q-analogues de deux problèmes de divisibilité via le lemme de Bailey.

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(*avec V. Guo et J. Zeng)

Origin of the problem (1)

The following integers are involved in Apery's proof of the irrationality of $\zeta(3)$ and are called Apery numbers

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

We have

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

proved by

- Schmidt (1992) through Legendre transform

- Strehl (1993) in 6 ways, among which combinatorics, hypergeometry or automatic verification by computer.

Origin of the problem (2)

The numbers a_n satisfy :

$$(n+1)^3 a_{n+1} - ((n+1)^3 + n^3 + 4(2n+1)^3)a_n + n^3 a_{n-1} = 0$$

which is hard to prove!
This relation is a crucial ingredient in Apery's proof of the irrationality of $\zeta(3)$.

Strehl calls the $f_n = \sum_{j=0}^{n} \binom{n}{j}^3$ Franel numbers, as Franel found (1895) :

$$(n+1)^2 f_{n+1} - (7n^2 + 7n + 2)f_n - 8n^2 f_{n-1} = 0$$

The previous identity is

$$a_n = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} f_k$$

which asserts that $(a_n)_n$ and $(\binom{2n}{n} f_n)_n$ are inverses under Legendre transform.

Zudilin's result

Zudilin proved the following generalization, raised by Schmidt :

Theorem (Zudilin, 2004)

For any integer $r \geq 2$*, define a sequence of numbers* $(c_k^{(r)})_{k\geq 0}$ *independant of* n*, by*

$$\sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k^{(r)},$$

then $c_k^{(r)} \in \mathbb{N}$.

Tools

- Legendre transform (also necessary for proving the existence of the $c_k^{(r)}$)
- the $q = 1$ specialization of a multiple series transformation due to G. Andrews

Special cases :

$$c_k^{(2)} = \sum_{j=0}^{k} \binom{k}{j}^3 \qquad \text{(Schmidt, Strehl)}$$

$$c_k^{(3)} = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}^2 \binom{2j}{k-j} \qquad \text{(Strehl, computer proof)}$$

Zudilin's question

Is it possible to find and prove a q-analogue of the previous theorem ?

q-shifted factorial : $(a)_0 = 1$ and

$$(a)_n = \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots, \\ ((1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n}))^{-1}, & n = -1, -2, \ldots. \end{cases}$$

compact notations :

$$(a_1, \ldots, a_m)_n := (a_1)_n \cdots (a_m)_n, \qquad (a_1, \ldots, a_m)_\infty := \lim_{n\to\infty} (a_1, \ldots, a_m)_n.$$

q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}} \in \mathbb{N}[q].$$

Since $\frac{1}{(q)_n} = 0$ if $n < 0$, we have $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $k > n$ or $k < 0$.

Note that

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$$

A solution

Theorem (Guo-J-Zeng, 2007)

For any integer $r \geq 1$, define rational fractions $c_k^{(r)}(q)$ of the variable q, independent of n, by writing

$$\sum_{k=0}^{n} q^{r\binom{n-k}{2}+(1-r)\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^r \begin{bmatrix} n+k \\ k \end{bmatrix}^r = \sum_{k=0}^{n} q^{\binom{n-k}{2}+(1-r)\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} c_k^{(r)}(q).$$

Then $c_k^{(r)}(q) \in \mathbb{N}[q]$.

The $r = 1$ case is trivial : we suppose that $r \geq 2$.

Tools

- q-Legendre transform
- Andrews' multiple extension of Bailey's lemma

q-Legendre transform

$$a_n = \sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n+k \\ n-k \end{bmatrix} b_k \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{1-q^{2k+1}}{1-q^{n+k+1}} \begin{bmatrix} 2n \\ n-k \end{bmatrix} a_k$$

Special case of Carlitz's q-Gould-Hsu inverse formula (1973).
Generalized in a matrix inversion by Krattenthaler (1996).

