How to Find Algebraic Relations?

Manuel Kauers
RISC-Linz, Austria
Let \( f_1(n), f_2(n), \ldots, f_m(n) \) be sequences in a field \( \mathbb{K} \).
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Suppose that $p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m]$ is such that

$$p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).$$
Algebraic Relations – Elementary Viewpoint

Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Suppose that $p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m]$ is such that

$$p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).$$

Then $p$ is called an \textit{algebraic relation} of the sequences $f_i(n)$. 
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Suppose that $p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m]$ is such that

$$p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).$$

Then $p$ is called an algebraic relation of the sequences $f_i(n)$.

Observations
Algebraic Relations – Elementary Viewpoint

Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Suppose that $p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m]$ is such that

$$p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).$$

Then $p$ is called an *algebraic relation* of the sequences $f_i(n)$.

Observations

- If $p$ and $q$ are algebraic relations, then so is $p + q$
Let \( f_1(n), f_2(n), \ldots, f_m(n) \) be sequences in a field \( \mathbb{K} \).

Suppose that \( p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m] \) is such that
\[
p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).
\]
Then \( p \) is called an \textit{algebraic relation} of the sequences \( f_i(n) \).

\textit{Observations}

- If \( p \) and \( q \) are algebraic relations, then so is \( p + q \)
- If \( p \) is an algebraic relation, then so is \( r \cdot p \) for any polynomial \( r \)
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Suppose that $p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m]$ is such that

$$p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).$$

Then $p$ is called an algebraic relation of the sequences $f_i(n)$.

Observations

- If $p$ and $q$ are algebraic relations, then so is $p + q$.
- If $p$ is an algebraic relation, then so is $r \cdot p$ for any polynomial $r$.
- If $p^s$ is an algebraic relation, then so is $p$.
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Suppose that $p(x_1, \ldots, x_m) \in \mathbb{K}[x_1, \ldots, x_m]$ is such that

$$p(f_1(n), f_2(n), \ldots, f_m(n)) = 0 \quad (n \geq 0).$$

Then $p$ is called an \textit{algebraic relation} of the sequences $f_i(n)$.

\textbf{Observations}

▶ If $p$ and $q$ are algebraic relations, then so is $p + q$
▶ If $p$ is an algebraic relation, then so is $r \cdot p$ for any polynomial $r$
▶ If $p^s$ is an algebraic relation, then so is $p$

\textbf{Consequence} The set of all algebraic relations forms a radical ideal.
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$. 
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Consider the ring homomorphism defined via

$$
\phi: \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}^\mathbb{N},
$$
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Consider the ring homomorphism defined via

$$
\phi: \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}^\mathbb{N},
$$

$$
c \mapsto (c)_{n \geq 0} \quad (c \in \mathbb{K}),
$$
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Consider the ring homomorphism defined via

$$
\phi: \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}^\mathbb{N},
$$

$$
c \mapsto (c)_{n \geq 0} \quad (c \in \mathbb{K}),
$$

$$
x_i \mapsto (f_i(n))_{n \geq 0} \quad (i = 1, \ldots, m).
$$
Let \( f_1(n), f_2(n), \ldots, f_m(n) \) be sequences in a field \( \mathbb{K} \).

Consider the ring homomorphism defined via

\[
\phi: \mathbb{K}[x_1, \ldots, x_m] \to \mathbb{K}^\mathbb{N},
\]

\[
c \mapsto (c)_{n \geq 0} \quad (c \in \mathbb{K}),
\]

\[
x_i \mapsto (f_i(n))_{n \geq 0} \quad (i = 1, \ldots, m).
\]

The ideal of algebraic relations among \( f_1(n), \ldots, f_m(n) \) is precisely the kernel of this map, \( \text{ker} \ \phi \).
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$. 
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Consider the set of points

$$P := \{(f_1(n), \ldots, f_m(n)) : n \in \mathbb{N}\} \subseteq \mathbb{K}^m.$$
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Consider the set of points

$$P := \{ (f_1(n), \ldots, f_m(n)) : n \in \mathbb{N} \} \subseteq \mathbb{K}^m.$$ 

The ideal of algebraic relations among $f_1(n), \ldots, f_m(n)$ is precisely the vanishing ideal of this set, $I(P)$. 
Let $f_1(n), f_2(n), \ldots, f_m(n)$ be sequences in a field $\mathbb{K}$.

Consider the set of points

$$P := \{(f_1(n), \ldots, f_m(n)) : n \in \mathbb{N}\} \subseteq \mathbb{K}^m.$$ 

The ideal of algebraic relations among $f_1(n), \ldots, f_m(n)$ is precisely the vanishing ideal of this set, $I(P)$.

**Summary**

$$\{p \in \mathbb{K}[x_1, \ldots, x_n] : p(f_1, \ldots, f_m) \equiv 0\} = \ker \phi = I(P).$$
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).
Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81 (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$. 
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

*Exercise 6.81* (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

*In other words* Find the ideal $\mathfrak{a} \trianglelefteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$. 
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81: (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, find the ideal $\alpha \triangleleft \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$.

Answer: $\alpha = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle$. 
Example: Fibonacci Numbers

\[ a = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle. \]
Example: Fibonacci Numbers

Answer \[ a = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle. \]

This is \( V(a) \).
Example: Fibonacci Numbers

Answer \[ a = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle. \]

This is \( V(\alpha) \).

It consists of two irreducible components.
Example: Fibonacci Numbers

Answer \( a = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle \).

This is \( V(a) \).

It consists of two irreducible components.

Each component carries “half” of the points \( (F_{n+1}, F_n) \).
Example: Fibonacci Numbers

Answer \[ \alpha = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle. \]

This is \( V(\alpha) \).

It consists of two irreducible components.

Each component carries “half” of the points \((F_{n+1}, F_n)\).

Based on the geometric interpretation, it is straightforward to prove that \( \alpha \) is really the ideal claimed above.
Relevance for Computer Algebra

We desire algorithms
Relevance for Computer Algebra

We desire algorithms

- proving
Relevance for Computer Algebra

We desire algorithms

- proving
- finding
We desire algorithms

- proving
- finding
- using
Relevance for Computer Algebra

We desire algorithms

- proving
- finding
- using

algebraic relations for specific classes of sequences.
Relevance for Computer Algebra

We desire algorithms

▶ proving
▶ finding
▶ using

algebraic relations for specific classes of sequences.

Problem: Given sequences $f_1(n), \ldots, f_m(n)$ and $p \in \mathbb{K}[x_1, \ldots, x_n]$, decide

$$\forall \ n \geq 0 : p(f_1(n), \ldots, f_m(n)) = 0$$
We desire algorithms

- proving
- finding
- using

algebraic relations for specific classes of sequences.

Problem: Given sequences $f_1(n), \ldots, f_m(n)$ and $p \in \mathbb{K}[x_1, \ldots, x_n]$, decide

$$\forall \ n \geq 0 : p(f_1(n), \ldots, f_m(n)) = 0$$

(e.g., gfun can do this for $f_i(n)$ P-finite.)
Relevance for Computer Algebra

We desire algorithms

- proving
- finding
- using

algebraic relations for specific classes of sequences.

Applications: Summation/Integration of special functions.
We desire algorithms

- proving
- finding
- using

algebraic relations for specific classes of sequences.

Applications: Summation/Integration of special functions.

We have an algorithm that can find identities like

\[ \sum_{k=0}^{n} ((k - \sqrt{k} + 1)H_k + 1)\sqrt{k!} = (1 + (n + 1)H_n)\sqrt{n!} \]

which depend on exploiting the relation \( \sqrt{n^2} - n = 0 \).
Relevance for Computer Algebra

We desire algorithms

- proving
- finding
- using

algebraic relations for specific classes of sequences.

