

Some Recent Results on Solving and Factoring Differential and Difference Equations

(Fully Integrable Systems and Ore Modules)

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A brief introduction to
Key Laboratory of Mathematics Mechanization

Beginning

Wentsun Wu: *On the Decision Problem and the Mechanization of Theorem-Proving in Elementary Geometry*, Scientia Sinica (1978), 159-172.

Re-published in *Automated Theorem Proving, after 25 years*, AMS (1984)

Key steps

Theorem = hypotheses + conclusion

1. Translate the hypotheses to a system H of polynomial equations, and the conclusion to a polynomial g .
2. Decompose $\mathbf{V}(H)$ into irreducible components

$$\mathbf{V}(H) = V_1 \cup \cdots \cup V_r,$$

where each V_i is represented by a set T_i of polynomials in triangular form.

3. Find the irreducible components corresponding to degenerating conditions.
4. Decide if g vanishes on the irreducible components that do not correspond to degenerating conditions.

Note: The algorithm used in step 2 is called **the Ritt-Wu** or **the characteristic set method**.

Mathematics Mechanization

From the preface of *Mathematics Mechanization* by Wu:

The subject tries to deal with mathematics in a constructive and algorithmic manner so that the reasonings become mechanical, automated, and as much as possible to be intelligence-lacking, with the results of lessening the painstaking heavy brain-labor.

Photos 1986, 2006

Projects related to differential and difference equations

1. Extending the characteristic set method to difference algebra
(*Gao and Yuan in collaboration with J. van der Hoeven*)
2. Finding algebraic solutions of first-order nonlinear ODE's with constant coefficients
(*Feng and Gao in collaboration with J. Cano*)
3. Decomposing linear partial differential and difference systems
(*Li and Wang in collaboration with M. Singer and M. Wu*)
4. Symbolic integration, decomposition of ODE's, rational solutions of $O\Delta E$, etc

Fully integrable systems and Ore modules

- Converting an integrable system to a fully integrable system
- Computing hyperexponential solutions of a fully integrable system
- Factoring reflexive Ore modules

Fully integrable systems

Linear ordinary difference equations

$F = \mathbb{C}(k)$, $\sigma : k \mapsto k + 1$ automorphism

Linear ordinary difference equation over F :

$$L(z) = \sigma^n(z) + a_{n-1}\sigma^{n-1}(z) + \cdots + a_1\sigma(z) + a_0 = 0, \quad a_i \in F$$

- $\dim_{\mathbb{C}} \text{sol}(L) = n \iff a_0 \neq 0$
- when $a_0 = 0$

$$L(z) = \sigma \left(\overbrace{\sigma^{n-1}(z) + \sigma^{-1}(a_{n-1})\sigma^{n-2}(z) + \cdots + \sigma^{-1}(a_1)}^{\tilde{L}(z)} \right) = 0$$

\Downarrow

$$\tilde{L}(z) = 0$$

\Downarrow

reducing the order of equation

\Downarrow

\vdots

Matrix forms

$F = \mathbb{C}(k)$, $\sigma : k \mapsto k + 1$ automorphism,

$$\sigma(\mathbf{z}) = A \mathbf{z}, \quad \text{with } \mathbf{z} = (z_1, \dots, z_n)^T \text{ and } A \in F^{n \times n}.$$

- $\dim_{\mathbb{C}} \text{sol}(\sigma(\mathbf{z}) = A \mathbf{z}) = n \iff A$ is invertible.
- when A is singular:

linear dep. among the row vectors of A

$$\Downarrow \sigma(\mathbf{z}) = A \mathbf{z}$$

linear relations in $\sigma(z_1), \dots, \sigma(z_n)$

$$\Downarrow \sigma^{-1}$$

linear relations in z_1, \dots, z_n , e.g. $(z_{d+1}, \dots, z_n)^T = Q(z_1, \dots, z_d)^T$,
substitute these relations into $\sigma(\mathbf{z}) = A \mathbf{z}$

$$\Downarrow$$

$$\sigma \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = B \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}, \quad R \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = 0$$

$$\Downarrow$$

\vdots

Example

$$F = \mathbb{C}(k), \quad \sigma : k \mapsto k + 1,$$

Consider the system

$$\sigma \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{k+1}{k} & 0 & 0 \\ \frac{k+1}{k} & 0 & 0 \\ \frac{k+1}{k} & -\frac{k}{k-1} & \frac{k}{k-1} \end{pmatrix}}_A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$|A| = 0 \implies$$

$$z_1 = z_2, \quad \sigma \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{k+1}{k} & 0 \\ -\frac{1}{k(k-1)} & \frac{k}{k-1} \end{pmatrix}}_B \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}$$

$|B| \neq 0 \implies$ the solution space of $\sigma(\mathbf{z}) = A\mathbf{z}$ has dimension 2.

