Integration of Algebraic Functions using Gröbner Bases

Manuel Kauers

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Subtalk

Closed Walks in the Quarter Plane

Manuel Kauers and Doron Zeilberger

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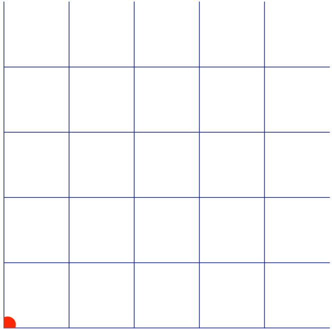

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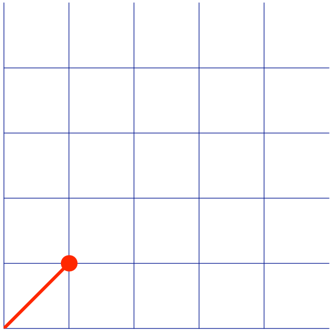

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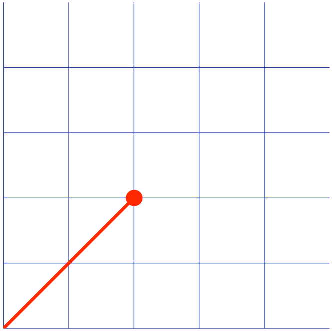

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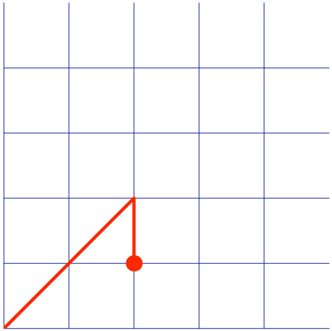

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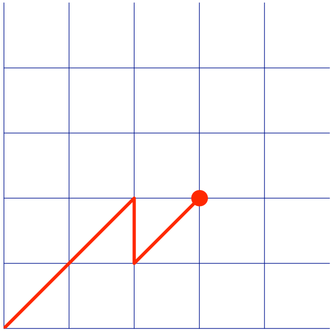

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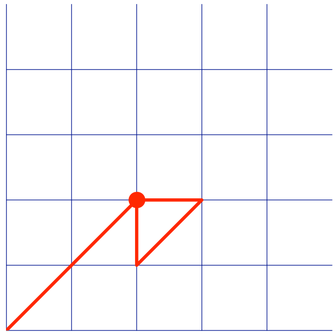

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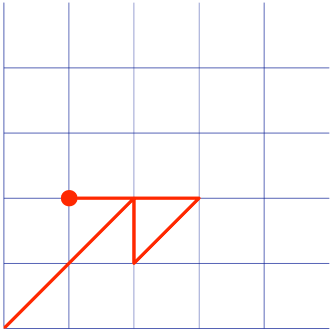

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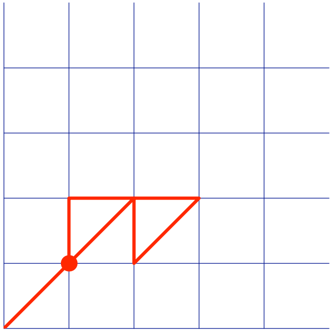

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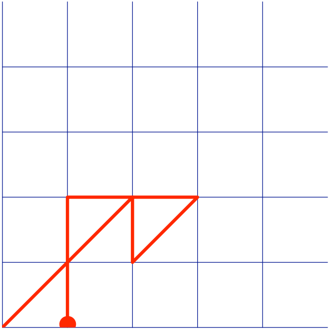

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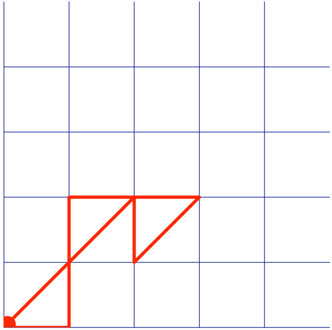

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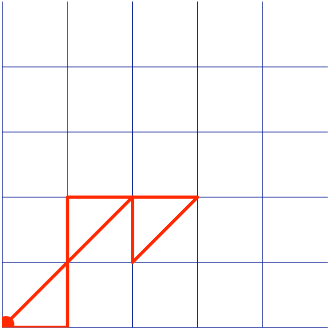

