Integration of Algebraic Functions
using Gröbner Bases

Manuel Kauers
Subtalk

Closed Walks in the Quarter Plane

Manuel Kauers and Doron Zeilberger
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▶ start at (0, 0)
Closed Walks in the Quarter Plane

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- end at $(0,0)$
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- start at \((0, 0)\)
- end at \((0, 0)\)
- do not leave \(\mathbb{N}^2\)
Closed Walks in the Quarter Plane

- start at \((0, 0)\)
- end at \((0, 0)\)
- do not leave \(\mathbb{N}^2\)
- consist only of steps from some fixed step set

\[ S \subseteq \{ \leftarrow, \searrow, \uparrow, \nearrow, \rightarrow, \swarrow, \downarrow, \nwarrow \} . \]
Closed Walks in the Quarter Plane

- start at $(0, 0)$
- end at $(0, 0)$
- do not leave $\mathbb{N}^2$
- consist only of steps from some fixed step set $S \subseteq \{←, ↖, ↑, ↗, →, ↘, ↓, ↙\}$.

**Question:** How many such walks of length $n$ are there?
For $S = \{\nearrow, \leftarrow, \downarrow\}$, this number is

$$f(n) = \begin{cases} \frac{4^k}{(k+1)(2k+1)} \binom{3k}{k} & \text{if } n = 3k \\ 0 & \text{if } 3 \nmid n \end{cases}$$
Kreweras’ Theorem

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a) with thinking
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a) with thinking

b) without thinking
Proving Kreweras’ Theorem without thinking

Let \( f(n; i, j) \) be the number of walks in \( \mathbb{N}^2 \) from \((0, 0)\) to \((i, j)\) with steps in \{\(\rightarrow\), \(\leftarrow\), \(\downarrow\}\).
Proving Kreweras’ Theorem without thinking

Let $f(n; i, j)$ be the number of walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$ with steps in $\{\uparrow, \leftarrow, \downarrow\}$.

Let $N, I, J$ denote the shift operators for $n, i, j$, respectively.
Proving Kreweras’ Theorem without thinking

Let \( f(n; i, j) \) be the number of walks in \( \mathbb{N}^2 \) from \((0, 0)\) to \((i, j)\) with steps in \{\(\uparrow, \leftarrow, \downarrow\}\).

Let \( N, I, J \) denote the shift operators for \( n, i, j \), respectively. Then \( f(n; i, j) \) is killed by an operator of the form

\[
P(N, I, J, n, i, j) = R(N, n) + iR_1(N, I, J, n, i, j) + jR_2(N, I, J, n, i, j)
\]

(found and proven by the computer).
Proving Kreweras’ Theorem without thinking

Let $f(n; i, j)$ be the number of walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$ with steps in $\{\uparrow, \leftarrow, \downarrow\}$.

Let $N, I, J$ denote the shift operators for $n, i, j$, respectively. Then $f(n; i, j)$ is killed by an operator of the form

$$
\mathcal{P}(N, I, J, n, i, j) = \mathcal{R}(N, n) + i\mathcal{R}_1(N, I, J, n, i, j) + j\mathcal{R}_2(N, I, J, n, i, j)
$$

(found and proven by the computer).

Therefore $\mathcal{R}(N, n)$ kills $f(n; 0, 0)$. 

Proving Kreweras’ Theorem without thinking

Let $f(n; i, j)$ be the number of walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$ with steps in $\{\nearrow, \leftarrow, \downarrow\}$.

Let $N, I, J$ denote the shift operators for $n, i, j$, respectively. Then $f(n; i, j)$ is killed by an operator of the form

$$P(N, I, J, n, i, j) = R(N, n)$$
$$+ iR_1(N, I, J, n, i, j)$$
$$+ jR_2(N, I, J, n, i, j)$$

(found and proven by the computer).

Therefore $R(N, n)$ kills $f(n; 0, 0)$.