Partial case

$F = \mathbb{C}(k_1, k_2)$, $\sigma_1 : k_1 \mapsto k_1 + 1, k_2 \mapsto k_2$ and $\sigma_2 : k_1 \mapsto k_1, k_2 \mapsto k_2 + 1$.

A first-order system \mathcal{A} :

$$\sigma_1(\mathbf{z}) = A_1\mathbf{z}, \quad \sigma_2(\mathbf{z}) = A_2\mathbf{z} \quad \text{with } \mathbf{z} = (z_1, \dots, z_n)^T, \quad A_1, A_2 \in F^{n \times n},$$

is

- *integrable* if $\sigma_1(A_2)A_1 = \sigma_2(A_1)A_2$ (derived from $\sigma_1\sigma_2(\mathbf{z}) = \sigma_2\sigma_1(\mathbf{z})$)
- *fully integrable* if it is integrable and A_1, A_2 are invertible.

Note: If \mathcal{A} is fully integrable, then its solution space has dimension n over \mathbb{C} .

From integrable to fully integrable

Idea: Let $\{\sigma_i(\mathbf{z}) = A_i \mathbf{z}\}_{i=1,2}$ be an integrable system where $A_i \in F^{n \times n}$

(1) If some A_j is singular, then

$$\text{solve } (y_1, \dots, y_n) A_j = 0 \implies \text{linear relations in } z_1, \dots, z_n, \text{ e.g. } \begin{pmatrix} z_{d+1} \\ \vdots \\ z_n \end{pmatrix} = Q \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}.$$

(2) Reduce $\{\sigma_i(\mathbf{z}) = A_i \mathbf{z}\}_{i=1,2}$ to get

$$\sigma_i \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = B_i \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix}, \quad R_i \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = 0, \quad i = 1, 2.$$

$\implies \dots$ Stop the reduction until both B_i are invertible and both $R_i = 0$.

Lemma: The difference system $\{\sigma_i(\mathbf{y}) = B_i \mathbf{y}\}_{i=1,2}$ is **integrable** if $R_1 = R_2 = 0$.

Example

Let $F = \mathbb{C}(n, k)$, $\sigma_n : n \mapsto n + 1$, $\sigma_k : k \mapsto k + 1$,

$$\{\sigma_n(\mathbf{z}) = A_n \mathbf{z}, \sigma_k(\mathbf{z}) = A_k \mathbf{z}\}$$

where $\mathbf{z} = (z_1, z_2, z_3, z_4)^T$ and

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-k-n-2}{k^2+k-1-2n-n^2} & 0 & \frac{-k^2+3n+2+n^2}{k^2+k-1-2n-n^2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-k^2+n^2-2k+1+3n}{k^2+k-1-2n-n^2} & 0 & \frac{-k+n+1}{k^2+k-1-2n-n^2} \end{pmatrix},$$

$$A_k = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k^2+3k+2-n^2}{-n^2+k^2+k} & 0 & -\frac{2n}{-n^2+k^2+k} & 0 \\ 0 & \frac{k^2-n^2+3k+1-2n}{k^2+k-1-2n-n^2} & 0 & \frac{-2n-2}{k^2+k-1-2n-n^2} \end{pmatrix}$$

integrable, A_n singular \implies not fully integrable.

Example (cont.)