Applications: Summation/Integration of special functions. We want to have an algorithm that can find identities like

\[
\sum_{k=0}^{n} \frac{((-1)^k - 1)x + (-1)^k + 1}{2U_k(x) + (-1)^k - 1} = \frac{(1 - 2x)U_n(x) + (-1)^n + U_{n+1}(x)}{2U_n(x) + (-1)^n - 1}
\]

which also depend on nontrivial relations.
Relevance for Computer Algebra

We desire algorithms

- proving
- finding
- using

algebraic relations for specific classes of sequences.

Today we discuss finding.
Problem Specification

GIVEN: Sequences $f_1(n), \ldots, f_m(n)$ in $\mathbb{Z}$
Problem Specification

GIVEN: Sequences \( f_1(n), \ldots, f_m(n) \) in \( \mathbb{K} \)
FIND: Polynomials \( b_1, \ldots, b_r \in \mathbb{K}[x_1, \ldots, x_m] \) such that

\[ \langle b_1, \ldots, b_r \rangle \subseteq \mathbb{K}[x_1, \ldots, x_m] \]

is the ideal of algebraic relations among the \( f_i(n) \).
Problem Specification

GIVEN: Sequences $f_1(n), \ldots, f_m(n)$ in $\mathbb{K}$
FIND: Polynomials $b_1, \ldots, b_r \in \mathbb{K}[x_1, \ldots, x_m]$ such that

$$\langle b_1, \ldots, b_r \rangle \leq \mathbb{K}[x_1, \ldots, x_m]$$

is the ideal of algebraic relations among the $f_i(n)$.

Of course, “given sequences” makes only sense when attention is restricted to particular classes of sequences that admit finitary representations, e.g., by defining recurrence equations.
A Brute Force Attack
A Brute Force Attack

Suppose we know how to prove algebraic relations for a certain class of sequences.
A Brute Force Attack

Suppose we know how to prove algebraic relations for a certain class of sequences.

Then we can also nd algebraic relations.
A Brute Force Attack

Suppose we know how to *prove* algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*. 
A Brute Force Attack

Suppose we know how to prove algebraic relations for a certain class of sequences.

Then we can also nd algebraic relations.

By just making an ansatz.

Consider the polynomial

\[ p(x_1, x_2) = a_0 x_1^2 + a_1 x_1 x_2 + a_2 x_2^2 + a_3 x_1 + a_4 x_2 + a_5 \]

with undetermined coefficients \( a_0, a_1, \ldots, a_5 \).
A Brute Force Attack

Suppose we know how to prove algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*.

For \( p(x_1, x_2) \) to be an algebraic relation of \( f_1(n), f_2(n) \), we must have

\[
\forall n \geq 0 : p(f_1(n), f_2(n)) = 0
\]
A Brute Force Attack

Suppose we know how to *prove* algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*.

In particular:

\[
p(f_1(0), f_2(0)) = 0 \\
p(f_1(1), f_2(1)) = 0 \\
p(f_1(2), f_2(2)) = 0 \\
\vdots \\
p(f_1(5), f_2(5)) = 0
\]
A Brute Force Attack

Suppose we know how to *prove* algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*.

In particular:

\[
\begin{align*}
a_0 f_1(0)^2 + a_1 f_1(0) f_2(0) + a_2 f_2(0)^2 + a_3 f_1(0) + a_4 f_2(0) + a_5 &= 0 \\
a_0 f_1(1)^2 + a_1 f_1(1) f_2(1) + a_2 f_2(1)^2 + a_3 f_1(1) + a_4 f_2(1) + a_5 &= 0 \\
a_0 f_1(2)^2 + a_1 f_1(2) f_2(2) + a_2 f_2(2)^2 + a_3 f_1(2) + a_4 f_2(2) + a_5 &= 0 \\
&\vdots \\
a_0 f_1(5)^2 + a_1 f_1(5) f_2(5) + a_2 f_2(5)^2 + a_3 f_1(5) + a_4 f_2(5) + a_5 &= 0
\end{align*}
\]
A Brute Force Attack

Suppose we know how to \textit{prove} algebraic relations for a certain class of sequences.

Then we can also \textit{nd} algebraic relations.

By just making an \textit{ansatz}.

Solutions of the linear system give relation \textit{candidates}. 
A Brute Force Attack

Suppose we know how to *prove* algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*.

Solutions of the linear system give relation *candidates*.

True relations can be detected by the prover.
A Brute Force Attack

Suppose we know how to prove algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*.

Solutions of the linear system give relation *candidates*.

True relations can be detected by the prover.

If there are fake ones, repeat with more sample points.
A Brute Force Attack

Suppose we know how to *prove* algebraic relations for a certain class of sequences.

Then we can also *nd* algebraic relations.

By just making an *ansatz*.

Solutions of the linear system give relation *candidates*.

True relations can be detected by the prover.

If there are fake ones, repeat with more sample points.

If sufficiently many points are taken into account, no fake relations will arise.
A Brute Force Attack

Suppose we know how to prove algebraic relations for a certain class of sequences.

Then we can also nd algebraic relations.

By just making an ansatz.

Solutions of the linear system give relation candidates.

True relations can be detected by the prover.

If there are fake ones, repeat with more sample points.

If sufficiently many points are taken into account, no fake relations will arise.

For speed-up, use the Buchberger-Möller algorithm.
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

**Exercise 6.81** (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n\geq0}$ and $(F_{n+1})_{n\geq0}$.
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

**Exercise 6.81** (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, find the ideal $\alpha \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$. 
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81: (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$.

Answer: $\mathfrak{a} \supseteq \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle$. 
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81: (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words: Find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$.

Answer: $\mathfrak{a} \supseteq \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle$.

Note: We can determine all algebraic relations up to a prescribed degree, but we get no information about existence/non-existence of higher degree relations.
Example: Somos Sequences

A sequence $C_n$ satisfying a nonlinear recurrence of the form

$$C_{n+r}C_n = \alpha_1 C_{n+r-1}C_{n+1} + \alpha_2 C_{n+r-2}C_{n+2} + \cdots$$

$$\cdots + \alpha_{\lfloor r/2 \rfloor} C_{n+r-\lfloor r/2 \rfloor}C_{n+\lfloor r/2 \rfloor}$$

with $r \in \mathbb{N}$ fixed and $\alpha_1, \ldots, \alpha_{\lfloor r/2 \rfloor}$ is called a Somos sequence of order $r$. 
Example: Somos Sequences

A sequence $C_n$ satisfying a nonlinear recurrence of the form

$$C_{n+r}C_n = \alpha_1 C_{n+r-1}C_{n+1} + \alpha_2 C_{n+r-2}C_{n+2} + \cdots$$

$$\cdots + \alpha_{\lfloor r/2 \rfloor} C_{n+r-\lfloor r/2 \rfloor}C_{n+\lfloor r/2 \rfloor}$$

with $r \in \mathbb{N}$ fixed and $\alpha_1, \ldots, \alpha_{\lfloor r/2 \rfloor}$ is called a Somos sequence of order $r$.

**Question** Can a given Somos sequence of order $r$ also be viewed as a Somos sequence for some different order $r'$?
Example: Somos Sequences

A sequence $C_n$ satisfying a nonlinear recurrence of the form

$$C_{n+r}C_n = \alpha_1 C_{n+r-1}C_{n+1} + \alpha_2 C_{n+r-2}C_{n+2} + \cdots + \alpha_{\lfloor r/2 \rfloor} C_{n+r-\lfloor r/2 \rfloor}C_{n+\lfloor r/2 \rfloor}$$

with $r \in \mathbb{N}$ fixed and $\alpha_1, \ldots, \alpha_{\lfloor r/2 \rfloor}$ is called a Somos sequence of order $r$.