As $R(N, n)$ also kills the RHS and initial values agree, we are done.
\[ \mathcal{P}(N, I, J, n, i, j) \]

\[-(21886i^4 - 118028ji^3 + 47480ni^3 + 244165i^3 + 218100j^2i^2 + 23980n^2i^2 - 802518ji^2 - 160152jni^2 + 219325ni^2 + 480993i^2 - 156416j^3i - 1632n^3i + 734634j^2i - 33556jn^2i - 50857n^2i - 418482ji + 151848j^2ni - 259606jni - 407910ni - 969246i + 34480j^4 - 18n^4 - 207332j^3 - 2040jn^3 - 457n^3 + 275232j^2 + 16996j^2n^2 - 20824jn^2 - 4167n^2 + 59382j - 42928j^3n + 143470j^2n - 41586jn - 15948n - 21060)N^6 - 6I(187i^3 + 884n^2i^2 + 14678ni^2 + 57967i^2 + 884n^3i - 1632j^2i - 1768jn^2i + 17095n^2i - 119975ji - 29768jni + 113065ni + 254940i + 1768j^3 + 26784j^2 - 800jn^2 - 56385j + 3776j^2n - 14103jn)N^5 + J(7696i^4 - 26936ji^3 + 135368ni^3 + 914918i^3 + 11544j^2i^2 + 128080n^2i^2 - 3123939ji^2 - 469848jni^2 + 1222792ni^2 + 2445516i^2 + 30784j^3i + 408n^3i + 3158151j^2i - 193984jn^2i - 124594n^2i - 2822667ji + 466152j^2ni - 1729387jni - 1379574ni - 3466116i - 15392j^4 - 927346j^3 - 6120jn^3 + 758460j^2 + 68104j^2n^2 - 34114jn^2 + 689364j - 135856j^3n + 568135j^2n + 149691jn)N^5 - 9IjJ(716n^2 + 8248n + 23381)N^4 - 54I^2(252in^2 - 272jn^2 + 2760in - 2776jn + 7377i - 6752j)N^4 + 3J^2(7288i^4 + 7288ji^3 - 5466j^2i^2 - 67072n^2i^2 - 840272ni^2 - 2631694i^2 - 364j^3i + 100780jn^2i + 72496n^2i + 3883759ji + 1256708jnw + 896840ni + 2717076i + 1822j^4 - 1342069j^2 + 35188j^2n^2 - 45644jn^2 - 1709010j - 436808j^2n - 563062jn)N^4 - 54(2i - j)(i + j + 2)J^3(n + 4)(34n + 211)N^3 + \ldots \]
\[ \mathcal{P}(N, I, J, n, i, j) \]

\[ \cdots + 162IJ^2(n + 4)(34n + 211)N^3 + 108I^3(i - 2j)(n + 4)(34n + 211)N^3 + 9(67888i^4 - 371924ji^3 + 142848ni^3 + 230800i^3 + 707820j^2i^2 + 71940n^2i^2 - 660372ji^2 - 482904jni^2 + 191808ni^2 + 130420i^2 - 540608j^3i - 4896n^3i + 443604j^2i - 100668jn^2i - 116667n^2i + 130316ji + 460440j^2ni - 99288jni - 381864ni - 181221i + 139120j^4 - 108n^4 - 76568j^3 - 6120jn^3 - 2013n^3 + 146560j^2 + 50988j^2n^2 - 16164jn^2 - 13788n^2 - 105576j - 132048j^3n + 122400j^2n - 25878jn - 41001n - 44478)N^3 + 162I(408i^3 + 884n^2i^2 - 1326ji^2 + 7988ni^2 + 15192i^2 + 884n^3i + 1020j^2i - 1768jn^2i + 10465n^2i - 28254ji - 15062jni + 45380ni + 63579i + 3372j^2 - 800jn^2 - 21018j + 1124j^2n - 10290jn)N^2 + 54J(204i^4 - 714ji^3 - 67480ni^3 - 200400i^3 + 306j^2i^2 - 64040n^2i^2 + 701100ji^2 + 235536jni^2 - 137462ni^2 + 159690i^2 + 816j^3i - 204n^3i - 701676j^2i + 96992jn^2i + 63827n^2i - 451641ji - 236136j^2ni + 137675jni + 202284ni + 29349i - 408j^4 + 207660j^3 + 3060jn^3 + 222159j^2 - 34052j^2n^2 - 5893jn^2 + 149382j + 69968j^3n - 26777j^2n + 5901jn)N^2 + 972IJ(n + 2)(179n + 384)N + 5832I^2(n + 2)(63ni + 104i - 68j - 68jn)N - 324J^2(n + 2)(204i^3 - 306ji^2 - 16768ni^2 - 50202i^2 - 306j^2i + 73800ji + 25195jni + 18124ni + 52536i + 204j^3 - 25524j^2 - 33009j - 8797j^2n - 11411jn)N + 49572(2i - j)(i + j + 2)J^3(n + 1)(n + 2) - 148716IJ^2(n + 1)(n + 2) - 99144I^3(i - 2j)(n + 1)(n + 2) + 1458(n + 1)(n + 2)(9n^2 + 80n + 136i - 68j + 159) \]
\[ \mathcal{R}(N, n) \]