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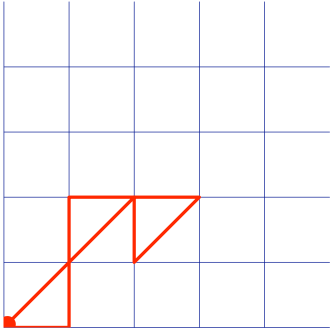

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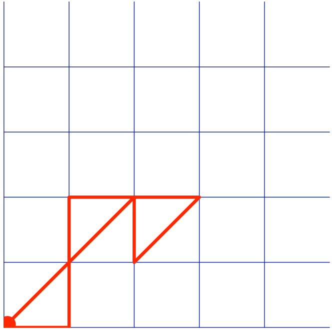

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- do not leave $\mathbb{N}^2$

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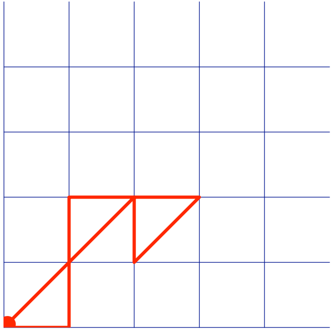

- start at $(0,0)$
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- consist only of steps from some fixed step set $S \subseteq \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$.

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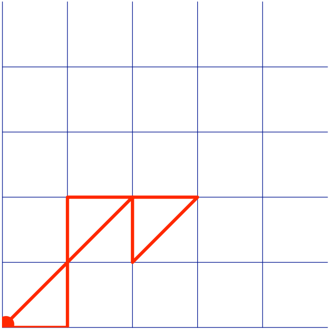

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- do not leave $\mathbb{N}^2$
- consist only of steps from some fixed step set $S \subseteq \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$.

Question: How many such walks of length n are there?

Kreweras' Theorem

For $S = \{\nearrow, \leftarrow, \downarrow\}$, this number is

$$f(n) = \begin{cases} \frac{4^k}{(k+1)(2k+1)}\binom{3k}{k} & \text{if } n = 3k \\ 0 & \text{if } 3 \nmid n \end{cases}$$

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Answer: Two possibilities

a) with thinking

b) without thinking

Proving Kreweras' Theorem without thinking

Let $f(n;i,j)$ be the number of walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j) with steps in $\{\nearrow, \leftarrow, \downarrow\}$.

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Then $f(n; i, j)$ is killed by an operator of the form

$$\begin{aligned}\mathcal{P}(N, I, J, n, i, j) &= \mathcal{R}(N, n) \\ &\quad + i\mathcal{R}_1(N, I, J, n, i, j) \\ &\quad + j\mathcal{R}_2(N, I, J, n, i, j)\end{aligned}$$

(found and proven by the computer).

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Therefore $\mathcal{R}(N,n)$ kills $f(n;0,0)$.

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(found and proven by the computer).

Therefore $\mathcal{R}(N,n)$ kills $f(n;0,0)$.

As $\mathcal{R}(N,n)$ also kills the RHS and initial values agree, we are done.