**Question** Can a given Somos sequence of order $r$ also be viewed as a Somos sequence for some different order $r'$?

**Example** Consider $C_n$ defined via

$$C_{n+4}C_n = C_{n+3}C_{n+1} + C_{n+2}^2, \quad C_0 = C_1 = C_2 = C_3 = 1.$$ 

Does this sequence satisfy a Somos-like recurrence of orders $5, 6, 7, 8$?
Example: Somos Sequences

Idea  Compute the algebraic relations of total degree $\leq 2$ among the terms

$$C_n, C_{n+1}, \ldots, C_{n+7}, C_{n+8}.$$
Example: Somos Sequences

Idea  Compute the algebraic relations of total degree $\leq 2$ among the terms $C_n, C_{n+1}, \ldots, C_{n+7}, C_{n+8}$.

Let $a = \langle p_1, \ldots, p_k \rangle \subseteq \mathbb{Q}[x_0, \ldots, x_8]$ be a Gröbner basis for the ideal generated by the quadratic relations.
Example: Somos Sequences

Idea  Compute the algebraic relations of total degree \( \leq 2 \) among the terms

\[ C_n, C_{n+1}, \ldots, C_{n+7}, C_{n+8}. \]

Let \( \mathfrak{a} = \langle p_1, \ldots, p_k \rangle \leq \mathbb{Q}[x_0, \ldots, x_8] \) be a Gröbner basis for the ideal generated by the quadratic relations.

Make an ansatz with undetermined coefficients for the desired relation, e.g.,

\[ C_{n+5}C_n = a_1 C_{n+4}C_{n+1} + a_2 C_{n+3}C_{n+2} \]
Example: Somos Sequences

Idea: Compute the algebraic relations of total degree \( \leq 2 \) among the terms

\[ C_n, C_{n+1}, \ldots, C_{n+7}, C_{n+8}. \]

Let \( \mathfrak{a} = \langle p_1, \ldots, p_k \rangle \subseteq \mathbb{Q}[x_0, \ldots, x_8] \) be a Gröbner basis for the ideal generated by the quadratic relations.

Make an ansatz with undetermined coefficients for the desired relation, e.g.,

\[ C_{n+5}C_n = a_1 C_{n+4}C_{n+1} + a_2 C_{n+3}C_{n+2} \]

Reduction modulo \( \mathfrak{a} \) gives

\[ x_5x_0 - a_1 x_4 x_1 - a_2 x_3 x_2 \longrightarrow \mathfrak{a} \left( 1 - \frac{1}{5} a_2 \right) x_0 x_5 - (a_1 + \frac{1}{5} a_2) x_1 x_4 \]
**Example: Somos Sequences**

**Idea**  Compute the algebraic relations of total degree $\leq 2$ among the terms

$$C_n, C_{n+1}, \ldots, C_{n+7}, C_{n+8}.$$  

Let $\mathfrak{a} = \langle p_1, \ldots, p_k \rangle \subseteq \mathbb{Q}[x_0, \ldots, x_8]$ be a Gröbner basis for the ideal generated by the quadratic relations.

Make an ansatz with undetermined coefficients for the desired relation, e.g.,

$$C_{n+5}C_n = a_1 C_{n+4}C_{n+1} + a_2 C_{n+3}C_{n+2}$$  

Reduction modulo $\mathfrak{a}$ gives

$$x_5x_0 - a_1x_4x_1 - a_2x_3x_2 \rightarrow_\mathfrak{a} (1 - \frac{1}{5}a_2)x_0x_5 - (a_1 + \frac{1}{5}a_2)x_1x_4$$

Comparing coefficients gives $a_1 = -1, a_2 = 5.$
What about a Degree Bound?

*Note* If the chosen degree bound $d$ is *sufficiently large* then we get a basis for the whole ideal.
What about a Degree Bound?

*Note* If the chosen degree bound $d$ is *sufficiently large* then we get a basis for the whole ideal.

*But* What does “sufficiently large” mean?
What about a Degree Bound?

*Note* If the chosen degree bound $d$ is *sufficiently large* then we get a basis for the whole ideal.

*But* What does “sufficiently large” mean?

*In particular* Can we compute a “sufficiently large” $d$?
What about a Degree Bound?

Note If the chosen degree bound $d$ is sufficiently large then we get a basis for the whole ideal.

But What does “sufficiently large” mean?

In particular Can we compute a “sufficiently large” $d$?

well, hardly ever...
What about a Degree Bound?

Theorem. If a class $C$ of sequences is such that
What about a Degree Bound?

**Theorem.** If a class $C$ of sequences is such that

$\Rightarrow n \in C$
Theorem. If a class $\mathcal{C}$ of sequences is such that

1. $n \in \mathcal{C}$
2. $f(n) \in \mathcal{C} \Rightarrow \prod_{k=0}^{n} f(k) \in \mathcal{C}$
What about a Degree Bound?

**Theorem.** If a class $C$ of sequences is such that

- $n \in C$
- $f(n) \in C \Rightarrow \prod_{k=0}^{n} f(k) \in C$
- There is an algorithm that produces for arbitrary given $f_1(n), \ldots, f_m(n) \in C$ a basis for their ideal of algebraic relations.
What about a Degree Bound?

Theorem. If a class $C$ of sequences is such that

- $n \in C$
- $f(n) \in C \Rightarrow \prod_{k=0}^{n} f(k) \in C$
- There is an algorithm that produces for arbitrary given $f_1(n), \ldots, f_m(n) \in C$ a basis for their ideal of algebraic relations.

Then there exists an algorithm that decides for arbitrary given $f(n) \in C$ whether $\exists n \geq 0 : f(n) = 0$. 
What about a Degree Bound?

**Theorem.** If a class $C$ of sequences is such that

- $n \in C$
- $f(n) \in C \Rightarrow \prod_{k=0}^{n} f(k) \in C$
- There is an algorithm that produces for arbitrary given $f_1(n), \ldots, f_m(n) \in C$ a basis for their ideal of algebraic relations.

Then there exists an algorithm that decides for arbitrary given $f(n) \in C$ whether $\exists n \geq 0 : f(n) = 0$.

For sufficiently rich classes $C$ there is no hope for such an algorithm.
What about a Degree Bound?

**Theorem.** If a class $C$ of sequences is such that

- $n \in C$
- $f(n) \in C \Rightarrow \prod_{k=0}^{n} f(k) \in C$
- There is an algorithm that produces for arbitrary given $f_1(n), \ldots, f_m(n) \in C$ a basis for their ideal of algebraic relations.

Then there exists an algorithm that decides for arbitrary given $f(n) \in C$ whether $\exists n \geq 0 : f(n) = 0$.

For sufficiently rich classes $C$ there is no hope for such an algorithm.

If we insist in a complete algorithm, we have to focus on smaller classes.
C-Finite Sequences
(joint work with B. Zimmermann)
C-finite Sequences

Recall $f(n)$ is C-finite if

$$f(n + r) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{r-1} f(n + r - 1)$$

for some constants $a_0, \ldots, a_{r-1} \in \mathbb{K}$. 
**C-finite Sequences**

*Recall* $f(n)$ is C-finite if

$$f(n + r) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{r-1} f(n + r - 1)$$

for some *constants* $a_0, \ldots, a_{r-1} \in \mathbb{K}$.