\[
(18n^4 + 457n^3 + 4167n^2 + 15948n + 21060)N^6 \\
- 9(108n^4 + 2013n^3 + 13788n^2 + 41001n + 44478)N^3 \\
+ 1458(n + 1)(n + 2)(n^2 + 80n + 159)
\]
## Analogs for Different Step Sets

<table>
<thead>
<tr>
<th>step set</th>
<th>number of closed paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 {↑, ↩, →}</td>
<td>( f(n, 0, 0) = 0 )</td>
</tr>
<tr>
<td>2 {↑, ↩, ↘}</td>
<td>( f(2n, 0, 0) = \frac{4^n(1/2)n}{(1)n+1} )</td>
</tr>
<tr>
<td>3 {↑, ↩, ↘}</td>
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<tr>
<td>5 {←, ↓, ↩}</td>
<td>( f(3n, 0, 0) = \frac{2·27^{n-1}(4/3)<em>{n-1}(5/3)</em>{n-1}}{(5/2)<em>{n-1}(3)</em>{n-1}} )</td>
</tr>
<tr>
<td>6 {↑, →, ↘}</td>
<td>( f(3n, 0, 0) = \frac{2·27^{n-1}(4/3)<em>{n-1}(5/3)</em>{n-1}}{(5/2)<em>{n-1}(3)</em>{n-1}} )</td>
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<td>( f(3n, 0, 0) = \frac{27^{n-1}(4/3)<em>{n-1}(5/3)</em>{n-1}}{(3)<em>{n-1}(4)</em>{n-1}} )</td>
</tr>
<tr>
<td>8 {↑, ↘, ↘}</td>
<td>( f(4n, 0, 0) = \frac{2·64^{n-1}(5/4)<em>{n-1}(3/2)</em>{n-1}(7/4)<em>{n-1}}{(2)</em>{n-1}(5/2)<em>{n-1}(3)</em>{n-1}} )</td>
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<tr>
<td>9 {←, ↘, ↘}</td>
<td>( f(4n, 0, 0) = \frac{2·64^{n-1}(5/4)<em>{n-1}(3/2)</em>{n-1}(7/4)<em>{n-1}}{(2)</em>{n-1}(5/2)<em>{n-1}(3)</em>{n-1}} )</td>
</tr>
<tr>
<td>10 {↖, ↑, ↘}</td>
<td>( f(n, 0, 0) = 0 )</td>
</tr>
<tr>
<td>11 {↖, ↗, ↘}</td>
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</table>
Gessel’s Conjecture

For $S = \{\leftarrow, \rightarrow, \uparrow, \downarrow\}$, it seems that

$$f(n) = \begin{cases} 
16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k} & \text{if } n = 2^k \\
0 & \text{if } n \text{ is odd}
\end{cases}$$
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We hoped to find a certifying operator $P(N, I, J, n, i, j)$ for proving this conjecture.
Gessel’s Conjecture

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16^k \frac{(5/6)_k}{(2)_k} \frac{(1/2)_k}{(5/3)_k} & \text{if } n = 2k \\
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We hoped to find a certifying operator $\mathcal{P}(N, I, J, n, i, j)$ for proving this conjecture.