As $\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} = \begin{bmatrix} 2k \\ k \end{bmatrix} \begin{bmatrix} n+k \\ n-k \end{bmatrix}$, this inversion gives

$$c_n^{(r)}(q) = q^{(r-1)\binom{n}{2}} \begin{bmatrix} 2n \\ n \end{bmatrix}^{-1} \sum_{j=0}^{n} \begin{bmatrix} 2j \\ j \end{bmatrix}^{r} t_{n,j}^{(r)}(q),$$

where

$$t_{n,j}^{(r)}(q) = q^{r\binom{j+1}{2}} \sum_{k=j}^{n} (-1)^{n-k} \frac{1-q^{2k+1}}{1-q^{n+k+1}} \begin{bmatrix} 2n \\ n-k \end{bmatrix} \begin{bmatrix} k+j \\ k-j \end{bmatrix}^{r} q^{\binom{k}{2}-rjk}.$$

Theorem (Guo-J-Zeng, 2007)

For $r \geq 2$ we have $q^{(r-1)\binom{n}{2}} \begin{bmatrix} 2j \\ j \end{bmatrix} \begin{bmatrix} 2n \\ n \end{bmatrix}^{-1} t_{n,j}^{(r)}(q) \in \mathbb{N}[q].$

Bailey's lemma

(α_n, β_n) is a Bailey pair related to a if

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}} \qquad \forall\ n \geq 0.$$

Theorem (Bailey's lemma)

If (α_n, β_n) is a Bailey pair related to a, then (α'_n, β'_n) is also a Bailey pair related to a, where

$$\alpha'_n = \frac{(b,c)_n}{(aq/b, aq/c)_n}(aq/bc)^n \alpha_n,$$

$$\beta'_n = \sum_{k=0}^{n} \frac{(b,c)_k (aq/bc)_{n-k}}{(q)_{n-k}(aq/b, aq/c)_n}(aq/bc)^k \beta_k.$$

Iterating gives the Bailey chain (Andrews) :

$$(\alpha_n, \beta_n) \longrightarrow (\alpha'_n, \beta'_n) \longrightarrow (\alpha''_n, \beta''_n) \longrightarrow \cdots$$

Some consequences

One Bailey pair $\Rightarrow$ infinitely many identities.

Most famous Bailey pair :

$$\alpha_n = (-1)^n q^{\binom{n}{2}} \frac{1-aq^{2n}}{1-a} \frac{(a)_n}{(q)_n}, \qquad \beta_n = \delta_{n,0}.$$

Two iterations of Bailey's lemma give Watson's transformation formula, a six parameter finite extension of the Rogers-Ramanujan identities :

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} = \frac{1}{(q,q^4;q^5)_\infty},$$

$$\sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q)_k} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

Andrews' consequence of the Bailey chain

Theorem (Andrews, 1975, 1986)

For every integers $m \geq 1$ and $N \geq 0$:

$$\sum_{k\geq 0} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, \ldots, b_m, c_m, q^{-N})_k}{(q, \sqrt{a}, -\sqrt{a}, aq/b_1, aq/c_1, \ldots, aq/b_m, aq/c_m, aq^{N+1})_k} \left(\frac{a^m q^{m+N}}{b_1 c_1 \cdots b_m c_m} \right)^k$$

$$= \frac{(aq, aq/b_m c_m)_N}{(aq/b_m, aq/c_m)_N} \sum_{l_1,\ldots,l_{m-1}\geq 0} \frac{(aq/b_1 c_1)_{l_1} \cdots (aq/b_{m-1} c_{m-1})_{l_{m-1}}}{(q)_{l_1} \cdots (q)_{l_{m-1}}}$$

$$\times \frac{(b_2, c_2)_{l_1} \ldots (b_m, c_m)_{l_1+\cdots+l_{m-1}}}{(aq/b_1, aq/c_1)_{l_1} \ldots (aq/b_{m-1}, aq/c_{m-1})_{l_1+\cdots+l_{m-1}}}$$

$$\times \frac{(q^{-N})_{l_1+\cdots+l_{m-1}}}{(b_m c_m q^{-N}/a)_{l_1+\cdots+l_{m-1}}} \frac{(aq)^{l_{m-2}+\cdots+(m-2)l_1} q^{l_1+\cdots+l_{m-1}}}{(b_2 c_2)^{l_1} \cdots (b_{m-1} c_{m-1})^{l_1+\cdots+l_{m-2}}}.$$

Special cases :

- for $m = 1$, Jackson's finite summation formula
- for $m = 2$, Watson's finite transformation formula

Application to our problem (1)