*Examples*
Recall $f(n)$ is C-finite if

$$f(n + r) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{r-1} f(n + r - 1)$$

for some constants $a_0, \ldots, a_{r-1} \in \mathbb{K}$.

Examples

- $n$, $n^2$, $n^3$, ...
C-finite Sequences

Recall $f(n)$ is C-finite if

$$f(n + r) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{r-1} f(n + r - 1)$$

for some constants $a_0, \ldots, a_{r-1} \in \mathbb{K}$.

Examples

- $n, n^2, n^3, \ldots$
- $2^n, 3^n, 4^n, \ldots$
C-finite Sequences

Recall $f(n)$ is C-finite if

$$f(n + r) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{r-1} f(n + r - 1)$$

for some constants $a_0, \ldots, a_{r-1} \in \mathbb{K}$.

Examples

- $n, n^2, n^3, \ldots$
- $2^n, 3^n, 4^n, \ldots$
- $F_n, U_n(x), \ldots$
C-finite Sequences

\textit{Recall} \( f(n) \) is C-finite if

\[ f(n + r) = a_0 f(n) + a_1 f(n + 1) + \cdots + a_{r-1} f(n + r - 1) \]

for some \textit{constants} \( a_0, \ldots, a_{r-1} \in \mathbb{K} \).

\textit{Recall} \( f(n) \) is C-finite if and only if

\[ f(n) = p_1(n)\phi_1^n + p_2(n)\phi_2^n + \cdots + p_s(n)\phi_s^n \quad (n \geq 0) \]

where \( \phi_i \) are the roots of the characteristic polynomial

\[ x^r - a_0 - a_1 x - a_2 x^2 - \cdots - a_{r-1} x^{r-1} \]

and \( p_i(n) \) is a polynomial whose degree is bounded by the multiplicity of the root \( \phi_i \).
Simple Examples

Example 1: $n$ and $2^n$ are algebraically independent.
Simple Examples

Example 1 $n$ and $2^n$ are algebraically independent.
(clear.)
Simple Examples

Example 1 \( n \) and \( 2^n \) are algebraically independent. (clear.)

Example 2 \( n^2 \) and \( n^3 \) are algebraically dependent.
Simple Examples

Example 1: \( n \) and \( 2^n \) are algebraically independent. (clear.)

Example 2: \( n^2 \) and \( n^3 \) are algebraically dependent. (by \( x_1^3 - x_2^2 \).)
Simple Examples

Example 1: \( n \) and \( 2^n \) are algebraically independent. (clear.)

Example 2: \( n^2 \) and \( n^3 \) are algebraically dependent. (by \( x_1^3 - x_2^2 \).)

Example 3: \((-1)^n\) is algebraically dependent.
Simple Examples

Example 1: \( n \) and \( 2^n \) are algebraically independent. (clear.)

Example 2: \( n^2 \) and \( n^3 \) are algebraically dependent. (by \( x_1^3 - x_2^2. \))

Example 3: \((-1)^n\) is algebraically dependent. (by \( x_1^2 - 1. \))
Simple Examples

Example 1: $n$ and $2^n$ are algebraically independent. (clear.)

Example 2: $n^2$ and $n^3$ are algebraically dependent. (by $x_1^3 - x_2^2$.)

Example 3: $(-1)^n$ is algebraically dependent. (by $x_1^2 - 1$.)

Example 4: $4^n$, $6^n$, $9^n$ are algebraically dependent.
Simple Examples

Example 1: $n$ and $2^n$ are algebraically independent. (clear.)

Example 2: $n^2$ and $n^3$ are algebraically dependent. (by $x_1^3 - x_2^2$.)

Example 3: $(-1)^n$ is algebraically dependent. (by $x_1^2 - 1$.)

Example 4: $4^n$, $6^n$, $9^n$ are algebraically dependent. (by $x_1 x_3 - x_2^2$.)
Simple Examples

Example 1: \( n \) and \( 2^n \) are algebraically independent. (clear.)

Example 2: \( n^2 \) and \( n^3 \) are algebraically dependent. (by \( x_1^3 - x_2^2 \).)

Example 3: \((-1)^n\) is algebraically dependent. (by \( x_1^2 - 1 \).)

Example 4: \( 4^n, 6^n, 9^n \) are algebraically dependent. (by \( x_1 x_3 - x_2^2 \).)

Example 5: \( 4^n, 7^n, 9^n \) are algebraically independent.
In General: Algebraic Relations of Exponentials

Theorem: Let $\phi_1, \ldots, \phi_m \in \mathbb{K}$
In General: Algebraic Relations of Exponentials

**Theorem**

Let \( \phi_1, \ldots, \phi_m \in \mathbb{K} \) and

\[
L := \{ (c_1, \ldots, c_m) : \phi_1^{c_1} \phi_2^{c_2} \cdots \phi_m^{c_m} = 1 \} \subseteq \mathbb{Z}^m.
\]
Let $\phi_1, \ldots, \phi_m \in \mathbb{K}$ and

$$L := \{ (c_1, \ldots, c_m) : \phi_1^{c_1} \phi_2^{c_2} \cdots \phi_m^{c_m} = 1 \} \subseteq \mathbb{Z}^m.$$

This is a lattice.
In General: Algebraic Relations of Exponentials

Theorem. Let $\phi_1, \ldots, \phi_m \in \mathbb{K}$ and

$$L := \{ (c_1, \ldots, c_m) : \phi_1^{c_1} \phi_2^{c_2} \cdots \phi_m^{c_m} = 1 \} \subseteq \mathbb{Z}^m.$$ 

This is a lattice. Let

$$I(L) := \langle x_1^{c_1} x_2^{c_2} \cdots x_m^{c_m} - 1 : (c_1, \ldots, c_m) \in L \rangle \subseteq \mathbb{K}[x_1, \ldots, x_m]$$

be the corresponding lattice ideal. (It is understood that “denominators are cleared”.)
In General: Algebraic Relations of Exponentials

Theorem Let $\phi_1, \ldots, \phi_m \in \mathbb{K}$ and

\[
L := \{ (c_1, \ldots, c_m) : \phi_1^{c_1} \phi_2^{c_2} \cdots \phi_m^{c_m} = 1 \} \subseteq \mathbb{Z}^m.
\]

This is a lattice. Let

\[
I(L) := \langle x_1^{c_1} x_2^{c_2} \cdots x_m^{c_m} - 1 : (c_1, \ldots, c_m) \in L \rangle \subseteq \mathbb{K}[x_1, \ldots, x_m]
\]

be the corresponding lattice ideal. (It is understood that “denominators are cleared”.)

Let $f_0(n) = n$ and $f_1(n) = \phi_1^n$, $f_2(n) = \phi_2^n$, \ldots, $f_m(n) = \phi_m^n$. 
In General: Algebraic Relations of Exponentials

Theorem: Let \( \phi_1, \ldots, \phi_m \in \mathbb{K} \) and

\[
L := \{ (c_1, \ldots, c_m) : \phi_1^{c_1} \phi_2^{c_2} \cdots \phi_m^{c_m} = 1 \} \subseteq \mathbb{Z}^m.
\]

This is a lattice. Let

\[
I(L) := \langle x_1^{c_1} x_2^{c_2} \cdots x_m^{c_m} - 1 : (c_1, \ldots, c_m) \in L \rangle \subseteq \mathbb{K}[x_1, \ldots, x_m]
\]

be the corresponding lattice ideal. (It is understood that “denominators are cleared”.)

Let \( f_0(n) = n \) and \( f_1(n) = \phi_1^n, f_2(n) = \phi_2^n, \ldots, f_m(n) = \phi_m^n \).