But the corresponding computations were too expensive.
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We hoped to find a certifying operator $\mathcal{P}(N, I, J, n, i, j)$ for proving this conjecture.

But the corresponding computations were too expensive.

The conjecture remains open.
Integration of Algebraic Functions
using Gröbner Bases

Manuel Kauers
Recall: Integration of Rational Functions

INPUT: $f \in k(x)$
Recall: Integration of Rational Functions

INPUT: \( f \in k(x) \)
OUTPUT: \( g \in k(x), \gamma_1, \ldots, \gamma_n \in \bar{k}, p_1, \ldots, p_n \in \bar{k}(x) \) with

\[
\int f = g + \gamma_1 \log(p_1) + \cdots + \gamma_n \log(p_n).
\]
Recall: Integration of Rational Functions

**INPUT:** $f \in k(x)$

**OUTPUT:** $g \in k(x)$, $\gamma_1, \ldots, \gamma_n \in \bar{k}$, $p_1, \ldots, p_n \in \bar{k}(x)$ with

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**THEOREM:** Such $g, \gamma_i, p_i$ exist for every $f$. 
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ALGORITHM (Sketch):
Polynomial Part $\rightarrow$ Rational Part $\rightarrow$ Logarithmic Part
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ALGORITHM (Sketch):

Polynomial Part $\rightarrow$ Rational Part $\rightarrow$ Logarithmic Part

Write $f = f_0 + \frac{a}{b}$ with $f_0, a, b \in k[x]$ and $\partial_x a < \partial_x b$. 
Recall: Integration of Rational Functions

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ALGORITHM (Sketch):

**Polynomial Part** → **Rational Part** → **Logarithmic Part**

Write \( f = f_0 + \frac{a}{b} \) with \( f_0, a, b \in k[x] \) and \( \partial_x a < \partial_x b \).

\( f_0 \) is easy to integrate. We are left with \( \frac{a}{b} \).
Recall: Integration of Rational Functions

INPUT: $f \in k(x)$
OUTPUT: $g \in k(x)$, $\gamma_1, \ldots, \gamma_n \in \bar{k}$, $p_1, \ldots, p_n \in \bar{k}(x)$ with

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ALGORITHM (Sketch):

Polynomial Part $\rightarrow$ Rational Part $\rightarrow$ Logarithmic Part

Find $g \in k(x)$ and $u, v \in k[x]$ with $v$ squarefree such that

$$\frac{a}{b} = Dg + \frac{u}{v}.$$
Recall: Integration of Rational Functions

INPUT: \( f \in k(x) \)
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\( g \) enters the rational part. We are left with \( \frac{u}{v} \).
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ALGORITHM (Sketch):
Polynomial Part → Rational Part → Logarithmic Part

Let \( \gamma_1, \ldots, \gamma_n \in \bar{k} \) be the distinct roots of

\[
\text{res}_x(u - tDv, v) \in k[t]
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and \( p_i = \gcd(u - \gamma_i Dv, v) \in \bar{k}[x] \).
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\]

and \( p_i = \gcd(u - \gamma_i Dv, v) \in \bar{k}[x] \). Then \( \frac{u}{v} = \sum_i \gamma_i \frac{Dp_i}{p_i} \).
Generalizations

\[ \int \frac{x^2 - x + 1}{x^3 + 2x - 3} \, dx = ? \]
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\[ \int \frac{x^2 - x + 1}{x^3 + 2x - 3} \, dx = ? \]
\[ \downarrow \]
\[ \int \frac{e^{x^2} - x + \log x}{x^3 - \log(x^2 + 1)} \, dx = ? \]