Singularity of $A_n \implies$

$$z_1 = -\frac{-n^2+k^2+k}{k+n+1}z_2 - \frac{k^2-n-n^2}{k+n+1}z_3, \quad z_4 = \frac{2k-n+1+k^2-n^2}{k+n+1}z_2 + \frac{k^2+k-1-2n-n^2}{k+n+1}z_3$$

Apply linear reduction to get

$$\sigma_n(z_2, z_3)^T = B_n(z_2, z_3)^T, \quad \sigma_k(z_2, z_3)^T = B_k(z_2, z_3)^T$$

where

$$B_n = \begin{pmatrix} \frac{-k-n-2}{k^2+k-1-2n-n^2} + \frac{(-k^2+3n+2+n^2)(2k-n+1+k^2-n^2)}{(k^2+k-1-2n-n^2)(k+n+1)} & \frac{-k^2+3n+2+n^2}{k+n+1} \\ \frac{2k-n+1+k^2-n^2}{k+n+1} & \frac{k^2+k-1-2n-n^2}{k+n+1} \end{pmatrix}$$

$$B_k = \begin{pmatrix} \frac{2k-n+1+k^2-n^2}{k+n+1} & \frac{k^2+k-1-2n-n^2}{k+n+1} \\ -\frac{k^2+3k+2-n^2}{k+n+1} & -\frac{(k^2+3k+2-n^2)(k^2-n-n^2)}{(-n^2+k^2+k)(k+n+1)} - \frac{2n}{-n^2+k^2+k} \end{pmatrix}$$

Both B_n and B_k are invertible. So the original system has a 2-dim solution space.

General setting

Let F be a field of characteristic zero.

Have

- $\delta_1, \dots, \delta_\ell$ derivations on F
- $\sigma_{\ell+1}, \dots, \sigma_m$ automorphisms of F .

Write $\Delta = \{\delta_1, \dots, \delta_\ell, \sigma_{\ell+1}, \dots, \sigma_m\}$

Assumption: All the maps in Δ commute pairwise.

Fully integrable systems

Consider a system \mathcal{A} :

$$\delta_i(\mathbf{z}) = A_i \mathbf{z}, \quad i = 1, \dots, \ell \quad \text{and} \quad \sigma_j(\mathbf{z}) = A_j \mathbf{z}, \quad j = \ell + 1, \dots, m,$$

where $A_i, A_j \in F^{n \times n}$. The system \mathcal{A} is **integrable** if

1. $\delta_i(A_j) = \delta_j(A_i) \quad (1 \leq i < j \leq \ell)$
2. $\sigma_i(A_j)A_i = \sigma_j(A_i)A_j \quad (\ell + 1 \leq i < j \leq m)$
3. $A_j A_i + \delta_i(A_j) = \sigma_j(A_i)A_j \quad (1 \leq i \leq \ell, \ell + 1 \leq j \leq m)$

The system \mathcal{A} is **fully integrable** if it is integrable and $A_{\ell+1}, \dots, A_m$ are invertible

Note: Can transform an integrable system to a fully integrable one in the same way.

Ore algebras

$$\Delta = \{\delta_1, \dots, \delta_\ell, \sigma_{\ell+1}, \dots, \sigma_m\}.$$

Let $\mathcal{O} = F[\partial_1, \dots, \partial_\ell, \partial_{\ell+1}, \dots, \partial_m]$ with commutation rules

1. $\partial_i \partial_j = \partial_j \partial_i$ for all $i, j \in \{1, \dots, m\}$
2. $\partial_i f = f \partial_i + \delta_i(f)$ for all $f \in F$ and $i \in \{1, \dots, \ell\}$
3. $\partial_j f = \sigma_j(f) \partial_j$ for all $f \in F$ and $j \in \{\ell + 1, \dots, m\}$

From fully integrable systems to \mathcal{O} -modules

System: a fully integrable system \mathcal{A} :

$$\delta_i(\mathbf{z}) = A_i \mathbf{z}, \quad i = 1, \dots, \ell \quad \text{and} \quad \sigma_j(\mathbf{z}) = A_j \mathbf{z} \quad j = \ell + 1, \dots, m,$$

where $A_i, A_j \in F^{n \times n}$.

Module: Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the canonical basis of F^n . Letting

$$\partial_i \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} = -A_i^T \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \quad i = 1, \dots, \ell \quad \text{and} \quad \partial_j \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} = (A_j^{-1})^T \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \quad j = \ell + 1, \dots, m.$$

Relation:

$(z_1, \dots, z_n)^T \in F^n$ is a solution of \mathcal{A}

\Updownarrow

$$\begin{cases} \partial_i(z_1 \mathbf{e}_1 + \dots + z_n \mathbf{e}_n) = 0 & i = 1, \dots, \ell, \\ \partial_j(z_1 \mathbf{e}_1 + \dots + z_n \mathbf{e}_n) = z_1 \mathbf{e}_1 + \dots + z_n \mathbf{e}_n, & j = \ell + 1, \dots, m \end{cases}$$

\mathcal{O} -modules associated with fully integrable systems

Let M be an \mathcal{O} -module.