$\mathcal{P}(N, I, J, n, i, j)$

$$-(21886i^4 - 118028ji^3 + 47480ni^3 + 244165i^3 + 218100j^2i^2 + 23980n^2i^2 - 802518ji^2 - 160152jni^2 + 219325ni^2 + 480993i^2 - 156416j^3i - 1632n^3i + 734634j^2i - 33556jn^2i - 50857n^2i - 418482ji + 151848j^2ni - 259606jni - 407910ni - 969246i + 34480j^4 - 18n^4 - 207332j^3 - 2040jn^3 - 457n^3 + 275232j^2 + 16996j^2n^2 - 20824jn^2 - 4167n^2 + 59382j - 42928j^3n + 143470j^2n - 41586jn - 15948n - 21060)N^6 - 6I(187i^3 + 884n^2i^2 + 14678ni^2 + 57967i^2 + 884n^3i - 1632j^2i - 1768jn^2i + 17095n^2i - 119975ji - 29768jni + 113065ni + 254940i + 1768j^3 + 26784j^2 - 800jn^2 - 56385j + 3776j^2n - 14103jn)N^5 + J(7696i^4 - 26936ji^3 + 135368ni^3 + 914918i^3 + 11544j^2i^2 + 128080n^2i^2 - 3123939ji^2 - 469848jni^2 + 1222792ni^2 + 2445516i^2 + 30784j^3i + 408n^3i + 3158151j^2i - 193984jn^2i - 124594n^2i - 2822667ji + 466152j^2ni - 1729387jni - 1379574ni - 3466116i - 15392j^4 - 927346j^3 - 6120jn^3 + 758460j^2 + 68104j^2n^2 - 34114jn^2 + 689364j - 135856j^3n + 568135j^2n + 149691jn)N^5 - 9IjJ(716n^2 + 8248n + 23381)N^4 - 54I^2(252in^2 - 272jn^2 + 2760in - 2776jn + 7377i - 6752j)N^4 + 3J^2(7288i^4 + 7288ji^3 - 5466j^2i^2 - 67072n^2i^2 - 840272ni^2 - 2631694i^2 - 3644j^3i + 100780jn^2i + 72496n^2i + 3883759ji + 1256708jni + 896840ni + 2717076i + 1822j^4 - 1342069j^2 - 35188j^2n^2 - 45644jn^2 - 1709010j - 436808j^2n - 563062jn)N^4 - 54(2i - j)(i + j + 2)J^3(n + 4)(34n + 211)N^3 + \dots$$

$$\mathcal{P}(N, I, J, n, i, j)$$

$$\cdots + 162IjJ^2(n+4)(34n+211)N^3 + 108I^3(i-2j)(n+4)(34n+211)N^3 + 9(67888i^4 - 371924ji^3 + 142848ni^3 + 230800i^3 + 707820j^2i^2 + 71940n^2i^2 - 660372ji^2 - 482904jni^2 + 191808ni^2 + 130420i^2 - 540608j^3i - 4896n^3i + 443604j^2i - 100668jn^2i - 116667n^2i + 130316ji + 460440j^2ni - 99288jni - 381864ni - 181221i + 139120j^4 - 108n^4 - 76568j^3 - 6120jn^3 - 2013n^3 + 146560j^2 + 50988j^2n^2 - 16164jn^2 - 13788n^2 - 105576j - 132048j^3n + 122400j^2n - 25878jn - 41001n - 44478)N^3 + 162I(408i^3 + 884n^2i^2 - 1326ji^2 + 7988ni^2 + 15192i^2 + 884n^3i + 1020j^2i - 1768jn^2i + 10465n^2i - 28254ji - 15062jni + 45380ni + 63579i + 3372j^2 - 800jn^2 - 21018j + 1124j^2n - 10290jn)N^2 + 54J(204i^4 - 714ji^3 - 67480ni^3 - 200400i^3 + 306j^2i^2 - 64040n^2i^2 + 701100ji^2 + 235536jni^2 - 137462ni^2 + 159690i^2 + 816j^3i - 204n^3i - 701676j^2i + 96992jn^2i + 63827n^2i - 451641ji - 236136j^2ni + 137675jni + 202284ni + 29349i - 408j^4 + 207660j^3 + 3060jn^3 + 222159j^2 - 34052j^2n^2 - 5893jn^2 + 149382j + 69968j^3n - 26777j^2n + 5901jn)N^2 + 972IjJ(n+2)(179n+384)N + 5832I^2(n+2)(63ni+104i-68j-68jn)N - 324J^2(n+2)(204i^3 - 306ji^2 - 16768ni^2 - 50202i^2 - 306j^2i + 73800ji + 25195jni + 18124ni + 52536i + 204j^3 - 25524j^2 - 33009j - 8797j^2n - 11411jn)N + 49572(2i-j)(i+j+2)J^3(n+1)(n+2) - 148716IjJ^2(n+1)(n+2) - 99144I^3(i-2j)(n+1)(n+2) + 1458(n+1)(n+2)(9n^2 + 80n + 136i - 68j + 159)$$

$\mathcal{R}(N,n)$

$$
\begin{aligned}
&(18n^4 + 457n^3 + 4167n^2 + 15948n + 21060)N^6 \\
&- 9(108n^4 + 2013n^3 + 13788n^2 + 41001n + 44478)N^3 \\
&+ 1458(n+1)(n+2)(n^2 + 80n + 159)
\end{aligned}
$$