Elementary transcendental functions: Risch’s algorithm
Generalizations

\[ \int \frac{x^2 - x + 1}{x^3 + 2x - 3} \, dx = ? \quad \rightarrow \quad \int \frac{x^2 - x + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \, dx = ? \]

\[ \int e^{x^2} - x + \log x \quad \frac{dx}{x^3 - \log(x^2 + 1)} = ? \]

Algebraic functions: Davenport's or Trager's algorithm
Generalizations

\[ \int \frac{x^2 - x + 1}{x^3 + 2x - 3} \, dx = ? \quad \longrightarrow \quad \int \frac{x^2 - x + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \, dx = ? \]

\[ \int e^{x^2} - x + \log x \frac{dx}{x^3 - \log(x^2 + 1)} = ? \quad \longrightarrow \quad \int \frac{e^{x^2} - x + \log x}{\sqrt{x^3 - \log(x^2 + 1)}} \, dx = ? \]

Arbitrary elementary functions: Bronstein’s algorithm
Generalizations

\[
\int \frac{x^2 - x + 1}{x^3 + 2x - 3} \, dx =? \quad \rightarrow \quad \int \frac{x^2 - x + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \, dx =? \\
\int \frac{e^{x^2} - x + \log x}{x^3 - \log(x^2 + 1)} \, dx =? \quad \rightarrow \quad \int \frac{e^{x^2} - x + \log x}{\sqrt{x^3 - \log(x^2 + 1)}} \, dx =?
\]

Today: algebraic functions
Algebraic Functions

Our domain of interest:

\[ k \]
Algebraic Functions

Our domain of interest:

\[ k(x) \]
Algebraic Functions

Our domain of interest:

\( k(x)[y] \)
Algebraic Functions

Our domain of interest:

\[ k(x)[y]/\langle m \rangle \]

\[ m \in k[x, y] \text{ irreducible over } \bar{k}(x) \]
Algebraic Functions

Our objects of interest:

\[ f \in k(x)[y]/\langle m \rangle \]

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Our objects of interest:

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\[ m \in k[x, y] \text{ irreducible over } \overline{k}(x) \]

Note: There are two algebraic functions:
Our objects of interest:

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\[ m \in k[x, y] \text{ irreducible over } \bar{k}(x) \]

Note: There are two algebraic functions:
y (the generator of the field) and
Algebraic Functions

Our objects of interest:

\[ f \in k(x)[y]/\langle m \rangle \]

\[ m \in k[x, y] \text{ irreducible over } \bar{k}(x) \]

Note: There are two algebraic functions:
\( y \) (the generator of the field) and \( f \) (expressed in terms of \( x \) and \( y \)).
Integration of Algebraic Functions

INPUT: $f \in k(x)[y]/\langle m \rangle$
Integration of Algebraic Functions

INPUT: $f \in k(x)[y]/\langle m \rangle$

OUTPUT: $g \in k(x)[y]/\langle m \rangle$, $\gamma_1, \ldots, \gamma_n \in \overline{k}$, $p_1, \ldots, p_n \in \overline{k}(x)[y]/\langle m \rangle$ such that

$$\int f = g + \gamma_1 \log(p_1) + \cdots + \gamma_n \log(p_n)$$

or $\perp$ if no such data exists.
Integration of Algebraic Functions

INPUT: \( f \in k(x)[y]/\langle m \rangle \)

OUTPUT: \( g \in k(x)[y]/\langle m \rangle, \gamma_1, \ldots, \gamma_n \in \overline{k}, p_1, \ldots, p_n \in \overline{k(x)[y]}/\langle m \rangle \) such that

\[
\int f = g + \gamma_1 \log(p_1) + \cdots + \gamma_n \log(p_n)
\]

or \( \perp \) if no such data exists.