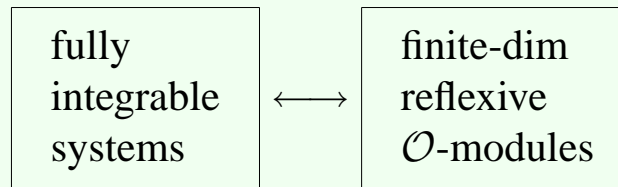
- A submodule N of M is **reflexive** if

$$\partial_j \mathbf{v} \in N \implies \mathbf{v} \in N$$

where $\ell + 1 \leq j \leq m$ and $\mathbf{v} \in M$.

- An \mathcal{O} -module is said to be **reflexive** if all its submodules are reflexive.

Proposition:



Δ -extensions

Let F be a field and $\Delta = \{\delta_1, \dots, \delta_\ell, \sigma_{\ell+1}, \dots, \sigma_m\}$

Let E be a commutative F -algebra. Extend the maps in Δ from F to E :

$$\text{derivations: } \delta_i : E \longrightarrow E \quad i = 1, \dots, \ell,$$

$$\text{automorphisms: } \sigma_j : E \longrightarrow E \quad j = \ell + 1, \dots, m.$$

(E, Δ) is called a Δ -extension of F if the extended maps commute pairwise.

Fundamental matrices

Definition: Let \mathcal{A} be a fully integrable system of size n . An $n \times n$ matrix U over some Δ -extension of F is a **fundamental matrix** for \mathcal{A} if

- (i) U is invertible
- (ii) Every column of U is a solution of \mathcal{A} .

Proposition: There exists a fundamental matrix for \mathcal{A} .

Correspondences

Assume

Fully integrable systems	\mathcal{A}	\mathcal{B}
Associated modules	$M_{\mathcal{A}}$	$M_{\mathcal{B}}$
Fundamental matrices	$U_{\mathcal{A}}$	$U_{\mathcal{B}}$

Definition: \mathcal{A} and \mathcal{B} are **equivalent**, denoted by $\mathcal{A} \sim \mathcal{B}$, if

$$\exists \text{ a square matrix } Q \text{ over } F, \quad U_{\mathcal{A}} = U_{\mathcal{B}}Q.$$

Proposition:

$$\begin{array}{l}
 M_{\mathcal{A}} \cong M_{\mathcal{B}} \quad \Leftrightarrow \quad \mathcal{A} \sim \mathcal{B} \\
 \hline
 0 \neq N \subsetneq M_{\mathcal{A}} \quad \Leftrightarrow \quad \mathcal{A} \sim \mathcal{B} \text{ with each } B_i = \begin{pmatrix} \square & \square \\ 0 & \square \end{pmatrix} \\
 \hline
 M_{\mathcal{A}} = N_1 \oplus N_2 \quad \Leftrightarrow \quad \mathcal{A} \sim \mathcal{B} \text{ with each } B_i = \begin{pmatrix} \square & 0 \\ 0 & \square \end{pmatrix} \\
 \hline
 \end{array}$$

where N, N_1, N_2 are submodules of $M_{\mathcal{A}}$.

Hyperexponential solutions of fully integrable systems

Hyperexponential elements

$\Delta = \{\delta_1, \dots, \delta_\ell, \sigma_{\ell+1}, \dots, \sigma_m\}$, (F, Δ) a Δ -field, (E, Δ) a Δ -ring extension of F

Definition: $0 \neq h \in E$ is *hyperexponential over F* if the $\delta_i(h)$ and $\sigma_j(h)$ are all linearly dependent on h over F .