Analogs for Different Step Sets

	step set	number of closed paths
1	$\{\uparrow, \nearrow, \rightarrow\}$	$f(n,0,0) = 0$
2	$\{\uparrow, \nearrow, \swarrow\}$	$f(2n,0,0) = \frac{4^n (1/2)_n}{(1)_{n+1}}$
3	$\{\uparrow, \nearrow, \nwarrow\}$	$f(n,0,0) = 0$
4	$\{\uparrow, \downarrow, \nwarrow\}$	$f(2n,0,0) = \frac{4^n (1/2)_n}{(1)_{n+1}}$
5	$\{\leftarrow, \downarrow, \nearrow\}$	$f(3n,0,0) = \frac{2 \cdot 27^{n-1} (4/3)_{n-1} (5/3)_{n-1}}{(5/2)_{n-1} (3)_{n-1}}$
6	$\{\uparrow, \rightarrow, \swarrow\}$	$f(3n,0,0) = \frac{2 \cdot 27^{n-1} (4/3)_{n-1} (5/3)_{n-1}}{(5/2)_{n-1} (3)_{n-1}}$
7	$\{\leftarrow, \uparrow, \nwarrow\}$	$f(3n,0,0) = \frac{27^{n-1} (4/3)_{n-1} (5/3)_{n-1}}{(3)_{n-1} (4)_{n-1}}$
8	$\{\uparrow, \swarrow, \nwarrow\}$	$f(4n,0,0) = \frac{2 \cdot 64^{n-1} (5/4)_{n-1} (3/2)_{n-1} (7/4)_{n-1}}{(2)_{n-1} (5/2)_{n-1} (3)_{n-1}}$
9	$\{\leftarrow, \nearrow, \nwarrow\}$	$f(4n,0,0) = \frac{2 \cdot 64^{n-1} (5/4)_{n-1} (3/2)_{n-1} (7/4)_{n-1}}{(2)_{n-1} (5/2)_{n-1} (3)_{n-1}}$
10	$\{\nwarrow, \uparrow, \nwarrow\}$	$f(n,0,0) = 0$
11	$\{\nwarrow, \nearrow, \nwarrow\}$	$f(n,0,0) = 0$

Gessel's Conjecture

For $S = \{\leftarrow, \rightarrow, \nearrow, \swarrow\}$, it seems that

$$f(n) = \begin{cases} 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

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But the corresponding computations were too expensive.

The conjecture remains open.

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Integration of Algebraic Functions using Gröbner Bases

Manuel Kauers

Recall: Integration of Rational Functions

INPUT: $f \in k(x)$

Recall: Integration of Rational Functions

INPUT: $f \in k(x)$
OUTPUT: $g \in k(x)$, $\gamma_1, \ldots, \gamma_n \in \bar{k}$, $p_1, \ldots, p_n \in \bar{k}(x)$ with

$$\int f = g + \gamma_1 \log(p_1) + \cdots + \gamma_n \log(p_n).$$

Recall: Integration of Rational Functions

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THEOREM: Such g, γ_i, p_i exist for every f.

Recall: Integration of Rational Functions

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ALGORITHM (Sketch):

Polynomial Part $\rightarrow$ Rational Part $\rightarrow$ Logarithmic Part

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ALGORITHM (Sketch):

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Write $f = f_0 + \frac{a}{b}$ with $f_0, a, b \in k[x]$ and $\partial_x a < \partial_x b$.

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Write $f = f_0 + \frac{a}{b}$ with $f_0, a, b \in k[x]$ and $\partial_x a < \partial_x b$.
f_0 is easy to integrate. We are left with $\frac{a}{b}$.

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ALGORITHM (Sketch):

Polynomial Part $\rightarrow$ Rational Part $\rightarrow$ Logarithmic Part

Find $g \in k(x)$ and $u, v \in k[x]$ with v squarefree such that

$$\frac{a}{b} = Dg + \frac{u}{v}.$$

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Polynomial Part $\to$ Rational Part $\to$ Logarithmic Part

Let $\gamma_1, \ldots, \gamma_n \in \bar{k}$ be the distinct roots of

$$\mathrm{res}_x(u - tDv, v) \in k[t]$$

and $p_i = \gcd(u - \gamma_i Dv, v) \in \bar{k}[x]$.