THEOREM (Liouville) \( g, \gamma_i, p_i \) exist \( \iff \) \( \int f \) is elementary
Integration of Algebraic Functions

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OUTPUT: \( g \in k(x)[y]/\langle m \rangle, \gamma_1, \ldots, \gamma_n \in \bar{k}, p_1, \ldots, p_n \in \bar{k}(x)[y]/\langle m \rangle \) such that

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ALGORITHM (Trager, sketch)
“Polynomial” Part \( \rightarrow \) Algebraic Part \( \rightarrow \) Logarithmic Part
Integration of Algebraic Functions

INPUT: \( f \in k(x)[y]/\langle m \rangle \)

OUTPUT: \( g \in k(x)[y]/\langle m \rangle, \gamma_1, \ldots, \gamma_n \in \bar{k}, \)
\( p_1, \ldots, p_n \in \bar{k}(x)[y]/\langle m \rangle \) such that

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\int f = g + \gamma_1 \log(p_1) + \cdots + \gamma_n \log(p_n)
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ALGORITHM (Trager, sketch)

“Polynomial” Part \( \rightarrow \) Algebraic Part \( \rightarrow \) Logarithmic Part

Choose a regular point \( x_0 \) and perform a change of variables

\[
x \mapsto \frac{1}{x' - x_0}, \quad dx \mapsto \frac{1}{(x' - x_0)^2} dx'
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Integration of Algebraic Functions

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(Undo the substitution in the end!)
Integration of Algebraic Functions

**INPUT:** $f \in k(x)[y]/\langle m \rangle$

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or $\perp$ if no such data exists.

**ALGORITHM** (Trager, sketch)

“Polynomial” Part $\rightarrow$ **Algebraic Part** $\rightarrow$ Logarithmic Part

Write the new integrand as $f = Dg + h$ such that $h$ only has simple poles
Integration of Algebraic Functions

INPUT: $f \in k(x)[y]/\langle m \rangle$

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ALGORITHM (Trager, sketch)

“Polynomial” Part $\rightarrow$ Algebraic Part $\rightarrow$ Logarithmic Part
The Logarithmic Part

Given $h \in k(x)[y]/\langle m \rangle$ with only simple poles.
The Logarithmic Part

Given $h \in k(x)[y]/\langle m \rangle$ with only simple poles.

- Write $h = \frac{u}{v}$ with $u \in k[x, y]$ and $v \in k[x]$ and consider

$$R(t) := \text{res}_x(\text{res}_y(u - tDv, m), v) \in k[t].$$
The Logarithmic Part

Given \( h \in k(x)[y]/\langle m \rangle \) with only simple poles.

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- Let \( \gamma_1, \ldots, \gamma_n \in \bar{k} \) be a basis for the splitting field of \( R \).
The Logarithmic Part

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- Let \( \gamma_1, \ldots, \gamma_n \in \bar{k} \) be a basis for the splitting field of \( R \).

- For each \( \gamma_i \), construct an associated divisor \( a_i \) specifying the singularities of a potential logand for \( \gamma_i \).
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- For each \( \gamma_i \), construct an associated divisor \( \alpha_i \) specifying the singularities of a potential logand for \( \gamma_i \).
- For each \( \alpha_i \), decide whether some power of it is principal.
- Any divisor with \( \alpha_i^r = \langle p \rangle \) gives rise to a contribution \( \frac{1}{r} \gamma_i \log(p) \) to the logarithmic part.
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- For each \( a_i \), decide whether some power of it is principal.

- Any divisor with \( a_i^r = \langle p \rangle \) gives rise to a contribution \( \frac{1}{r} \gamma_i \log(p) \) to the logarithmic part.

- If some of the divisors do not have a principal power, then \( \int h \) is not elementary.
Hardly any CAS has a complete implementation of this
Problem

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- Because the details are difficult and time consuming
- Is there a simple alternative?
- How about the rational case?
Czichowski’s Observation