Example:

	base field	rational	radical	transcendental
Partial differential	$\mathbb{C}(x, y)$	$\frac{x-y}{x^4y}$	$\frac{2y+x^2}{\sqrt{xy}}$	$e^{x^3-y^2}$
Partial difference	$\mathbb{C}(m, n)$	$\frac{1}{m+n}$	$(\sqrt{-1})^m (-1)^n$	$\binom{m}{n}$
Differential-difference	$\mathbb{C}(x, n)$	$n + \frac{1}{x}$	$(-1)^n \sqrt{x}$	x^n

Definition: A nonzero vector $\mathbf{h} \in E^n$ is *hyperexp* if

$$\mathbf{h} = h\mathbf{v}, \quad \text{where } h \in E \text{ is hyperexp. and } \mathbf{v} \in F^n$$

Problem and result

Problem: Given a fully integrable system, compute all its hyperexp sols.

Result: A method, which is recursive on the set $\Delta = \{\delta_1, \dots, \delta_\ell, \sigma_{\ell+1}, \dots, \sigma_m\}$.

Note: The method becomes an algorithm when

1. $F = \bar{\mathbb{Q}}(x_1, \dots, x_\ell, x_{\ell+1}, \dots, x_m)$
2. $\delta_i = \frac{\partial}{\partial x_i}$ and $\sigma_j : x_j \mapsto x_j + 1$.

Structure of hyperexp sols

Proposition:

All hyperexp sols of a fully integrable system can be partitioned into

$$\{h_1 V_1 \mathbf{d}_1\} \cup \cdots \cup \{h_s V_s \mathbf{d}_s\},$$

where

- h_i is hyperexp;
- V_i is a matrix over F ;
- \mathbf{d}_i is an arbitrary nonzero vector over C_F .

Extensible hyperexponential elements

Let $\Delta_1, \Delta_2 \subset \Delta$ be nonempty and disjoint. Let E_1 be a Δ_1 -extension of F . Let $h_1 \in E_1$ be hyperexponential w.r.t. Δ_1 .

Definition: We say that h_1 is **extensible with respect to Δ_2** if there exist

- a $(\Delta_1 \cup \Delta_2)$ -extension E_2 of F , and
- an element $h_2 \in E_2$, which is hyperexponential w.r.t. $(\Delta_1 \cup \Delta_2)$

such that

$$\frac{\theta_1(h_2)}{h_2} = \frac{\theta_1(h_1)}{h_1} \quad \text{for all } \theta_1 \in \Delta_1.$$

Example

Let $F = \mathbb{C}(x, y)$ and $\Delta = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$.

Then

$$h = \exp \left(\int f(x, y) dx \right) \quad \text{where } f \in F,$$

is extensible w.r.t $\frac{\partial}{\partial y}$ iff

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial y}$$

has a solution in F .

If $f = \frac{y}{x}$, then $\frac{\partial z}{\partial x} = \frac{1}{x}$ has no solution in F , and so

$$h = \exp \left(\int \frac{y}{x} dx \right) = x^y$$

is not extensible w.r.t. $\frac{\partial}{\partial y}$.

Computing hyperexp sols of $\{\theta(\mathbf{z})=A_{\theta}\mathbf{z}\}_{\theta\in\Delta}$

1. If $|\Delta| = 1$: known methods. Otherwise partition $\Delta = \Delta_1 \cup \Delta_2$.
2. Find hyperexp sols of $\{\theta_1(\mathbf{z})=A_{\theta_1}\mathbf{z}\}_{\theta_1\in\Delta_1}$

$$\begin{array}{c}
 (\{h_1V_1\mathbf{d}_1\} \cup \dots, \{h_sV_s\mathbf{d}_s\} \cup \{h_{s+1}V_{s+1}\mathbf{d}_{s+1}\} \cup \dots \cup \{h_tV_t\mathbf{d}_t\}) \\
 \downarrow \text{ext.} \qquad \qquad \qquad \downarrow \text{ext.} \\
 \bar{h}_1, \qquad \dots, \qquad \bar{h}_s
 \end{array} \tag{1}$$

3. Determine extensibility w.r.t. Δ_2 .
4. Substitute $\mathbf{z} = \bar{h}_iV_i\mathbf{d}_i, i = 1, \dots, s$, into $\{\theta_2(\mathbf{z})=A_{\theta_2}\mathbf{z}\}_{\theta_2\in\Delta_2}$.
[Note: The substitution neither misses any hyperexp. sols nor introduces any coefficients outside F due to the replacement of h_i by \bar{h}_i]
5. Transform new systems in \mathbf{d}_i to fully integ. systems $\{\theta_2(\mathbf{y})=B_{\theta_2}^{(i)}\mathbf{y}\}_{\theta_2\in\Delta_2}$ over $\text{const}_{\Delta_1}(F)$, where $i = 1, \dots, s$.
6. Compute hyperexp sols of $\{\theta_2(\mathbf{y})=B_{\theta_2}^{(i)}\mathbf{y}\}_{\theta_2\in\Delta_2}$ and combine with (1).

