Multivariate generalizations of the Foata-Schützenberger equidistribution

Fourth Colloquium in Mathematics and Computer Science

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Overline

1. Motivation
2. Combinatorial background
3. Cayley trees
4. From trees to a permutation statistic
5. Descent classes and codes
6. Conclusion

Initial motivation and results

- Better understanding of Pre-Lie algebras,
- Relate P-L with combinatorics, algorithmics.

A different conclusion:

Analysis of *Cayley's trees-formula* for integrating vector field

$$\Downarrow$$

A pure *combinatorial construction*, namely, a new permutation statistic, coming from trees!

$$\Downarrow$$

A *multivariate equirepartition theorem* of the number of inversion and the inverse Mac-Mahon index on permutations of a given descent class

Inversions and Icode

Definitions

An *inversion* of a word $w = w_1 w_2 \dots w_n$ is a pair (i, j) such that

$$i < j \quad \text{and} \quad w_i > w_j \,. \tag{1}$$

The *inversion number* is denoted by $Inv(w)$.

Separate the set of inversions by the value of w_j (inverse Lehmer code).

σ	3	6	8	1	5	2	9	7	4
	1	2	3	4	5	6	7	8	9
Icode	3	4	0	5	2	0	2	0	0

Descents and the major index

Definitions

A *descent* of a word $w = w_1 w_2 \dots w_n$ is an integer i such that

$$w_i > w_{i+1}\,. \tag{2}$$

A *descent class* is the set of permutations with given descents.
The *major index* Maj of a word is the sum of its descents.

σ	3	6	8	1	5	2	9	7	4
descent position			3		5		7	8	

$$\mathrm{Maj}(368152974) = 3 + 5 + 7 + 8 = 23.$$

Inversions vs descents

Theorem (MacMahon, 1913)

Over the symmetric group, the generating series of the number of inversions is equal to the g. s. of the major index.

Theorem (Foata-Schützenberger, 1970)

Over any descent class of the symmetric group, the same result holds.

A computation problem in integration (Cayley 1857)

Problem

Knowing the speed V as a function of the distance x, compute the distance x as a function of the time t, that is solve

$$x(0) = 0 \qquad \textit{and} \qquad x'(t) = V(x(t))\,. \tag{3}$$

Formal (algebraic way): compute the Taylor series of $x(t)$ from the Taylor series of $V(x)$.

$$x(t) = 0 + x'(0)\,t + x^{(2)}(0)\,\frac{t^2}{2!} + x^{(3)}(0)\,\frac{t^3}{3!} + \cdots \tag{4}$$

The derivatives of $x(t)$

$$x'(t) = V(x(t)) = (V \circ x)(t)$$

Using the derivative of compose functions

$$x^{(2)} = \left(\frac{dV}{dx}\right)_{x(t)} \cdot x'(t) = \left(\frac{dV}{dx}\right)_{x(t)} \cdot V_{x(t)} =: V_{10}$$

$$x^{(3)} = \left(\frac{d^2V}{dx^2}\right)_{x(t)} \cdot V_{x(t)}^2 + \left(\frac{dV}{dx}\right)_{x(t)}^2 \cdot V_{x(t)} = V_{200} + V_{110}$$

$$x^{(4)} = V_{3000} + 4V_{2100} + V_{1110}$$

$$x^{(5)} = V_{40000} + 7V_{31000} + 4V_{22000} + 11V_{21100} + V_{11110}$$

$$x^{(6)} = V_{500000} + 11V_{410000} + 15V_{320000} + 32V_{311000} + 34V_{221000} + 26V_{211100} + V_{111110}$$

A combinatorial interpretation

Observation

$$x^{(n)} = \sum_{\sigma \in \mathfrak{S}_{n-1}} V_{\mathrm{Sort}(\mathrm{Eval}(\mathrm{Code}(\sigma)))} \cdot$$

σ	3	6	8	1	5	2	9	7	4
Code	2	5	5	0	1	0	2	1	0
Eval	0^3	1^2	2^2	3^0	4^0	5^2	6^0	7^0	8^0
Sort	3	2	2	2	0	0	0	0	0

Combinatorial interpretation (2)

$$x^{(4)} = V_{3000} + 4V_{2100} + V_{1110}$$

permutation	code	multiplicities 0123	sort
123	000	3000	3000
132	010	2100	2100
213	100	2100	2100
231	110	1200	2100
312	200	2010	2100
321	210	1110	1110