Let $u/v \in k(x)$ be such that $\deg u < \deg v$ and $v$ is squarefree.
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Let $u/v \in k(x)$ be such that $\deg u < \deg v$ and $v$ is squarefree. Let $G$ be the Gröbner basis of $\langle v, u - tDv \rangle \subseteq k[x, t]$ wrt. an order eliminating $x$. 
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Then \( G \), when sorted according to ascending leading terms, has the form

\[
\{ c_0(t), c_1(t)p_1(t, x), c_2(t)p_2(t, x), \ldots, c_r(t)p_r(t, x) \}
\]

for some square-free polynomials \( c_i, p_i \) with
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\begin{aligned}
&\quad c_{i+1}(t) | c_i(t) \text{ for } i = 0, \ldots, r - 1, \text{ say} \\
&\bar{c}_i(t) = c_i(t)/c_{i+1}(t) \in k[t] \quad (i = 0, \ldots, r - 1).
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\[
\int \frac{u}{v} = \sum_{i=0}^{r-1} \sum_{\gamma: \bar{c}_i(\gamma) = 0} \gamma \log(p_{i+1}(\gamma, x))
\]
Example

Let \[ \frac{u}{v} = \frac{4x^2 - x - 12}{x^3 - x^2 - 4x + 4} \].
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Then \( G = \{(t - 3)(2t - 1), (2t - 1)(x - 1), 5x^2 + 6t - 23\}. \)
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Logands: 
\[
p_1 = x - 1, \quad p_2 = 5x^2 + \frac{1}{2}6 - 23 = 5(x^2 - 4)
\]

Solution: 
\[
\int \frac{u}{v} = 3 \log(x - 1) + \frac{1}{2} \log(x^2 - 4)
\]
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What is the variety of \( \langle G \rangle = \langle v, u - tDv \rangle \subseteq k[x, t] \)?
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What is the variety of \( \langle G' \rangle = \langle v, u - tDv \rangle \subseteq k[x, t] \)?
Example

Let \( \frac{u}{v} = \frac{4x^2 - x - 12}{x^3 - x^2 - 4x + 4} \).

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Note:
Example

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Note: \( \text{Gb}(\langle G \rangle + \langle t - 3 \rangle) = \{t - 3, x - 1\} \)
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In general, if \( c \) is an irreducible factor of \( c_0 \), then

\[ Gb(\langle c, v, u - tDv \rangle) = \{ c(t), p(x, t) \} \leq k[x, t] \]

for \( p \) such that \( p(x, \gamma) = \gcd(v, u - \gamma Dv) \) when \( c(\gamma) = 0 \).
Obvious Question:
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- Does this work also for algebraic functions?
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- Does this work also for algebraic functions?
- Answer: Not always, but often.
Example 1

\[ \int \frac{\sqrt{x^2 + 1}}{x^4 + 1} \, dx = ? \]
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\[ f = \frac{u}{v} = \frac{y}{x^4 + 1} \in \mathbb{Q}(x)[y]/\langle y^2 - (x^2 + 1) \rangle. \]
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\textbf{Want:} polynomials \( c \in k[t] \) and \( p \in k[x, y, t] \) such that

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*Indeed:*

\[ \langle u - tDv, v, m \rangle = \langle 128t^4 + 16t^2 + 1, x - 32t^3y, y^2 - (x^2 + 1) \rangle. \]
Example 1

\[ \int \frac{\sqrt{x^2 + 1}}{x^4 + 1} \, dx = \sum_{\gamma: 128\gamma^4 + 16\gamma^2 + 1 = 0} \gamma \log(x - 32\gamma^3 \sqrt{x^2 + 1}) \]

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**Note:** The Gröbner basis of \( \langle u - tDv, v, m \rangle \) \((x > y > t)\) is

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\int \frac{\sqrt{x^2 + 1}}{x^4 + 1} \, dx = \frac{1}{2} \left( \sqrt{1 + i \tan \frac{i\sqrt{x^2 + 1}}{\sqrt{1 + i x}} + \sqrt{1 - i \tan \frac{i\sqrt{x^2 + 1}}{\sqrt{1 - i x}}} \right)
\]

\[f = \frac{u}{v} = \frac{y}{x^4 + 1} \in \mathbb{Q}(x)[y]/\langle y^2 - (x^2 + 1) \rangle.\]

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- If not, and if $c$ is not irreducible, it is promising to consider $\langle q, v, u - tDv, m \rangle$ for each $q \mid c$ separately.
Observation