Example

$F = \mathbb{Q}(e)(x, y)$, $\sigma(x) = x + 1$, $\delta = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Compute hyperexp sols of

$$\sigma(\mathbf{z}) = \begin{pmatrix} 0 & \frac{1}{y} & -xe & e+1 \\ -ye & e+1 & 0 & ye \\ 0 & 0 & 0 & \frac{1}{x+1} \\ 0 & 0 & -xe & e+1 \end{pmatrix} \mathbf{z}, \quad \delta(\mathbf{z}) = \begin{pmatrix} -\frac{1}{y} & -\frac{4}{2y-1} & \frac{x}{y} & \frac{2y-1+4y^2}{y(2y-1)} \\ 0 & 0 & 0 & 1 \\ -\frac{4y}{x(2y-1)} & 0 & \frac{4yx-2y+1}{x(2y-1)} & \frac{4y}{x(2y-1)} \\ 0 & -\frac{4}{2y-1} & 0 & \frac{4y}{2y-1} \end{pmatrix} \mathbf{z}.$$

$$\mathbf{z}_1 = \mathbf{1} \cdot \begin{pmatrix} \frac{1}{y} & 1 \\ 1 & 0 \\ 0 & \frac{1}{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \xrightarrow{\text{subs}} \delta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{4}{2y-1} & \frac{4y}{2y-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$\mathbf{z}_2 = e^x \cdot \begin{pmatrix} \frac{1}{ye} & 1 \\ 1 & 0 \\ 0 & \frac{1}{xe} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \xrightarrow{\text{subs}} \delta \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -\frac{4}{2y-1} & \frac{2y+1}{2y-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Note: c_1 and c_2 are free of x .

Solutions:

$$\begin{pmatrix} 2 \\ y \\ \frac{1}{x} \\ 1 \end{pmatrix}, e^{2y} \begin{pmatrix} \frac{1}{y} + 2 \\ 1 \\ \frac{2}{x} \\ 2 \end{pmatrix}, e^{x+y} \begin{pmatrix} \frac{1}{y} + 2e \\ e \\ \frac{2}{x} \\ 2e \end{pmatrix}, e^{x-y} \begin{pmatrix} 1+e \\ ye \\ \frac{1}{x} \\ e \end{pmatrix}.$$

Determining submodules of finite-dimensional reflexive Ore modules

Problem and result

Let M be a reflexive and finite-dimensional \mathcal{O} -module with an F -basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, and.

$$\partial_i \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_m \end{pmatrix} = B_i \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_m \end{pmatrix} \quad B_i \in F^{n \times n}, i = 1, \dots, m$$

Want: Determine all the submodules of M .

Result: A generalization of Beke's method for determining the submodules of M .

Generalization

Let M be a reflexive \mathcal{O} -module of finite dimension n over F .

Proposition A: Let $\mathbf{v}_1, \dots, \mathbf{v}_d \in M$ be F -linearly independent. Then

$F\mathbf{v}_1 + \dots + F\mathbf{v}_d$ is a submodule $\Leftrightarrow F(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_d)$ is a submodule of $\wedge^d M$.

Proposition B: Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be an F -basis of M and \mathcal{A} be the associated system. Let

$$\mathbf{w} = f_1\mathbf{e}_1 + \dots + f_n\mathbf{e}_n$$

with $f_i \in F$. Then

$F\mathbf{w}$ is a submodule $\Leftrightarrow \exists h$ hyperexponential $h(f_1, \dots, f_n)^T$ solves \mathcal{A} .

Determine all d -dimensional submodules of M

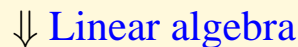
Form $\wedge^d M$ and an associated system \mathcal{A}



hyperexponential solutions of \mathcal{A}



1-dim submodules of $\wedge^d M$



1-dim submodules of the form $F(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_d)$



d -dim submodules of M

Example

Let $F = \mathbb{C}(x, k)$, $\Delta = \left\{ \frac{d}{dx}, \sigma_k : k \mapsto k + 1 \right\}$, and $\mathcal{O} = F[\partial_x, \partial_k]$.