Then
In General: Algebraic Relations of Exponentials

Theorem Let $\phi_1, \ldots, \phi_m \in \mathbb{K}$ and

$$L := \{ (c_1, \ldots, c_m) : \phi_1^{c_1} \phi_2^{c_2} \cdots \phi_m^{c_m} = 1 \} \subseteq \mathbb{Z}^m.$$ 

This is a lattice. Let

$$I(L) := \langle x_1^{c_1} x_2^{c_2} \cdots x_m^{c_m} - 1 : (c_1, \ldots, c_m) \in L \rangle \subseteq \mathbb{K}[x_1, \ldots, x_m]$$

be the corresponding lattice ideal. (It is understood that "denominators are cleared").

Let $f_0(n) = n$ and $f_1(n) = \phi_1^n$, $f_2(n) = \phi_2^n$, $\ldots$, $f_m(n) = \phi_m^n$.

Then

$$I(L)\mathbb{K}[x_0, \ldots, x_m]$$

is the ideal of algebraic relations among $f_0(n), \ldots, f_m(n)$.
Relation to Summation-Theory

Remark (for people familiar with difference fields)
Relation to Summation-Theory

Remark (for people familiar with difference fields)

If $\mathbb{F}$ is a difference field and $t_1, t_2, t_3$ are formal sums over $\mathbb{F}$, then a new sum $t_4 = \sum r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

\[
\sigma(g) - g = c_1(\sigma(t_1) - t_1) + c_2(\sigma(t_2) - t_2)
\]
\[
+ c_3(\sigma(t_3) - t_3) + c_4(r(t_1, t_2, t_3))
\]

has no solution $(c_1, c_2, c_3, c_4; g) \in \mathbb{K}^3 \times \mathbb{F}$. 
Relation to Summation-Theory

*Remark* (for people familiar with difference fields)

If $\mathbb{F}$ is a difference field and $t_1, t_2, t_3$ are formal sums over $\mathbb{F}$, then a new sum $t_4 = \Sigma r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

$$
\sigma(g) - g = c_1(\sigma(t_1) - t_1) + c_2(\sigma(t_2) - t_2) + c_3(\sigma(t_3) - t_3) + c_4(r(t_1, t_2, t_3))
$$

has no solution $(c_1, c_2, c_3, c_4; g) \in \mathbb{K}^3 \times \mathbb{F}$. (Creative Telescoping.)
Relation to Summation-Theory

Remark (for people familiar with difference fields)

If $\mathbb{F}$ is a difference field and $t_1, t_2, t_3$ are formal products over $\mathbb{F}$, then a new product $t_4 = \prod r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

$$\frac{\sigma(g)}{g} = \left(\frac{\sigma(t_1)}{t_1}\right)^{c_1} \left(\frac{\sigma(t_2)}{t_2}\right)^{c_2} \left(\frac{\sigma(t_3)}{t_3}\right)^{c_3} \left(r(t_1, t_2, t_3)\right)^{c_4}$$

has no solution $(c_1, c_2, c_3, c_4; g) \in \mathbb{Z}^3 \times \mathbb{F}$. 
Relation to Summation-Theory

Remark (for people familiar with difference fields)

If $\mathbb{F}$ is a difference field and $t_1, t_2, t_3$ are formal products over $\mathbb{F}$, then a new product $t_4 = \Pi r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

$$\frac{\sigma(g)}{g} = \left(\frac{\sigma(t_1)}{t_1}\right)^{c_1} \left(\frac{\sigma(t_2)}{t_2}\right)^{c_2} \left(\frac{\sigma(t_3)}{t_3}\right)^{c_3} \left(r(t_1, t_2, t_3)\right)^{c_4}$$

has no solution $(c_1, c_2, c_3, c_4; g) \in \mathbb{Z}^3 \times \mathbb{F}$. (Multiplicative Creative Telescoping.)
Remark (for people familiar with difference fields)

If $\mathbb{F}$ is a difference field and $t_1, t_2, t_3$ are formal products over $\mathbb{F}$, then a new product $t_4 = \prod r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

$$\frac{\sigma(g)}{g} = \left(\frac{\sigma(t_1)}{t_1}\right)^{c_1} \left(\frac{\sigma(t_2)}{t_2}\right)^{c_2} \left(\frac{\sigma(t_3)}{t_3}\right)^{c_3} \left(r(t_1, t_2, t_3)\right)^{c_4}$$

has no solution $(c_1, c_2, c_3, c_4; g) \in \mathbb{Z}^3 \times \mathbb{F}$. (Multiplicative Creative Telescoping.)

The theorem of the previous slide may be viewed as a corollary to the second case, with $\mathbb{F} = \mathbb{K}$ (constants) and the $t_i$ being exponentials.
Relation to Summation-Theory

Remark (for people familiar with difference fields)

If $\mathbb{F}$ is a difference field and $t_1, t_2, t_3$ are formal products over $\mathbb{F}$, then a new product $t_4 = \prod r(t_1, t_2, t_3)$ can be adjoined to $\mathbb{F}(t_1, t_2, t_3)$ only if

$$\frac{\sigma(g)}{g} = \left( \frac{\sigma(t_1)}{t_1} \right)^{c_1} \left( \frac{\sigma(t_2)}{t_2} \right)^{c_2} \left( \frac{\sigma(t_3)}{t_3} \right)^{c_3} \left( r(t_1, t_2, t_3) \right)^{c_4}$$

has no solution $(c_1, c_2, c_3, c_4; g) \in \mathbb{Z}^3 \times \mathbb{F}$. (Multiplicative Creative Telescoping.)

The theorem of the previous slide may be viewed as a corollary to the second case, with $\mathbb{F} = \mathbb{K}$ (constants) and the $t_i$ being exponentials. But it can be proven also by an elementary argument.
Arbitrary C-Finite Sequences

Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \subseteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.
Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \triangleleft \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

1. Write the sequences in the form

$$f_i(n) = p_{i,0}(n)\phi_1^n + \cdots + p_{i,l}(n)\phi_l^n$$

for certain numbers $\phi_j$ and polynomials $p_{i,l}(n)$. 
Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \trianglelefteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

1. Write the sequences in the form

$$f_i(n) = p_{i,0}(n) \phi_1^n + \cdots + p_{i,l}(n) \phi_l^n$$

for certain numbers $\phi_j$ and polynomials $p_{i,l}(n)$.

This is easy.
Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \subseteq K[x_1, \ldots, x_m]$ of their algebraic relations.

Compute the ideal

$$\langle b_1, \ldots, b_r \rangle \subseteq K[y_0, y_1, \ldots, y_l]$$

of all relations among $n, \phi_1^n, \ldots, \phi_l^n$. 

\textit{Arbitrary C-Finite Sequences}
Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \trianglelefteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

2. Compute the ideal

$$\langle b_1, \ldots, b_r \rangle \trianglelefteq \mathbb{K}[y_0, y_1, \ldots, y_l]$$

of all relations among $n, \phi_1^n, \ldots, \phi_l^n$.

This only requires finding a lattice basis of

$$L = \{(c_1, \ldots, c_l) : \phi_1^{c_1} \cdots \phi_l^{c_l} = 1\}.$$
Let \( f_1(n), \ldots, f_m(n) \) be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal \( \mathfrak{a} \trianglelefteq \mathbb{K}[x_1, \ldots, x_m] \) of their algebraic relations.

2. Compute the ideal

\[
\langle b_1, \ldots, b_r \rangle \trianglelefteq \mathbb{K}[y_0, y_1, \ldots, y_l]
\]

of all relations among \( n, \phi_1^n, \ldots, \phi_l^n \).