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- If this also does not help, give up.
Pseudocode

\[ G := \text{GröbnerBasis}\left\{ v, u - tDv, m \right\}; \quad int = 0 \]
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\[ A := \{1\} \]
G := GröbnerBasis(\{v, u - tDv, m\}); int = 0
for all irreducible factors q of min G do
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    for n from 1 to 12 do
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for \( n \) from 1 to 12 do
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for all \( p \) in \( A \) do
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\quad for \ n \ from 1 \ to \ 12 \ do

\quad \quad A := \text{GröbnerBasis}\left((A \cdot G) \cup \{q, m\}\right)

\quad for all p in A do

\quad \quad if A = \text{GröbnerBasis}\left(\{q, p, m\}\right) \ then
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    for $n$ from 1 to 12 do
        $A := \text{GröbnerBasis}((A \cdot G) \cup \{q, m\})$
        for all $p$ in $A$ do
            if $A = \text{GröbnerBasis}(\{q, p, m\})$ then
                $int := int + \sum_{\gamma : q(\gamma) = 0} \frac{\gamma}{n} \log(p(x, y, \gamma))$
            next $p$
next $q$
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return \( \text{int} \)
Example 2

\[ \int \frac{\sqrt{x^2 + 1}}{x^3 + 1} \, dx = ? \]
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\[ f = \frac{u}{v} = \frac{y}{x^3 + 1} \in \mathbb{Q}(x)[y]/\langle y^2 - (x^2 + 1) \rangle. \]
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The Gröbner basis of \( \langle v, u - tDv, m \rangle \) (block drl \([x, y] > [t]\)) is

\( \{(9t^2 + 1)(9t^2 - 2), y + 9yt^2 - 27t^3 - 3t, x + 3t - 1, y^2 + 3yt - 9t^2 - 2\} \).
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For \( q = 9t^2 + 1 \), we find
\[ \langle q, m \rangle + \langle v, u - tDv \rangle = \langle 9t^2 + 1, x + 3ty - 1, y^2 + 3ty - 1 \rangle \]
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\[
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\]

\[
\langle q, m \rangle + (\langle v, u - tDv \rangle)^2 = \langle 9t^2 - 2, x + 3ty - 1, y^2 - 6ty + 2 \rangle
\]
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\[
\int \frac{\sqrt{x^2 + 1}}{x^3 + 1} \, dx = \sum_{\gamma : 9\gamma^2 + 1 = 0} \gamma \log(x + 3\gamma \sqrt{x^2 + 1} - 1) \\
+ \sum_{\gamma : 9\gamma^2 - 2 = 0} \frac{\gamma}{2} \log(x + 3\gamma \sqrt{x^2 + 1} - 1)
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- A typical integrand is

$$ f = \left( 360x^5 + 86x^4 + 256x^3 + 1218x^2 + 128x + 640 
- (96x^3 + 1024x^2 + 64x + 664) \sqrt{x^3 + 1} \right) / 
\left( 96x^6 + 40x^5 + 64x^4 + 568x^3 + 40x^2 + 64x + 472 
- (64x^4 + 336x^3 + 24x^2 + 48x + 488) \sqrt{x^3 + 1} \right). $$
Performance

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- We applied the integrators to 1000 integrals involving \( y = \sqrt{x^3 + 1} \) and 1000 integrals involving \( y = (x^2 + 1)^{1/3} \).
- A typical integral is

\[
\int f = \frac{1}{4} (1 - i\sqrt{47}) \log \left( x - \frac{1}{4} (1 - i\sqrt{47}) \sqrt{x^3 + 1} - \frac{1}{4} (3 + i\sqrt{47}) \right) + \frac{1}{4} (1 + i\sqrt{47}) \log \left( x - \frac{1}{4} (1 + i\sqrt{47}) \sqrt{x^3 + 1} - \frac{1}{4} (3 - i\sqrt{47}) \right) + 2 \log (x - 2\sqrt{x^3 + 1 + 3})
\]
Integrals involving $\sqrt{x^3 + 1}$
Integrals involving \((x^2 + 1)^{1/3}\)