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and $p_i = \gcd(u - \gamma_i Dv, v) \in \bar{k}[x]$. Then $\frac{u}{v} = \sum_i \gamma_i \frac{Dp_i}{p_i}$.

Generalizations

$$\int \frac{x^2 - x + 1}{x^3 + 2x - 3} dx = ?$$

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$$\downarrow$$

$$\int \frac{e^{x^2} - x + \log x}{x^3 - \log(x^2 + 1)} dx = ?$$

Elementary transcendental functions: Risch's algorithm

Generalizations

$$\int \frac{x^2 - x + 1}{x^3 + 2x - 3} dx = ? \qquad \longrightarrow \qquad \int \frac{x^2 - x + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} dx = ?$$

$$\downarrow$$

$$\int \frac{e^{x^2} - x + \log x}{x^3 - \log(x^2 + 1)} dx = ?$$

Algebraic functions: Davenport's or Trager's algorithm

Generalizations

$$\begin{array}{ccc}
\displaystyle\int \frac{x^2 - x + 1}{x^3 + 2x - 3} dx = ? & \longrightarrow & \displaystyle\int \frac{x^2 - x + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} dx = ? \\
\downarrow & & \downarrow \\
\displaystyle\int \frac{e^{x^2} - x + \log x}{x^3 - \log(x^2 + 1)} dx = ? & \longrightarrow & \displaystyle\int \frac{e^{x^2} - x + \log x}{\sqrt{x^3 - \log(x^2 + 1)}} dx = ?
\end{array}$$

Arbitrary elementary functions: Bronstein's algorithm

Generalizations

$$\begin{array}{ccc}
\displaystyle\int \frac{x^2 - x + 1}{x^3 + 2x - 3}dx =? & \longrightarrow & \displaystyle\int \frac{x^2 - x + \sqrt{1-x^2}}{1 + \sqrt{1-x^2}}dx =? \\
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Today: algebraic functions

Algebraic Functions

Our domain of interest:

k

Algebraic Functions

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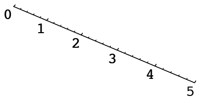
$k(x)$



Algebraic Functions

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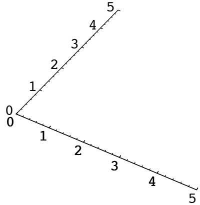

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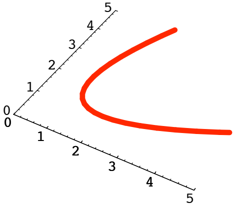

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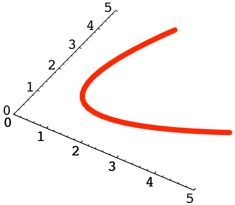
Algebraic Functions



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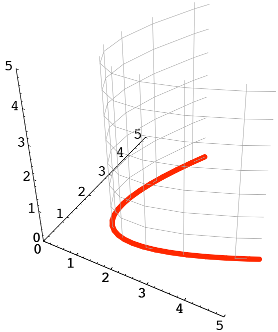

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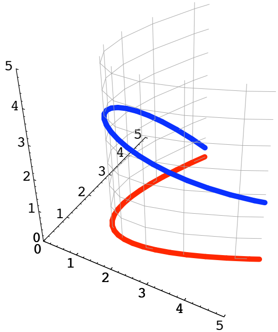

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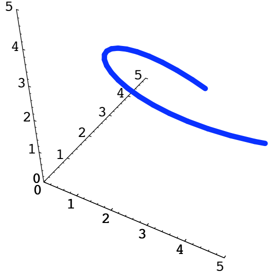

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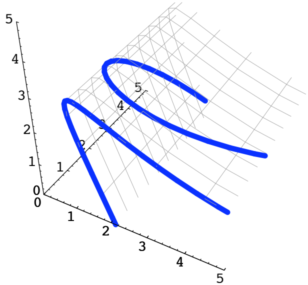

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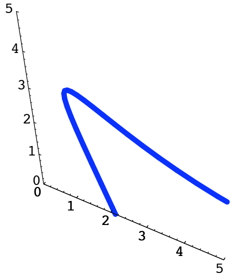

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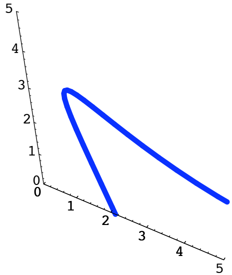

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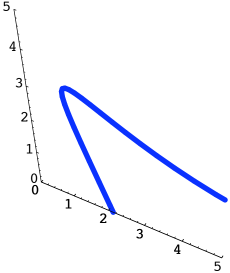

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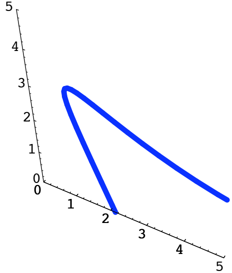

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$$\int f = g + \gamma_1 \log(p_1) + \cdots + \gamma_n \log(p_n)$$

or $\perp$ if no such data exists.