This only requires finding a lattice basis of

\[
L = \{ (c_1, \ldots, c_l) : \phi_1^{c_1} \cdots \phi_l^{c_l} = 1 \}.
\]

For \( \phi_j \in \mathbb{Q} \) this is easy.
Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \leq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

Compute the ideal

$$\langle b_1, \ldots, b_r \rangle \leq \mathbb{K}[y_0, y_1, \ldots, y_l]$$

of all relations among $n, \phi_1^n, \ldots, \phi_l^n$.

This only requires finding a lattice basis of

$$L = \{ (c_1, \ldots, c_l) : \phi_1^{c_1} \cdots \phi_l^{c_l} = 1 \}.$$ 

For $\phi_j \in \overline{\mathbb{Q}}$ this can be done with LLL and diophantine approximation (Ge’s algorithm).
Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \subseteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

3. Form the ideal

$$\mathfrak{a} := \langle x_1 - (p_{1,0}(y_0)y_1 + \cdots + p_{1,l}(y_0)y_l),$$
$$x_2 - (p_{2,0}(y_0)y_1 + \cdots + p_{2,l}(y_0)y_l),$$
$$\ldots$$
$$x_m - (p_{m,0}(y_0)y_1 + \cdots + p_{m,l}(y_0)y_l),$$
$$b_1, \ldots, b_r \rangle \subseteq \mathbb{K}[x_1, \ldots, x_m, y_0, \ldots, y_l]$$
Arbitrary C-Finite Sequences

Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $a \subseteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

Return

$$a \cap \mathbb{K}[x_1, \ldots, x_m]$$
Arbitrary C-Finite Sequences

Let $f_1(n), \ldots, f_m(n)$ be C-finite sequences (given via recurrence and initial values). We wish to compute the ideal $\mathfrak{a} \subseteq \mathbb{K}[x_1, \ldots, x_m]$ of their algebraic relations.

Return

$$\mathfrak{a} \cap \mathbb{K}[x_1, \ldots, x_m]$$

These are precisely the desired relations.
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81 (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words Find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$. 
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81: (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, Find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$.
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

Exercise 6.81: (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations among $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$.

Answer: $\mathfrak{a} = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle$. 
Example: Fibonacci Numbers

Let $F_n$ denote the $n$th Fibonacci number ($n \in \mathbb{Z}$).

**Exercise 6.81** (Graham/Knuth/Patashnik) Let $P(x, y)$ be a polynomial in $x$ and $y$ with integer coefficients. Find a necessary and sufficient condition that $P(F_{n+1}, F_n) = 0$ for all $n \geq 0$.

In other words, find the ideal $\mathfrak{a} \subseteq \mathbb{C}[x, y]$ of algebraic relations along $(F_n)_{n \geq 0}$ and $(F_{n+1})_{n \geq 0}$.

**Answer** $\mathfrak{a} = \langle (x^2 - xy - y^2 - 1)(x^2 - xy - y^2 + 1) \rangle$.

**Note** We can determine all algebraic relations with this algorithm.
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]
Example: Fibonacci Numbers

**en detail**

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

1. \[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

1. \[
F_{n+1} = \frac{5 + \sqrt{5}}{10} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{5 - \sqrt{5}}{10} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

1.

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[ F_{n+1} = \frac{5 + \sqrt{5}}{10} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{5 - \sqrt{5}}{10} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

So \( \phi_1 = \frac{1}{2}(1 + \sqrt{5}) \) and \( \phi_2 = \frac{1}{2}(1 - \sqrt{5}) \).
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

\[ \phi_1 \phi_2 = -1 \]
Example: Fibonacci Numbers

*en detail*

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

\[ ? \text{ We have } \phi_1 \phi_2 = -1, \text{ so } \phi_1^2 \phi_2^2 = 1 \]
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

We have \( \phi_1 \phi_2 = -1 \), so \( \phi_1^2 \phi_2^2 = 1 \), so

\[ \langle y_1^2 y_2^2 - 1 \rangle \leq \overline{\mathbb{Q}}[y_1, \ldots, y_l] \]

are the relations among \( n, \phi_1^n, \phi_2^n \).
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

3. Next we set

\[
\mathbf{a} := \langle x_1 - \left( \frac{1}{\sqrt{5}} y_1 - \frac{1}{\sqrt{5}} y_2 \right), \\
x_2 - \left( \frac{5+\sqrt{5}}{10} y_1 + \frac{5-\sqrt{5}}{10} y_2 \right), \\
y_1^2 y_2^2 - 1 \rangle \leq \bar{K}[x_1, x_2, y_0, y_1, y_2]
\]
Example: Fibonacci Numbers

\( F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \)

We obtain

\[ \alpha \cap \bar{\mathbb{Q}}[x_1, x_2] \]
Example: Fibonacci Numbers

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

We obtain

\[ a \cap \overline{\mathbb{Q}}[x_1, x_2] = \langle (x_1^2 - x_1 x_2 - x_2^2 - 1)(x_1^2 - x_1 x_2 - x_2^2 + 1) \rangle. \]
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

We obtain

\[ \alpha \cap \overline{\mathbb{Q}}[x_1, x_2] \]
\[ = \langle (x_1^2 - x_1x_2 - x_2^2 - 1)(x_1^2 - x_1x_2 - x_2^2 + 1) \rangle. \]

Note Intermediate algebraic field extensions always cancel out in the final result.
Example: Fibonacci Numbers

*en detail*

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

We obtain

\[ a \cap \overline{Q}[x_1, \ldots, x_m] = \langle (x_1^2 - x_1 x_2 - x_2^2 - 1)(x_1^2 - x_1 x_2 - x_2^2 + 1) \rangle. \]

*Note* In this example the final elimination just amounts to a linear transform:
Example: Fibonacci Numbers

en detail

\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

We obtain

\[
\mathfrak{a} \cap \overline{\mathbb{Q}}[x_1, \ldots, x_m]
\]

\[
= \langle (x_1^2 - x_1 x_2 - x_2^2 - 1)(x_1^2 - x_1 x_2 - x_2^2 + 1) \rangle.
\]

Note In this example the final elimination just amounts to a linear transform:

\[ V(y_1^2 y_2^2 - 1) \rightarrow V(\mathfrak{a}) \]
Example: Fibonacci Numbers

detail
\[ F_{n+2} = F_n + F_{n+1}, \quad F_0 = 0, F_1 = 1. \]

We obtain
\[
\mathfrak{a} \cap \overline{\mathbb{Q}}[x_1, \ldots, x_m]
\]
\[
= \langle (x_1^2 - x_1 x_2 - x_2^2 - 1)(x_1^2 - x_1 x_2 - x_2^2 + 1) \rangle.
\]

Note In this example the final elimination just amounts to a linear transform:

\[
V(y_1^2 y_2^2 - 1)
\] \[\rightarrow\]

\[
V(\mathfrak{a})
\]
Multivariate Sequences

A sequence $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ is \textit{ulti-C-finite} if it satisfies a C-finite recurrence equation in every direction.
Multivariate Sequences

A sequence \( f : \mathbb{Z}^d \rightarrow \mathbb{C} \) is \textit{ulti-C-finite} if it satisfies a C-finite recurrence equation in every direction.

Multi-C-finite sequences also have closed forms in terms of polynomials and exponentials.
Multivariate Sequences

A sequence $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ is \textit{ulti-C-finite} if it satisfies a C-finite recurrence equation in every direction.

Multi-C-finite sequences also have closed forms in terms of polynomials and exponentials.

