $n \to \infty$
  - exact-size: bounded by $C |D|.n$

where $C_1, C_2 > 0$ are constants.
General scheme

Generation of a plane partition of size $n$ (resp. $\sim n$), with a targeted sampler, i.e., with a parameter tuned such that $\mathbb{E}(N_x) = n$.

\[
\text{mean cost} = \text{cost of one call to } \Gamma M \times \text{expected number of calls} + \text{cost of Pak’s algorithm}
\]

1. cost of one call to $\Gamma M$: $O(n^{\frac{2}{3}})$
2. expected number of calls to the sampler:
   - approximate size sampler: $O(1)$
   - exact size sampler: $O(n^{\frac{2}{3}})$
3. expected complexity of Pak’s algorithm applied to a diagram of size $n$: $O(n \ln(n)^3)$
complexity of the free Boltzmann sampler, as \( x \to 1^- \):

\[
\Lambda P(x) = \Lambda M(x) + \mathbb{E}_x[\text{PakAlgo}](x)
\]

\[
\Lambda M(x) = \sum_{i \geq 1} \mathbb{E} \left( \text{Pois} \left( \frac{A(x^i)}{i} \right) \right) \Lambda A(x^i) = \sum_{i \geq 1} \frac{A(x^i)}{i} \Lambda A(x^i)
\]

using Mellin transform:

\[
\Lambda M(x) = \mathcal{O} \left( \frac{1}{(1-x)^2} \right)
\]

length of the bounding rectangle of a multiset drawn under Boltzmann model: \( \mathcal{O}((1-x)^{-1} \ln((1-x)^{-1})) \) as \( x \to 1^- \):

\[
\mathbb{E}_x[\text{PakAlgo}](x) = \mathcal{O} \left( \frac{1}{(1-x)^3} \ln \left( \frac{1}{1-x} \right)^3 \right) = \Lambda P(x)
\]
Details – targeted sampler

using Mellin transform:

\[
\mathbb{E}(N_x) = \frac{2\zeta(3)}{(1 - x)^3} + \mathcal{O}_{x \to 1^-} \left( \frac{1}{(1 - x)^2} \right)
\]

\[
\mathbb{V}(N_x) = \frac{6\zeta(3)}{(1 - x)^4} + \mathcal{O}_{x \to 1^-} \left( \frac{1}{(1 - x)^3} \right)
\]

tuned parameter: \( \xi_n := 1 - (2\zeta(3)/n)^{1/3} \)

expected complexity of \( \Gamma M(\xi_n) \) and Pak’s algorithm under the uniform distribution at a fixed size \( n \):

\[
\Lambda M(\xi_n) = \mathcal{O}(n^{2/3}), \quad \mathbb{E}_n[\text{Pak}] = \mathcal{O}(n \log(n)^3)
\]

probability that the output of \( \Gamma P(\xi_n) \) has size \( n \):

- using Chebyshev inequality: \( \pi_{n,\epsilon} \xrightarrow[n \to \infty]{} 1 \)

- using Mellin transform and the saddle-point method:

\[
\pi_n \xrightarrow[n \to \infty]{} c n^{2/3}, \quad \text{with } c \approx 0.1023
\]
Details – boxed, skew

sampler for \((a \times b)\)-boxed plane partitions:

\[ \xi_{n}^{a,b} := 1 - \frac{ab}{n} \]

\[ \pi_{n,\epsilon} \sim O(1), \quad \pi_{n} \sim O(n) \]

\( \Gamma P_{a,b}(x) \) is of constant complexity \( C \cdot a \cdot b \)

expected complexity of the approximate-size sampler:

\[ \Lambda P_{a,b}(\xi_{n})/\pi_{n,\epsilon} \sim C \cdot ab \]

expected complexity of the exact-size sampler:

\[ \Lambda P_{a,b}(\xi_{n})/\pi_{n} \sim C \cdot abn \]
Bibliography

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- Random generation under Boltzmann model

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