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THEOREM (Liouville) g, γ_i, p_i exist $\iff$ $\int f$ is elementary

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Write the new integrand as $f = Dg + h$ such that h only has simple poles

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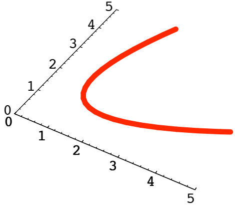

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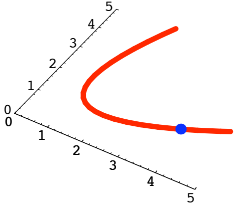

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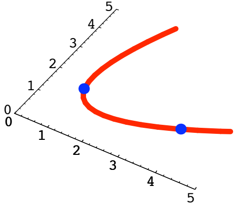

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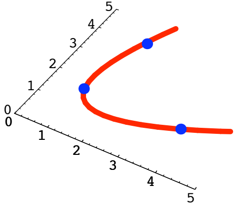

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- $\displaystyle\int \frac{u}{v} = \sum_{i=0}^{r-1} \sum_{\gamma:\bar{c}_i(\gamma)=0} \gamma \log(p_{i+1}(\gamma, x))$

Example

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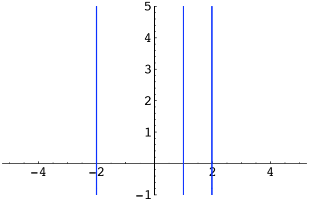

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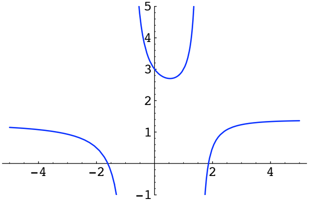

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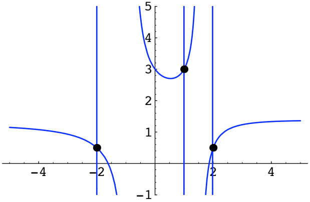

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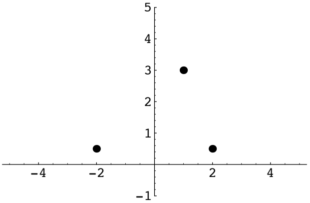

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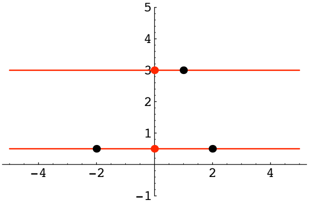

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In general, if c is an irreducible factor of c_0, then

$$\mathrm{Gb}(\langle c, v, u - tDv \rangle) = \{c(t), p(x,t)\} \trianglelefteq k[x,t]$$

for p such that $p(x,\gamma) = \gcd(v, u - \gamma Dv)$ when $c(\gamma) = 0$.

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- A typical integrand is

$$f = \Big(360x^5 + 86x^4 + 256x^3 + 1218x^2 + 128x + 640 - \left(96x^3 + 1024x^2 + 64x + 664\right)\sqrt{x^3+1}\Big) \Big/ \Big(96x^6 + 40x^5 + 64x^4 + 568x^3 + 40x^2 + 64x + 472 - \left(64x^4 + 336x^3 + 24x^2 + 48x + 488\right)\sqrt{x^3+1}\Big).$$

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$$+ \frac{1}{4}(1 + i\sqrt{47}) \log\left(x - \frac{1}{4}(1 + i\sqrt{47})\sqrt{x^3+1} - \frac{1}{4}(3 - i\sqrt{47})\right)$$
$$+ 2\log(x - 2\sqrt{x^3+1} + 3)$$

Integrals involving $\sqrt{x^3 + 1}$

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