For example, if $f(n, m)$ is such that $f(0, 0) = f(0, 1) = 1$ and

\begin{align*}
  f(n + 2, m) &= 4f(n + 1, m) - 4f(n, m) \\
  f(n, m + 1) &= 5f(n, m),
\end{align*}

then

$$f(n, m) = (1 - \frac{1}{2}n)2^n5^m$$
Multivariate Sequences

A sequence $f: \mathbb{Z}^d \rightarrow \mathbb{C}$ is \textit{ulti-C-finite} if it satisfies a C-finite recurrence equation in every direction.

Multi-C-finite sequences also have closed forms in terms of polynomials and exponentials.

For example, if $f(n, m)$ is such that $f(0, 0) = f(0, 1) = 1$ and

\begin{align*}
f(n + 2, m) &= 4f(n + 1, m) - 4f(n, m) \\
f(n, m + 1) &= 5f(n, m),
\end{align*}

then

$$f(n, m) = (1 - \frac{1}{2}n)2^n5^m$$

Algebraic relations among multi-C-finite sequences can be found in very much the same way as for univariate sequences.
Some Funny Applications
Some Funny Applications

Proving Identities

For deciding $p(F_n, F_{n+1}) = 0$ for a given polynomial $p$, compute a normal form of $p$ wrt. a Gröbner basis of the ideal of algebraic relations.
Some Funny Applications

Proving Identities

For deciding $p(F_n, F_{n+1}) = 0$ for a given polynomial $p$, compute a normal form of $p$ wrt. a Gröbner basis of the ideal of algebraic relations.
(This makes only sense if you have many $p$ for the same sequences.)
Express something in terms of something else

Given $f(n)$ and $g_1(n), \ldots, g_m(n)$, is there a formula

$$f(n) = A(g_1(n), \ldots, g_m(n)) \quad (n \geq 0)$$

for some polynomial (or rational function, or algebraic function) $A$?
Some Funny Applications

Express something in terms of something else

Given $f(n)$ and $g_1(n), \ldots, g_m(n)$, is there a formula

$$f(n) = A(g_1(n), \ldots, g_m(n)) \quad (n \geq 0)$$

for some polynomial (or rational function, or algebraic function) $A$?

Compute the algebraic relations among $f(n)$ and the $g_i(n)$.
Express something in terms of something else

Given $f(n)$ and $g_1(n), \ldots, g_m(n)$, is there a formula

$$f(n) = A(g_1(n), \ldots, g_m(n)) \quad (n \geq 0)$$

for some polynomial (or rational function, or algebraic function) $A$?

Compute the algebraic relations among $f(n)$ and the $g_i(n)$.

Consider a Gröbner basis with respect to a block ordering that assigns highest weight to the variable corresponding to $f(n)$. 

Some Funny Applications
Express something in terms of something else

Given \( f(n) \) and \( g_1(n), \ldots, g_m(n) \), is there a formula

\[
f(n) = A(g_1(n), \ldots, g_m(n)) \quad (n \geq 0)
\]

for some polynomial (or rational function, or algebraic function) \( A \)?

Compute the algebraic relations among \( f(n) \) and the \( g_i(n) \).

Consider a Gröbner basis with respect to a block ordering that assigns highest weight to the variable corresponding to \( f(n) \).

Inspect polynomial with least degree with respect to that variable appearing in the Gröbner basis.
Some Funny Applications

Express something in terms of something else

Let \( f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{n+k} \). (This is C-finite, order 2)
Some Funny Applications

*Express something in terms of something else*

Let \( f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{n+k} \). (This is C-finite, order 2)

The ideal of relations among \( f(n), F_n, F_{n+1} \) contains

\[
x_0 - x_1(2x_1^2 - 3x_1x_2 + 3x_2^2)
\]
Some Funny Applications

Express something in terms of something else

Let $f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}$. (This is C-finite, order 2)
The ideal of relations among $f(n), F_n, F_{n+1}$ contains

$$x_0 - x_1(2x_1^2 - 3x_1x_2 + 3x_2^2)$$

Consequently,

$$f(n) = F_n(2F_n^2 - 3F_nF_{n+1} + 3F_{n+1}^2).$$
Express something in terms of something else

Let $f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{2k}$. (This is C-finite, order 2)
Some Funny Applications

Express something in terms of something else

Let \( f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{2k} \). (This is C-finite, order 2)

No polynomial in the Gr"obner basis for the ideal of relations among \( f(n), F_n, F_{n+1} \) involves \( x_0 \).
Some Funny Applications

Express something in terms of something else

Let \( f(n) = \sum_{k=0}^{n} \binom{n}{k} F_{2k} \). (This is C-finite, order 2)

No polynomial in the Gröbner basis for the ideal of relations among \( f(n), F_n, F_{n+1} \) involves \( x_0 \).

Consequently, the sum \( f(n) \) has no closed form in terms of Fibonacci numbers.
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $F_{3n+1} \equiv F_{n+1}^3 \mod F_n$ (n ≥ 0)?
Some Funny Applications

Divisibility Properties and Modular Identities

How to show \( F_{3n+1} \equiv F_{n+1}^3 \mod F_n \) (\( n \geq 0 \))?

Compute the ideal \( \alpha \) of all relations among \( F_{3n+1}, F_{n+1}, F_n \).
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $F_{3n+1} \equiv F_{n+1}^3 \mod F_n \ (n \geq 0)$?

Compute the ideal $\mathfrak{a}$ of all relations among $F_{3n+1}, F_{n+1}, F_n$.

Next,

$$\mathfrak{a} + \langle x_2 \rangle = \langle x_2, x_1x_3 - 1, x_3^2 - x_1^2, x_1^3 - x_3 \rangle.$$
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $F_{3n+1} \equiv F_{n+1}^3 \mod F_n$ ($n \geq 0$)?

Compute the ideal $\mathfrak{a}$ of all relations among $F_{3n+1}, F_{n+1}, F_n$.

Next,

$$\mathfrak{a} + \langle x_2 \rangle = \langle x_2, x_1 x_3 - 1, x_3^2 - x_1^2, x_1^3 - x_3 \rangle.$$

The desired identity follows from the last generator.
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $F_{3n+1} \equiv F_{n+1}^3 \mod F_n$ ($n \geq 0$)?

Compute the ideal $\mathfrak{a}$ of all relations among $F_{3n+1}, F_{n+1}, F_n$. Next,

\[ \mathfrak{a} + \langle x_2 \rangle = \langle x_2, x_1x_3 - 1, x_3^2 - x_1^2, x_1^3 - x_3 \rangle. \]

The desired identity follows from the last generator.

The second gives another identity for free: $F_{3n+1}F_{n+1} \equiv 1 \mod F_n$. 
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $\gcd(L_n, F_{n+1}) = 1$ ($L_n$ being the $n$th Lucas number)?
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $\gcd(L_n, F_{n+1}) = 1$ ($L_n$ being the $n$th Lucas number)?

Compute the relations among $L_n$ and $F_{n+1}$:

$$a = \langle x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1 \rangle.$$
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $\gcd(L_n, F_{n+1}) = 1$ ($L_n$ being the $n$th Lucas number)?
Compute the relations among $L_n$ and $F_{n+1}$:

$$a = \langle x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1 \rangle.$$

Next, use Gröbner bases to determine $u, v, w$ with

$$1 = ux_1 + vx_2 + w(x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1).$$
Some Funny Applications

**Divisibility Properties and Modular Identities**

How to show $\gcd(L_n, F_{n+1}) = 1$ ($L_n$ being the $n$th Lucas number)?