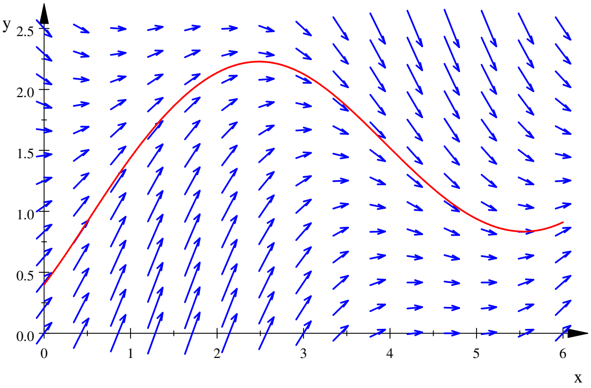
$$x^{(5)} = V_{40000} + 7V_{31000} + 4V_{22000} + 11V_{21100} + V_{11110}$$

perm.	code	mult.	sort	perm.	code	mult.	sort
1234	0000	40000	40000	1432	0210	21100	21100
1243	0010	31000	31000	2413	1200	21100	21100
1324	0100	31000	31000	2431	1210	12100	21100
1423	0200	30100	31000	3142	2010	21100	21100
2134	1000	31000	31000	3214	2100	21100	21100
2341	1110	13000	31000	3241	2110	12100	21100
3124	2000	30100	31000	3421	2210	11200	21100
4123	3000	30010	31000	4132	3010	21010	21100
1342	0110	22000	22000	4213	3100	21010	21100
2143	1010	22000	22000	4231	3110	12010	21100
2314	1100	22000	22000	4312	3200	20110	21100
3412	2200	20200	22000	4321	3210	11110	11110

Better understanding ? Add dimensions !

Given a *vector field* $\vec{V}_x$ for $x \in \mathbb{R}^d$, find the *flow* integrating the vector field, *i.e.*, find $x(t)$ such that

$$x(0) = x_0 \qquad \text{and} \qquad x'(t) = \vec{V}_{x(t)} \tag{5}$$



The differential of a vector field

Definition

Let $\vec{V}$ and $\vec{U}_1, \ldots \vec{U}_k$ be some vector fields. Then the *k-th differential* $D^k\vec{V}$ of $\vec{V}$ is defined by

$$[D^k\vec{V}(\vec{U}_1, \ldots \vec{U}_k)]_i := \sum_{j_1 \ldots j_k = 1}^{d} \frac{\partial^k [\vec{V}]_i}{\partial x_{j_1} \ldots \partial x_{j_k}} [\vec{U}_1]_{j_1} \ldots [\vec{U}_k]_{j_k}\,, \tag{6}$$

where $[\vec{W}]_i$ denotes the i-th coordinate of the vector field $\vec{W}$.

- This definition is independent of the coordinate system.
- The point x where the vector fields are taken is implicit.

The derivatives of $x(t)$

$$x'(t) = \vec{V}_{x(t)} = (\vec{V} \circ x)(t)$$

Using the derivative of compose functions

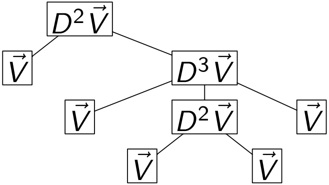
$$x^{(2)} = D\vec{V}_x(x') = D\vec{V}_x(\vec{V}_x)$$

Third and fourth derivative with implicit $x(t)$:

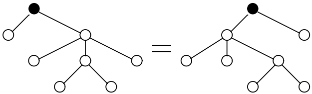
$$\begin{aligned} x^{(3)} =& D^2\vec{V}(\vec{V},\vec{V}) + D\vec{V}(D\vec{V}(\vec{V})) \\ x^{(4)} =& D^3\vec{V}(\vec{V},\vec{V},\vec{V}) + 3\,D^2\vec{V}(\vec{V},D\vec{V}(\vec{V})) + \\ & D\vec{V}(D^2\vec{V}(\vec{V},\vec{V})) + D\vec{V}(D\vec{V}(D\vec{V}(\vec{V}))) \end{aligned}$$

A better notation: expression trees (Cayley)

$$D^2\vec{V}(\vec{V}, D^3\vec{V}(\vec{V}, D^2\vec{V}(\vec{V}, \vec{V}), \vec{V})) =$$



Clairaut's theorem $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$: rooted topological (Cayley) trees

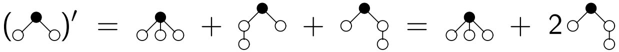


Compose derivative formula

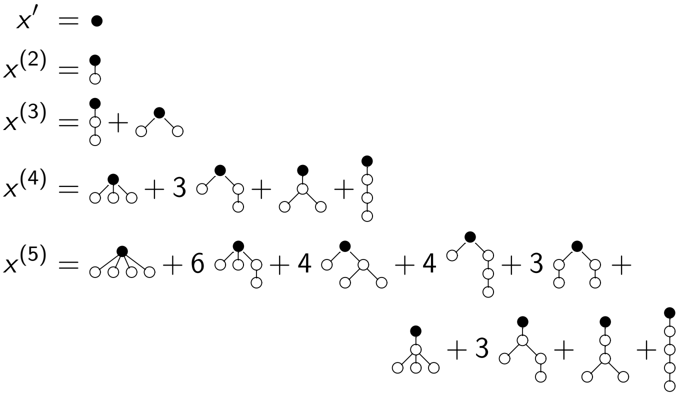
Proposition

$$(D_T V)' = \sum_{T'} D_{T'} V \qquad (7)$$

where T' runs over set of trees obtained by adding a leaf to each node of T.