For $r = 2$, apply Andrews's formula with $m = 1$, $a = q^{-(2n+1)}$, $N = n - j$, $b_1 = q^{-n}$ and $c_1 = q^{-(n-j)}$. This gives

$$t_{n,j}^{(2)}(q) = \frac{(q)_{2n}(q)_j^2}{(q)_n(q)_{2j}(q)_{2j-n}(q)_{n-j}^2} q^{2\binom{n-j}{2}-\binom{n}{2}},$$

thus

$$\begin{bmatrix} 2j \\ j \end{bmatrix} \begin{bmatrix} 2n \\ n \end{bmatrix}^{-1} q^{\binom{n}{2}} t_{n,j}^{(2)}(q) = \begin{bmatrix} n \\ n-j, n-j, 2j-n \end{bmatrix} q^{2\binom{n-j}{2}} \in \mathbb{N}[q].$$

We have more : $c_n^{(2)}(q) = \sum_{j=0}^{n} \begin{bmatrix} 2j \\ n \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}^2 q^{2\binom{n-j}{2}}$.

They are q-analogues of Franel numbers :

$$c_n^{(2)}(1) = \sum_{j=0}^{n} \binom{2j}{n} \binom{n}{j}^2 = \sum_{j=0}^{n} \binom{n}{j}^3 = f_n.$$

Application to our problem (2)

For $r = 3$, apply Andrews's formula with $m = 1$, $a = q^{-(2n+1)}$, $N = n - j$ and $b_1 = c_1 = q^{-(n-j)}$:

$$t_{n,j}^{(3)}(q) = \frac{(q)_{2n}}{(q)_{3j-n}(q)_{n-j}^3} q^{3\binom{n-j}{2}-2\binom{n}{2}},$$

which shows that

$$\begin{bmatrix} 2j \\ j \end{bmatrix} \begin{bmatrix} 2n \\ n \end{bmatrix}^{-1} q^{2\binom{n}{2}} t_{n,j}^{(3)}(q) = \begin{bmatrix} n \\ j \end{bmatrix}^2 \begin{bmatrix} 2j \\ n-j \end{bmatrix} q^{3\binom{n-j}{2}} \in \mathbb{N}[q].$$

Treat similarly the cases $r = 2s \geq 4$ and $r = 2s + 1 \geq 5$ by specializing Andrews' formula.

A result of Calkin

Calkin (1998) proved by arithmetical techniques that for $m \geq 1$:

$$c_m(n) := \binom{2n}{n}^{-1} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^m \in \mathbb{Z}$$

By the binomial theorem, $c_1(n) = 0$.

By Kummer's formula, $c_2(n) = 1$.

By Dixon's formula, $c_3(n) = \binom{3n}{n}$.

De Bruijn (1981) had shown by asymptotic techniques that for $m \geq 4$ there is no closed form for $c_m(n)$.

Finite forms of the Rogers-Ramanujan identities (1)

As a consequence of Bailey's lemma :

Theorem (Guo-J-zeng, 2007)

If at least one of a, b *and* c *is of the form* q^n, $n = 1, 2, \dots,$ *then :*

$$\sum_{k=-n}^{n} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d, e)_k} (abcdeq^{-3})^k = \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{n} \frac{(q/a, q/b, q/c, de/q)_k}{(q, q^3/abc, d, e)_k} q^k,$$

$$\sum_{k=-n}^{n} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(aq, bq, cq, dq, eq)_k} (abcdeq^{-1})^k = \frac{(q, ab, bc, ac)_\infty}{(aq, bq, cq, abc/q)_\infty} \sum_{k=0}^{n} \frac{(q/a, q/b, q/c, de)_k}{(q, q^2/abc, dq, eq)_k} q^k.$$

These are finite extensions of the Rogers-Ramanujan identities, extending recent work of Liu (2003).

Finite forms of the Rogers-Ramanujan identities (2)

As a consequence of the Bailey lattice from Agarwal, Andrews and Bressoud (1987), we proved :

Theorem (Guo-J-zeng, 2007)

If at least one of a, b and c is of the form q^n, $n = 1, 2, \ldots,$ then :

$$\sum_{k=-n}^{n} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d/q, e/q)_k} (abcdeq^{-3})^k = \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{n} \frac{(q/a, q/b, q/c, de/q^2)_k}{(q, q^3/abc, d, e)_k} q^k,$$

$$\sum_{k=-n}^{n} \frac{(q/a, q/b, q/c, q/d, q/e)_k}{(a, b, c, d/q, e/q)_k} (abcdeq^{-4})^k = \frac{(q, ab/q, bc/q, ac/q)_\infty}{(a, b, c, abc/q^2)_\infty} \sum_{k=0}^{n} \frac{(q/a, q/b, q/c, de/q^2)_k}{(q, q^3/abc, d, e)_k} q^{2k}.$$