M is an \mathcal{O} -module with an F -basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

$$\partial_x \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-x^3 - x^2k + 2x^2 + xk + k^2x + k^2 + k^3}{x^2(-x+k)} & \frac{2(x^2 - x - k^2)}{(x-k)x} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-x^3 - x^2k + x^2 + 3xk + 2x + k^2x + 4k^2 + 5k + 2 + k^3}{x^2(-x+k+1)} & -\frac{2(-x^2 + x + k^2 + 2k + 1)}{(-x+k+1)x} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}$$

and

$$\partial_k \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(x-k)x^2}{x-k-2} & 0 & \frac{2x(x-k-1)}{x-k-2} & 0 \\ -\frac{2x(x^2 - 2xk - 3x + k^2 + 2k)}{(x-k-2)^2} & -\frac{(x-k)x^2}{x-k-2} & \frac{2(x^2 - 2xk - 4x + k^2 + 3k + 2)}{(x-k-2)^2} & \frac{2x(x-k-1)}{x-k-2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}.$$

Find all two-dimensional submodules of M .

$$N_i = \{a_1 \mathbf{u}_{i,1} + a_2 \mathbf{u}_{i,2} \mid a_1, a_2 \in F\}, \quad i = 1, 2, 3, 4.$$

For N_1 ,

$$\begin{aligned} \mathbf{u}_{1,1} = & \left((2c_1x^2k + c_1k^2x + c_4x^5 + c_1x^2 + 2c_1x^4 + c_2x^2k - 2x^3c_2k - 3x^3c_1k - 2x^4c_4k \right. \\ & + x^2c_2k^2 + xk^3c_1 + x^4c_2 + c_4x^3k^2 - c_2x^3 - 2c_4x^4 - 3c_1x^3 + c_4x^3 + 2c_4x^3k) / \\ & (x(2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k \\ & - c_1k^2 + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2)) \mathbf{e}_1 \\ & - \left((c_4x^3 + c_1x^2 - kc_4x^2 - c_4x^2 - c_1xk - c_1x)(x - k) / \right. \\ & (2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 \\ & \left. + 2k^2c_4 - 2c_1k - c_1k^2 + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2) \right) \mathbf{e}_2 + \mathbf{e}_3, \end{aligned}$$

$$\begin{aligned} \mathbf{u}_{1,2} = & \left((c_1x^4 - 2c_1x^2 + c_4x^5 + c_1x - 2c_4x^3 - c_1x^3 - c_4x^4 + c_2x^3 - 2c_2x^2 \right. \\ & + 2c_4x^2 + c_1k^2 + 2c_1k^3 + c_1k^4 - 2c_2x^2k + 3c_1k^2x - 4c_1x^2k + 5kc_4x^2 + k^2c_4x + c_2xk \\ & + 3c_1xk - 4c_4x^3k - 2c_4x^3k^2 - 2c_1x^2k^2 + 2c_4xk^3 + 3c_4x^2k^2 + c_4xk^4 + c_2xk^2) / \\ & (x(2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 \\ & + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2)) \mathbf{e}_1 \\ & - \left((c_4x^3 - c_2x^2 + kc_4x^2 + c_2xk - 2k^2c_4x + c_2x - 2xc_4 + c_1xk - 4kc_4x - c_1 - c_1k^2 - 2c_1k)(x - k) / \right. \\ & \left. ((2c_1x - c_1x^2 + 2c_1xk - 2c_2k - c_2k^2 + k^3c_4 + 2c_2x - c_2x^2 + 2k^2c_4 - 2c_1k - c_1k^2 \right. \\ & \left. + 2c_2xk - 2kc_4x - 2k^2c_4x + kc_4x^2 + kc_4 - c_1 - c_2)) \right) \mathbf{e}_2 + \mathbf{e}_4, \end{aligned}$$

Summary

- (i) **From integrable systems to fully integrable ones**
 - *understand the algorithm from a module-theoretic viewpoint*
- (ii) **Hyperexponential solutions of fully integrable systems**
 - *develop efficient algorithms for ordinary cases*
- (iii) **Factoring fully integrable systems (module-theoretic version)**
 - *organize submodules with respect to isomorphisms*