Compute the relations among $L_n$ and $F_{n+1}$:

$$a = \langle x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1 \rangle.$$

Next, use Gröbner bases to determine $u, v, w$ with

$$1 = ux_1 + vx_2 + w(x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1).$$

The choice $u = x_1^3, v = -10x_1 + 35x_1^2x_2 - 50x_1x_2^2 + 25x_2^3, w = -1$ will do.
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $\gcd(L_n, F_{n+1}) = 1$ ($L_n$ being the $n$th Lucas number)?
Compute the relations among $L_n$ and $F_{n+1}$:

$$a = \langle x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1 \rangle.$$

Next, use Gröbner bases to determine $u, v, w$ with

$$1 = ux_1 + vx_2 + w(x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1).$$

The choice $u = x_1^3, v = -10x_1 + 35x_1^2x_2 - 50x_1x_2^2 + 25x_2^3, w = -1$ will do.

It follows that $1 = p(n)L_n + q(n)F_{n+1}$ for some $p(n), q(n) \in \mathbb{Z}$. 
Some Funny Applications

Divisibility Properties and Modular Identities

How to show $\gcd(L_n, F_{n+1}) = 1$ ($L_n$ being the $n$th Lucas number)? Compute the relations among $L_n$ and $F_{n+1}$:

$$a = \langle x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1 \rangle.$$

Next, use Gröbner bases to determine $u, v, w$ with

$$1 = ux_1 + vx_2 + w(x_1^4 - 10x_1^3x_2 + 35x_1^2x_2 - 50x_1x_2^3 + 25x_2^4 - 1).$$

The choice $u = x_1^3$, $v = -10x_1 + 35x_1^2x_2 - 50x_1x_2^2 + 25x_2^3$, $w = -1$ will do.

It follows that $1 = p(n)L_n + q(n)F_{n+1}$ for some $p(n), q(n) \in \mathbb{Z}$. Therefore $\gcd(L_n, F_{n+1}) \mid 1$. This suffices.
Summary
Summary

Knowing algebraic relations is useful.
Knowing algebraic relations is useful.

There is an algorithm which computes the algebraic relations among some given C-finite sequences.
Knowing algebraic relations is useful.

There is an algorithm which computes the algebraic relations among some given C-finite sequences.

All these relations are consequences of multiplicative relations among the roots of the characteristic polynomial.
Knowing algebraic relations is useful.

There is an algorithm which computes the algebraic relations among some given C-finite sequences.

All these relations are consequences of multiplicative relations among the roots of the characteristic polynomial.

A lattice basis for these relations can be computed with a number-theoretic algorithm. The rest can be done with Gröbner bases.
What’s next?
$P$-finite sequences?
Note:

\[
\frac{1}{1 - az} \circ \frac{1}{1 - bz} = \frac{1}{1 - (ab)z}.
\]
Note:

\[
\frac{1}{1 - az} \circ \frac{1}{1 - bz} = \frac{1}{1 - (ab)z}.
\]

The algorithm presented earlier uses multiplicative relations among the singularities of the generating functions in order to make all the singularities cancel.
Generating Functions

Note:

\[
\frac{1}{1 - az} \odot \frac{1}{1 - bz} = \frac{1}{1 - (ab)z}.
\]

The algorithm presented earlier uses multiplicative relations among the singularities of the generating functions in order to make all the singularities cancel.

Does this only work for rational generating functions (i.e. C-finite sequences)?
Generating Functions

Note:

\[
\frac{1}{1 - az} \odot \frac{1}{1 - bz} = \frac{1}{1 - (ab)z}.
\]

The algorithm presented earlier uses multiplicative relations among the singularities of the generating functions in order to make all the singularities cancel.

Does this only work for rational generating functions (i.e. C-finite sequences)?

Consider some examples...
Example 1

Let \( f(n) \) be defined by

\[
4(2n + 3)(4n^2 - 1) f(n) + 4(2n + 3)(n + 1) f(n + 1) - (n + 1)(n + 2)(2n - 3) f(n + 2) = 0,
\]

\[
f(0) = f(1) = 1.
\]
Example 1

Let \( f(n) \) be defined by

\[
4(2n + 3)(4n^2 - 1) f(n) + 4(2n + 3)(n + 1) f(n + 1) \\
- (n + 1)(n + 2)(2n - 3) f(n + 2) = 0,
\]

\[
f(0) = f(1) = 1.
\]

There exist nontrivial algebraic relations among \( f(n) \) and \( f(n + 1) \), e.g.,

\[
((4n + 2) f(n) - (n - 4) f(n + 1)) \\
\times ((4n^2 - 10n - 6) f(n) - n(n + 1) f(n + 1)) = 0.
\]
Example 1

Let $f(n)$ be defined by

$$4(2n + 3)(4n^2 − 1) f(n) + 4(2n + 3)(n + 1) f(n + 1) - (n + 1)(n + 2)(2n − 3) f(n + 2) = 0,$$

$$f(0) = f(1) = 1.$$

This is in accordance with the generating function

$$\sum_{n=0}^{\infty} f(n) z^n = \frac{1}{12} \left( \frac{i}{(4z - 1)^{3/2}} + \frac{5i}{\sqrt{4z - 1}} - \frac{4}{\sqrt{4z + 1}} \right)$$

having the singularities $\frac{1}{4}$ and $-\frac{1}{4}$, which have a multiplicative relation.
Example 2

Let $f(n)$ be defined by

$$2n(n + 3)f(n) - (3n + 8)(n + 1)f(n + 1) + (3n + 8)(n + 2)f(n + 2) - (n + 3)(n + 2)f(n + 3) = 0,$$

$f(0) = 1$, $f(1) = 3$, $f(2) = \frac{9}{2}$. 
Example 2

Let $f(n)$ be defined by

\[
2n(n + 3)f(n) - (3n + 8)(n + 1)f(n + 1) + (3n + 8)(n + 2)f(n + 2) - (n + 3)(n + 2)f(n + 3) = 0, \\
f(0) = 1, f(1) = 3, f(2) = \frac{9}{2}.
\]

There are nontrivial relations among $f(n), f(n + 1), f(n + 2)$ (too long to fit on this slide).
Example 2

Let $f(n)$ be defined by

$$2n(n + 3)f(n) - (3n + 8)(n + 1)f(n + 1) + (3n + 8)(n + 2)f(n + 2) - (n + 3)(n + 2)f(n + 3) = 0,$$

$f(0) = 1$, $f(1) = 3$, $f(2) = \frac{9}{2}$.

This is in accordance with the generating function

$$\sum_{n=0}^{\infty} f(n)z^n = \frac{1}{1 - 2z} + \frac{1}{9} \sqrt{3} \left( \pi - 6 \arctan\left( \frac{1 - 2z}{\sqrt{3}} \right) \right)$$

having the singularities $\frac{1}{2}$ and $(1 + i\sqrt{3})/2$, $(1 - i\sqrt{3})/2$, the latter two bearing a multiplicative relation.
Example 3

Let $f(n)$ be defined as

$$f(n) = H_n(x)/n!$$

where $H_n(x)$ is the $n$th Hermite polynomial.
Example 3

Let $f(n)$ be defined as

$$f(n) = H_n(x)/n!$$

where $H_n(x)$ is the $n$th Hermite polynomial.

There do not seem to exist algebraic relations among $f(n)$, $f(n+1)$. 

Example 3

Let \( f(n) \) be defined as

\[
f(n) = \frac{H_n(x)}{n!}
\]

where \( H_n(x) \) is the \( n \)th Hermite polynomial.

This is consistent with the generating function

\[
\sum_{n=0}^{\infty} f(n)z^n = \exp(2xz - z^2)
\]

having no singularities.
Can we construct a complete algorithm for finding algebraic relations among $P$-finite sequences using singularity analysis?