Link with Calkin's result

Recall Calkin's result :

$$c_m(n) := \binom{2n}{n}^{-1} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^m \in \mathbb{Z}$$

Specializing the first of the 4 previous finite forms of the Rogers-Ramanujan identities we stumbled across :

$$c_4(n) = \sum_{k=0}^{n} \binom{2n+k}{k} \binom{2n}{n+k}^2$$

and

$$c_5(n) = \sum_{k=0}^{n} \binom{3n-k}{n-k} \binom{2n+k}{k} \binom{2n}{n+k}^2$$

Is there a q-analogue of Calkin's result, also implying $c_m(n) \in \mathbb{N}$ for $m \geq 2$?

Extension

By specializing Andrews' formula, we can prove :

Theorem (Guo-J-Zeng, 2007)

For $m \geq 3$ and all positive integers $n_1, \ldots, n_m$:

$$\sum_{k=-n_1}^{n_1} (-1)^k q^{(m-1)k^2+\binom{k}{2}} \prod_{i=1}^{m} \begin{bmatrix} n_i + n_{i+1} \\ n_i + k \end{bmatrix} = \begin{bmatrix} n_1 + n_m \\ n_1 \end{bmatrix} \sum_{\lambda} \prod_{i=1}^{m-2} q^{\lambda_i^2} \begin{bmatrix} \lambda_{i-1} \\ \lambda_i \end{bmatrix} \begin{bmatrix} n_{i+1} + n_{i+2} \\ n_{i+1} - \lambda_i \end{bmatrix},$$

where $n_{m+1} = \lambda_0 = n_1$ and the sum is over all sequences $\lambda = (\lambda_1, \ldots, \lambda_{m-2})$ of nonnegative integers such that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{m-2}$.

Theorem (Guo-J-Zeng, 2007)

For all positive integers $n_1, \ldots, n_m$, $n_{m+1} = n_1$ and $0 \leq j \leq m-1$:

$$\begin{bmatrix} n_1 + n_m \\ n_1 \end{bmatrix}^{-1} \sum_{k=-n_1}^{n_1} (-1)^k q^{jk^2+\binom{k}{2}} \prod_{i=1}^{m} \begin{bmatrix} n_i + n_{i+1} \\ n_i + k \end{bmatrix} \in \mathbb{N}[q].$$

Special cases

Setting $q = 1$ in the first theorem :

Corollary

For $m \geq 3$ and all positive integers $n_1, \ldots, n_m$:

$$\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{m} \binom{n_i + n_{i+1}}{n_i + k} = \binom{n_1 + n_m}{n_1} \sum_{\lambda} \prod_{i=1}^{m-2} \binom{\lambda_{i-1}}{\lambda_i} \binom{n_{i+1} + n_{i+2}}{n_{i+1} - \lambda_i},$$

where $n_{m+1} = \lambda_0 = n_1$ and the sum is over all sequences $\lambda = (\lambda_1, \ldots, \lambda_{m-2})$ of nonnegative integers such that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{m-2}$.

Setting $n_1 = \cdots = n_m = n$ in the second theorem :

Corollary

For all positive m, n and $0 \leq j \leq m-1$,

$$\begin{bmatrix} 2n \\ n \end{bmatrix}^{-1} \sum_{k=-n}^{n} (-1)^k q^{jk^2 + \binom{k}{2}} \begin{bmatrix} 2n \\ n+k \end{bmatrix}^{m} \in \mathbb{N}[q]$$

Some conjectures

Based on computer experiments :

Conjecture

Let $\gcd(a_1, a_2, \ldots)$*=greatest common divisor of integers* $a_1, a_2, \ldots$.
For all positive m and n, we have

$$\gcd\left(\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}^r,\ r=m,m+1,\ldots\right)=\binom{2n}{n}.$$

Conjecture

For all positive r, s, t and n,

$$\sum_{k=-n}^{n}(-1)^k\binom{8n}{4n+k}^r\binom{4n}{2n+k}^s\binom{2n}{n+k}^t$$

is divisible by $2\binom{8n}{3n}$.