The derivatives of $x(t)$ (continued)



$\dots$

Closed formula

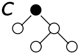
Theorem

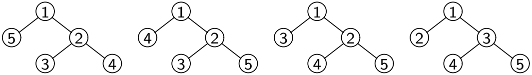
The n-th derivative of $x(t)$ is given by

$$x^{(n)} = \sum_{T:\ \text{tree of size } n} c_T \, T \tag{8}$$

where c_T is the number of standard increasing labellings of T.

Example:

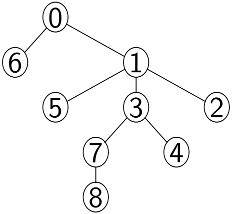
c  $= 4$:



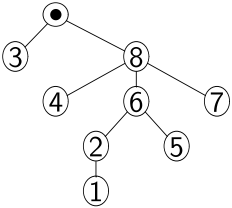
From trees to a permutation statistic

Bijections

code $\leftrightarrows$ increasing trees $\leftrightarrows$ permutations



$\equiv$

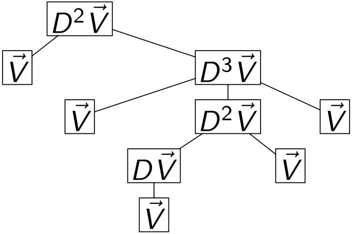


$\equiv$ 38462157

$$\text{Scode} = \begin{array}{cccccccc} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 3 & 0 & 1 & 3 & 1 & 1 & 0 \end{array}$$

Back to dimension $d = 1$

The n-th differential becomes multiplication by the n-th derivative; therefore one has to record the *arity* of the nodes:



$$\longmapsto \quad V_{322100000}$$

$$\mathrm{Eval}(73013110) \;=\; 0^2 1^3 2^0 3^2 4^0 5^0 6^0 7^1 8^0$$

Back to codes

The Icode and the Scode share the property that

$$x^{(n)} = \sum_{\sigma \in \mathfrak{S}_{n-1}} V_{\mathrm{Sort(Eval(I\ or\ S(}\sigma\mathrm{)))}} \cdot$$

Obvious since $\{I(\sigma)\} = \{S(\sigma)\}$.

Proof "natural" from the S point of view.

What about a finer result? $\longrightarrow$ Descent classes.

What about the major index? $\longrightarrow$ The majcode.

The majcode

Same operation as in the Icode case:
If $w^{(i)}$ is obtained from w by erasing $w_k < i$, cut the major index into parts as the sequence $\mathrm{Maj}(w^{(i)}) - \mathrm{Maj}(w^{(i+1)})$.

σ							Maj	*majcode*
$\sigma^{(1)}$	3	6•	1	5•	4•	2	11	2
$\sigma^{(2)}$	3	6•		5•	4•	2	9	4
$\sigma^{(3)}$	3	6•		5•	4		5	2
$\sigma^{(4)}$		6•		5•	4		3	2
$\sigma^{(5)}$		6•		5			1	1
$\sigma^{(6)}$		6					0	0
majcode	2	4	2	2	1	0		

Main result: inversions vs descents

Theorem

Over any descent class of the symmetric group, the inverse icode, the inverse majcode and the inverse Scode have same distribution, up to order.

Descent class: $\{2,4\}$ of $\mathfrak{S}_5$.

perm.	IIcode	Imajcode	IScode
13254	01010	22110	03010
14253	02010	10110	01010
14352	02110	22010	03110
15243	03010	23110	02010
15342	03110	23010	02110
23154	11010	12110	33010
24153	12010	20110	31010
24351	12110	12010	13110

perm.	IIcode	Imajcode	IScode
25143	13010	33110	32010
25341	13110	13010	12110
34152	22010	00110	11010
34251	22110	02010	33110
35142	23010	13110	12010
35241	23110	33010	32110
45132	33010	03110	22010
45231	33110	03010	22110

Conclusion and open questions

- Combine statistics with the number of descents: Euler-Mahonian.
- Many new statistics obtained by an equivalent process.
- Bijective proof?
- Back to Pre-Lie algebras?

Thank you!

Some algebraic structure

The classical Lie bracket on vector field

$$[U, V] = DU(V) - DV(U) \qquad (9)$$

can be encoded on trees where

$$DT_1(T_2) = \sum_{n\text{:node of } T_1} \text{grafting of the root of } T_2 \text{ on } n \qquad (10)$$

For example: