Dessin de triangulations: algorithmes, combinatoire, et analyse

Éric Fusy

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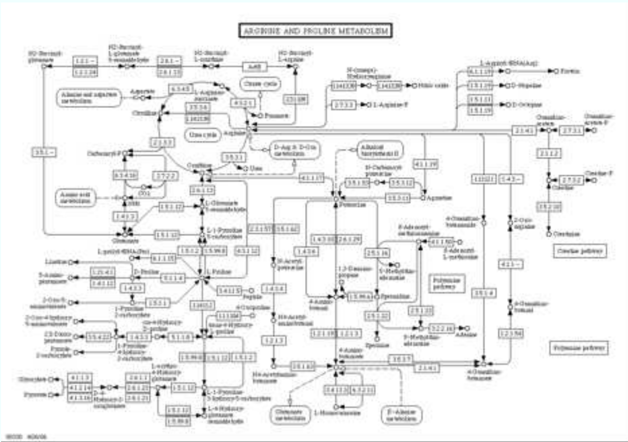
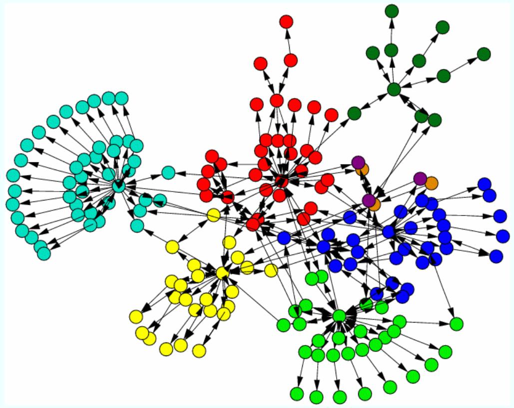
Motivations

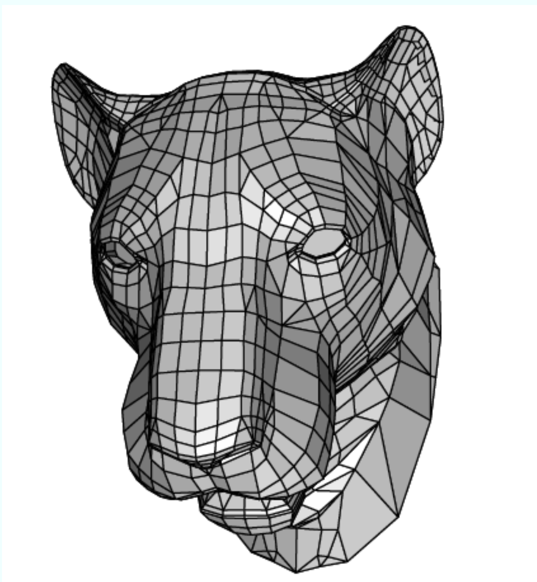
Display of large structures on a planar surface



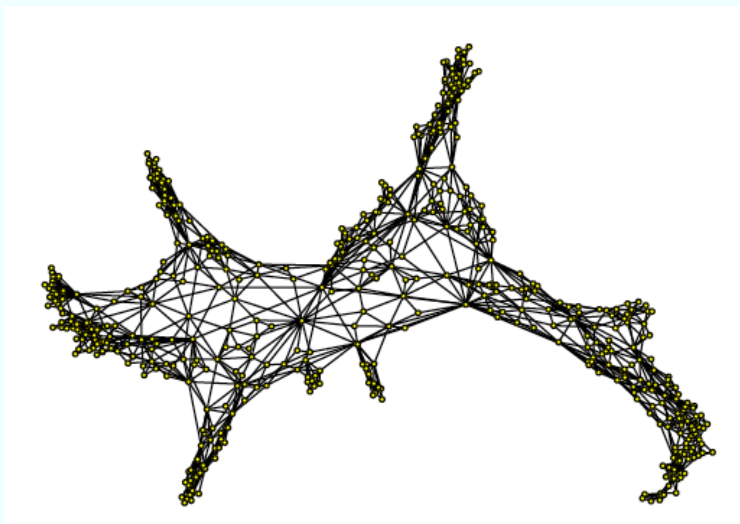
Metabolism



Web site



Meshes

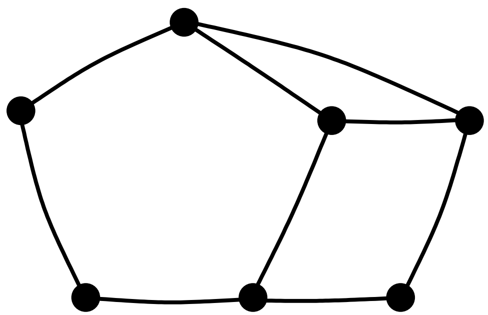


Discrete surfaces

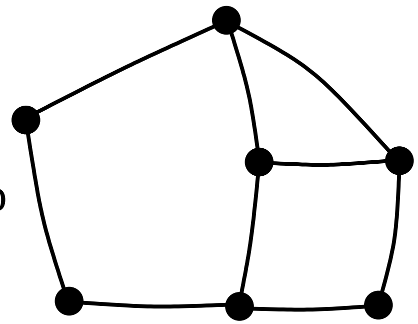
Planar maps

- A planar map is obtained by embedding a planar graph in the plane without edge crossings.
- A planar map is defined up to continuous deformation

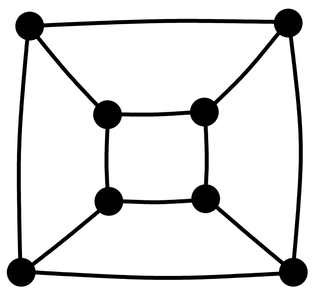
Planar map



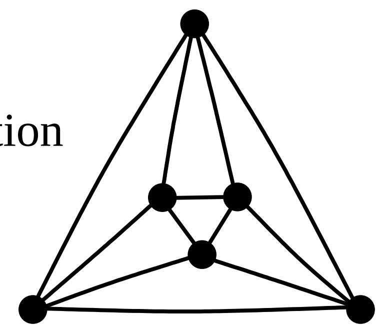
The same planar map



Quadrangulation



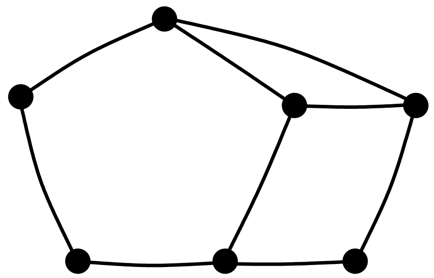
Triangulation



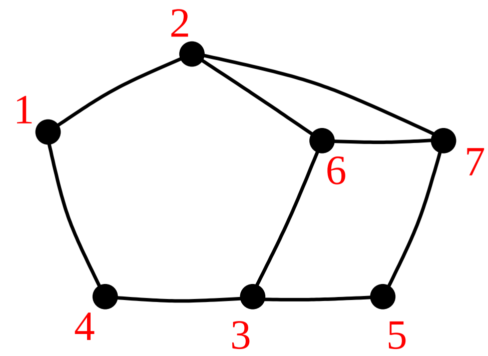
Planar maps

- Planar maps are combinatorial objects
- They can be encoded without dealing with coordinates

Planar map



Choice of labels



Encoding: to each vertex is associated the (cyclic) list of its neighbours in clockwise order

1: (2, 4)
2: (1, 7, 6)
3: (4, 6, 5)
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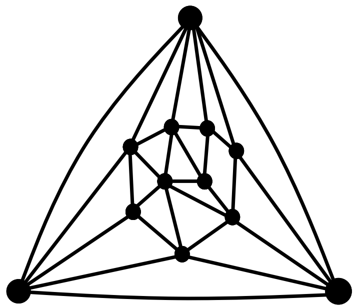
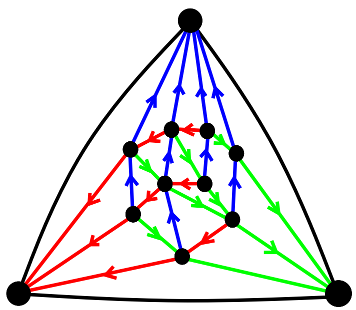
Combinatorics of maps

Triangulations



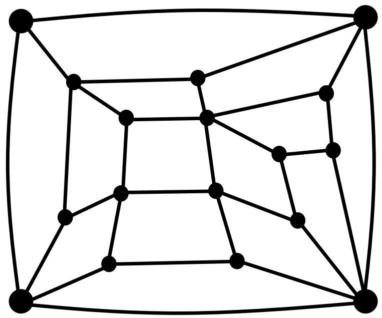
$\Downarrow$

3 spanning trees



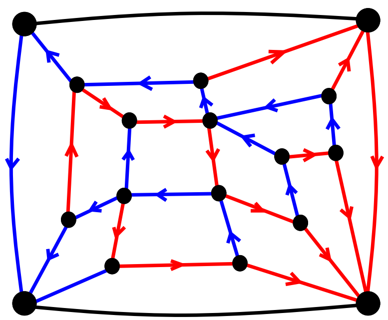
$$|\mathcal{T}_n| = \frac{2(4n-3)!}{n!(3n-2)!}$$

Quadrangulations



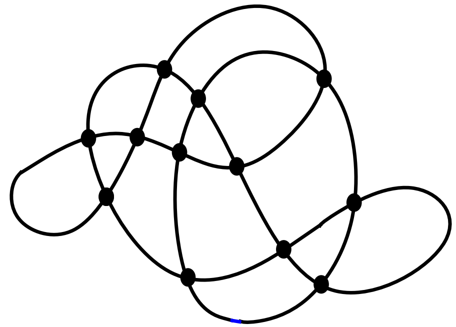
$\Downarrow$

2 spanning trees



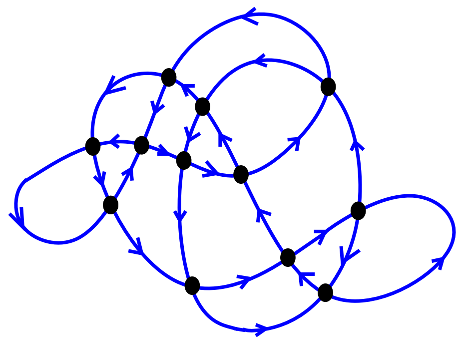
$$|\mathcal{Q}_n| = \frac{2(3n-3)!}{n!(2n-2)!}$$

Tetravalent



$\Downarrow$

eulerian orientation

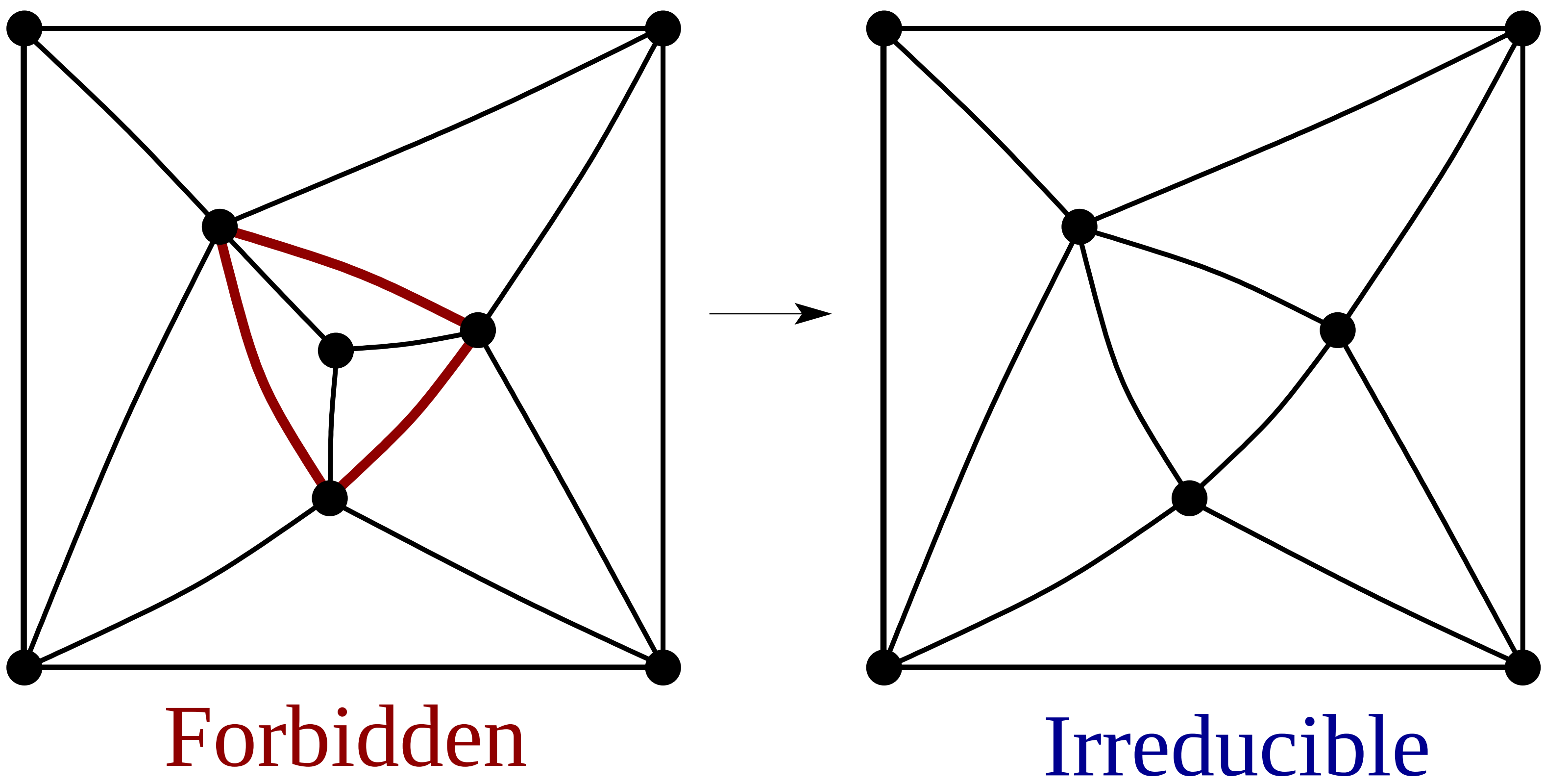


$$|\mathcal{E}_n| = \frac{2\cdot 3^n(2n)!}{n!(n+2)!}$$

$\Rightarrow$ Tutte, Schaeffer, Schnyder, De Fraysseix et al...

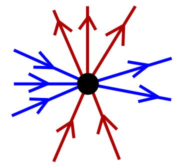
A particular family of triangulations

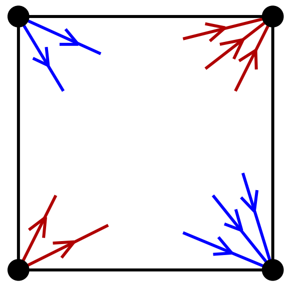
- We consider triangulations of the 4-gon (the outer face is a quadrangle)
- Each 3-cycle delimits a face (irreducibility)

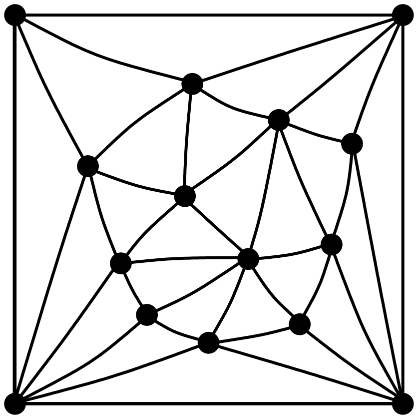
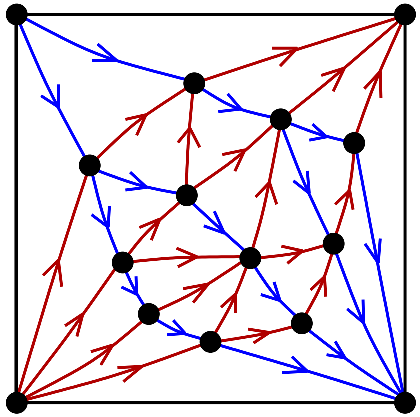


Transversal structures

We define a transversal structure using local conditions
Inner edges are colored blue or red and oriented:

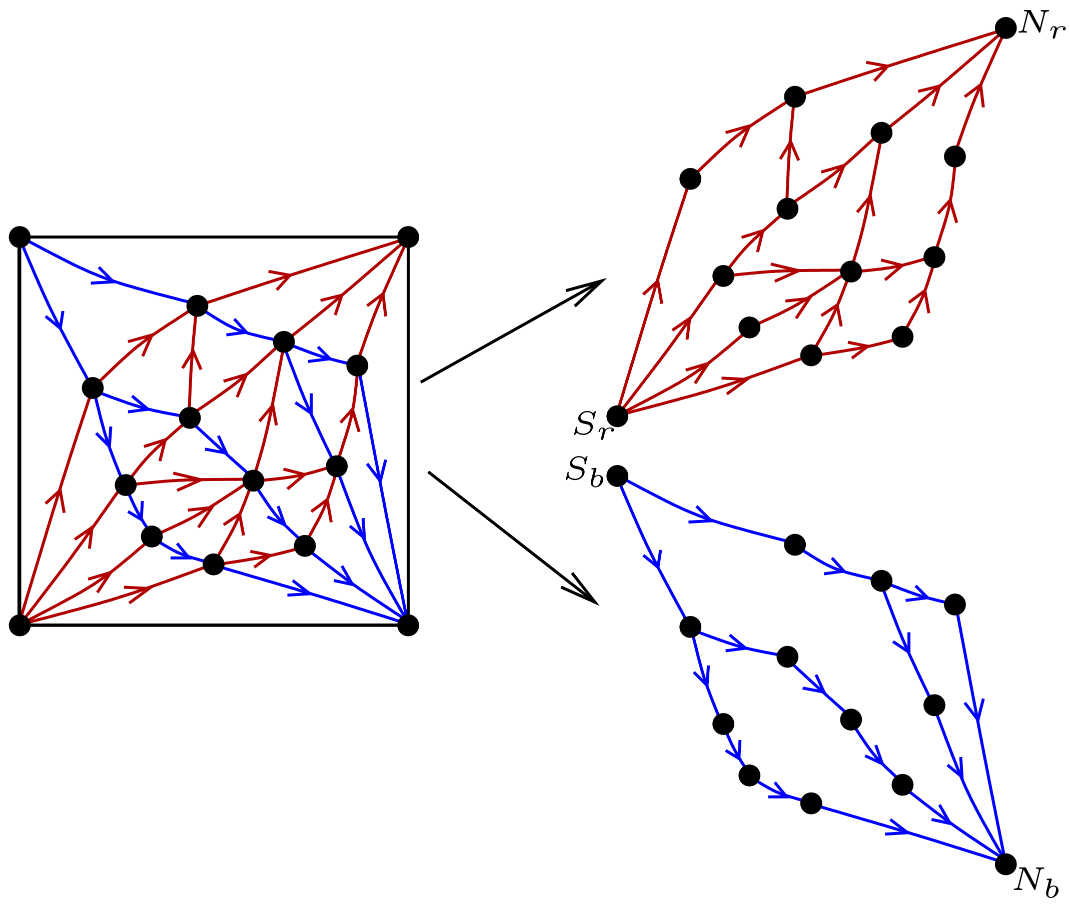
Inner vertex ⇒ 

Border vertices ⇒ 

Example:  ⇒ 

Link with bipolar orientations

bipolar orientation = **acyclic** orientation with a unique minimum and a unique maximum
The blue (resp. red) edges give a bipolar orientation
The two bipolar orientations are transversal



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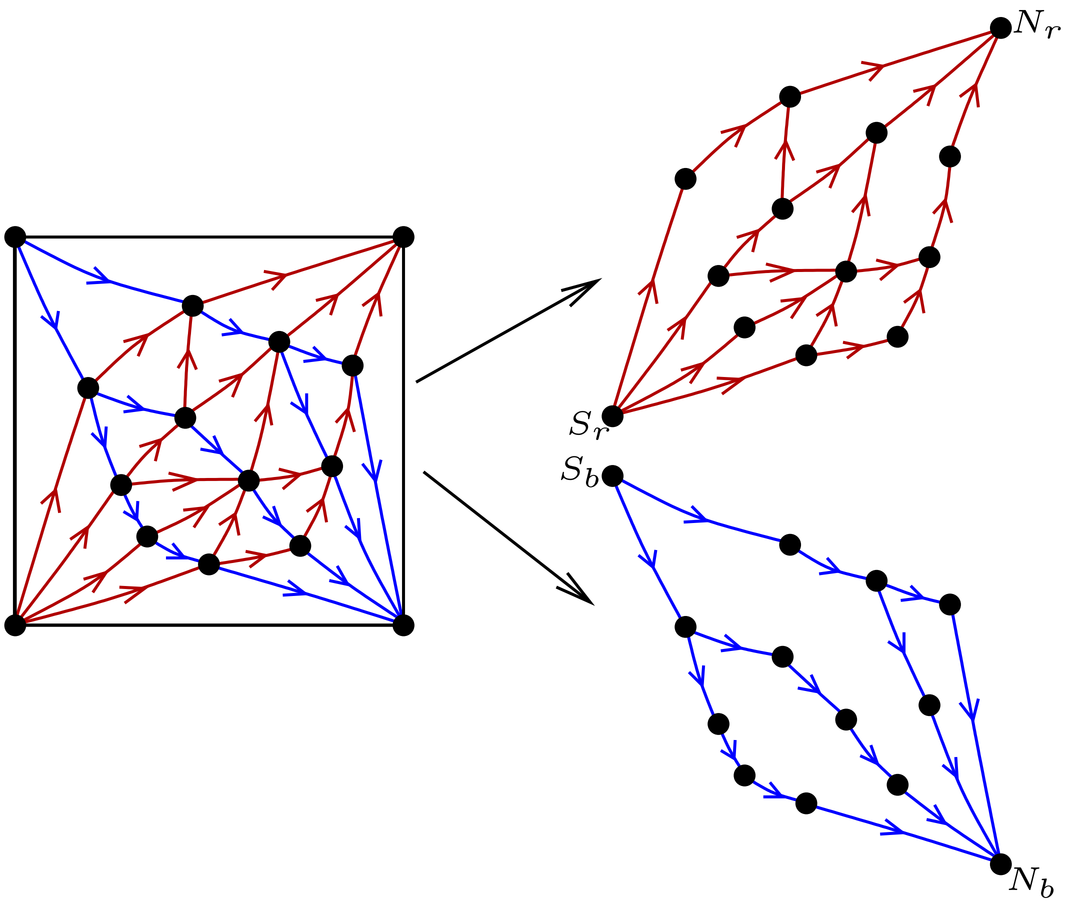

Definition and properties of transversal structures on triangulations

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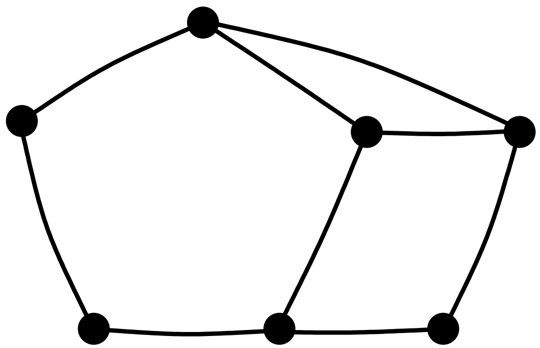

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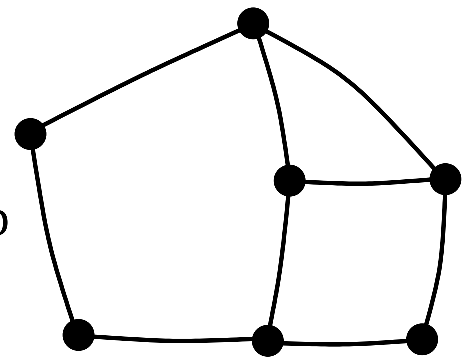

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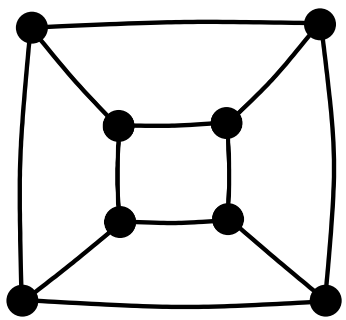
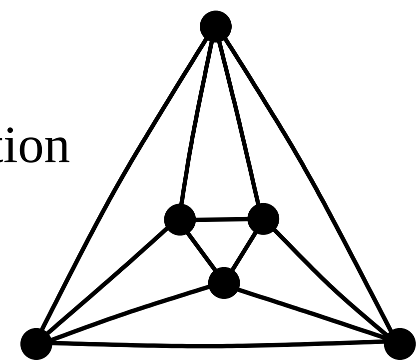


Triangulation



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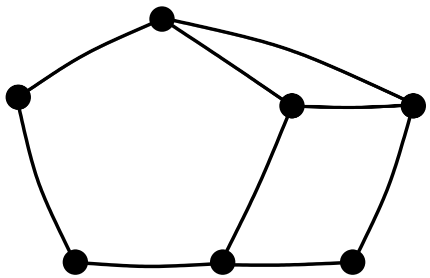

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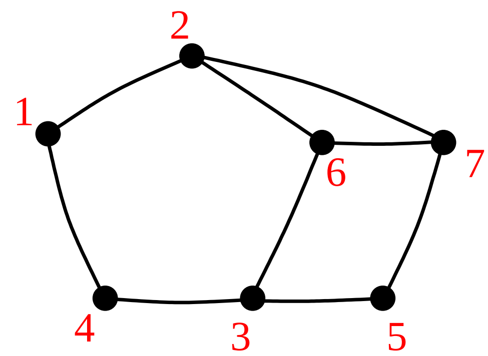

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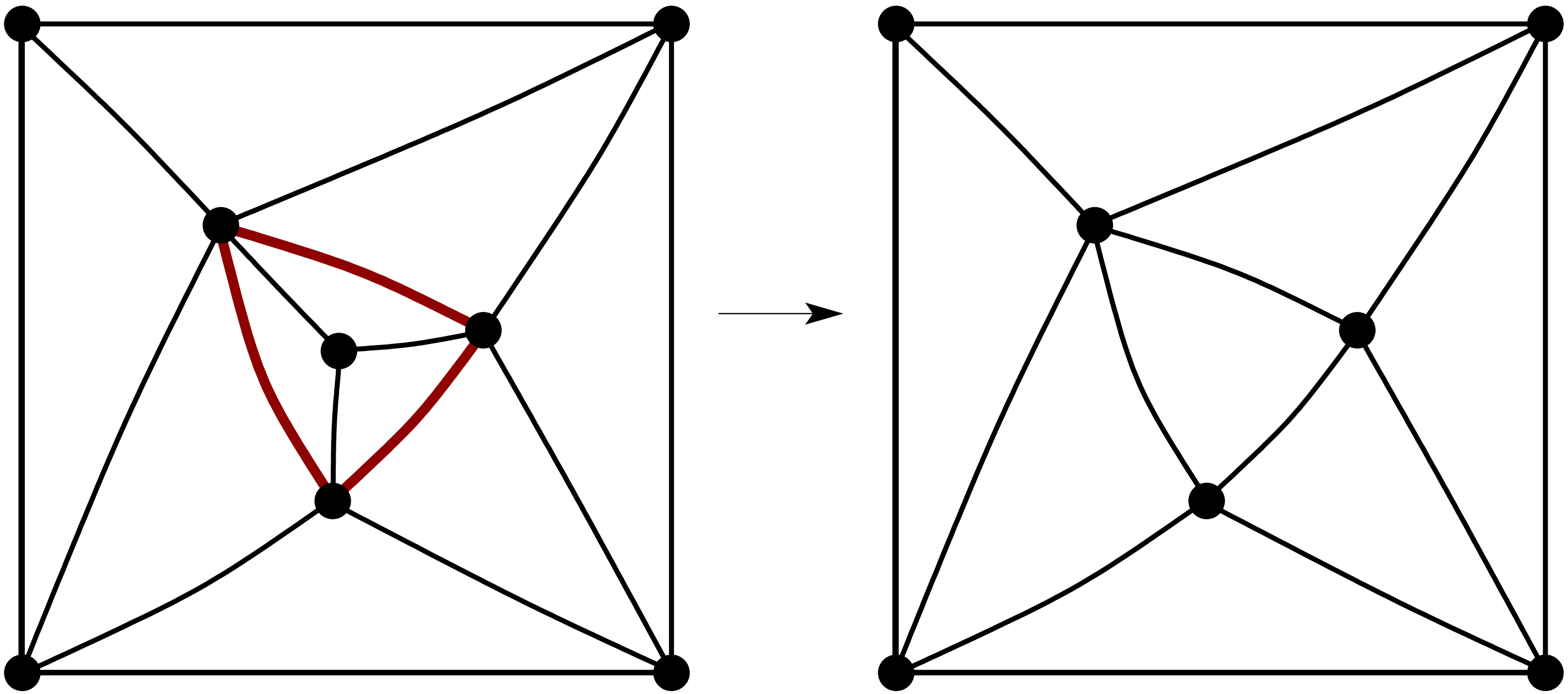

Forbidden   Irreducible

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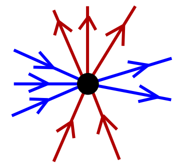

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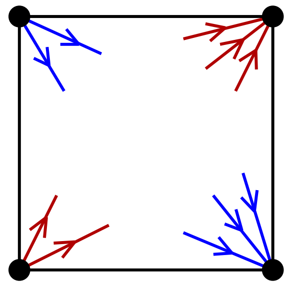
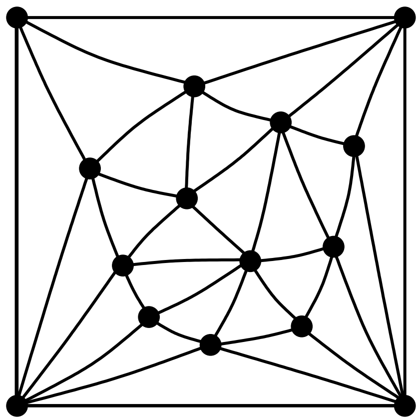
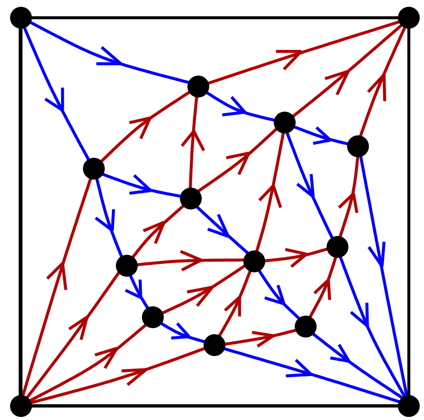


Example:



⇒



cf Regular edge 4-labelings (Kant, He)

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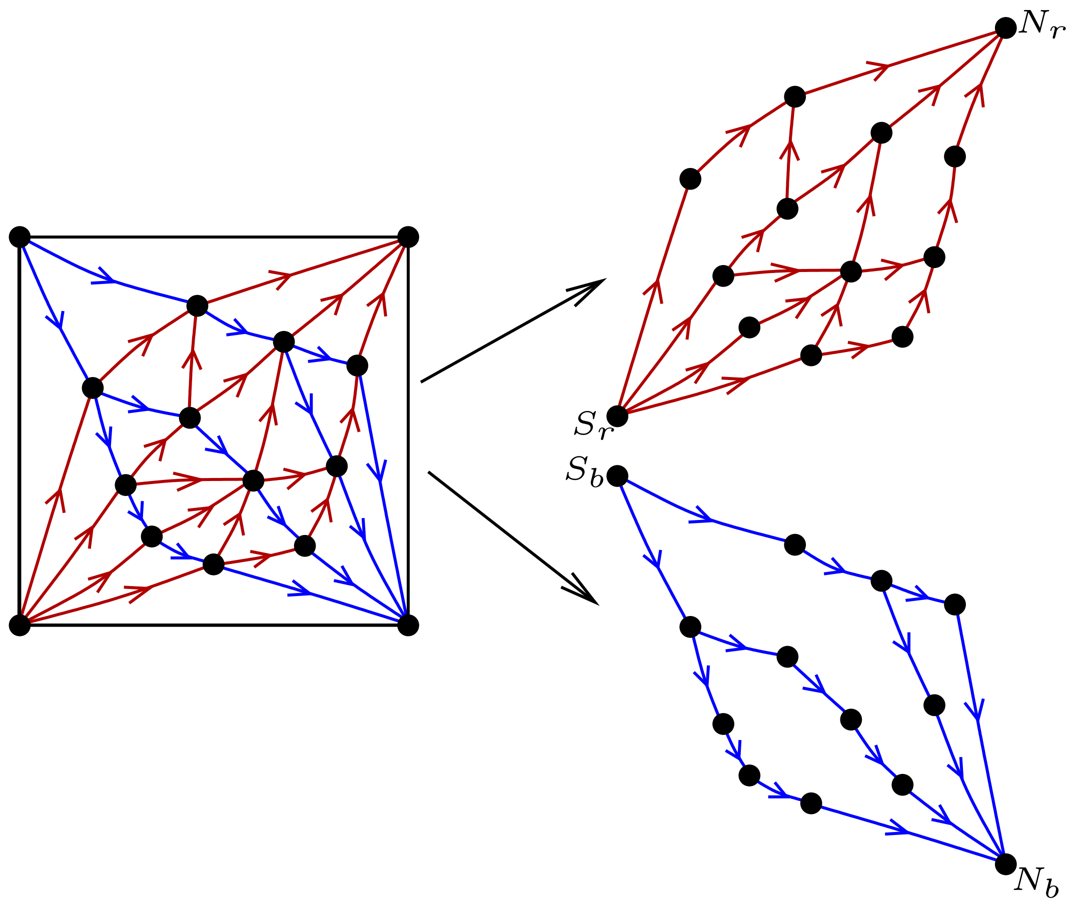
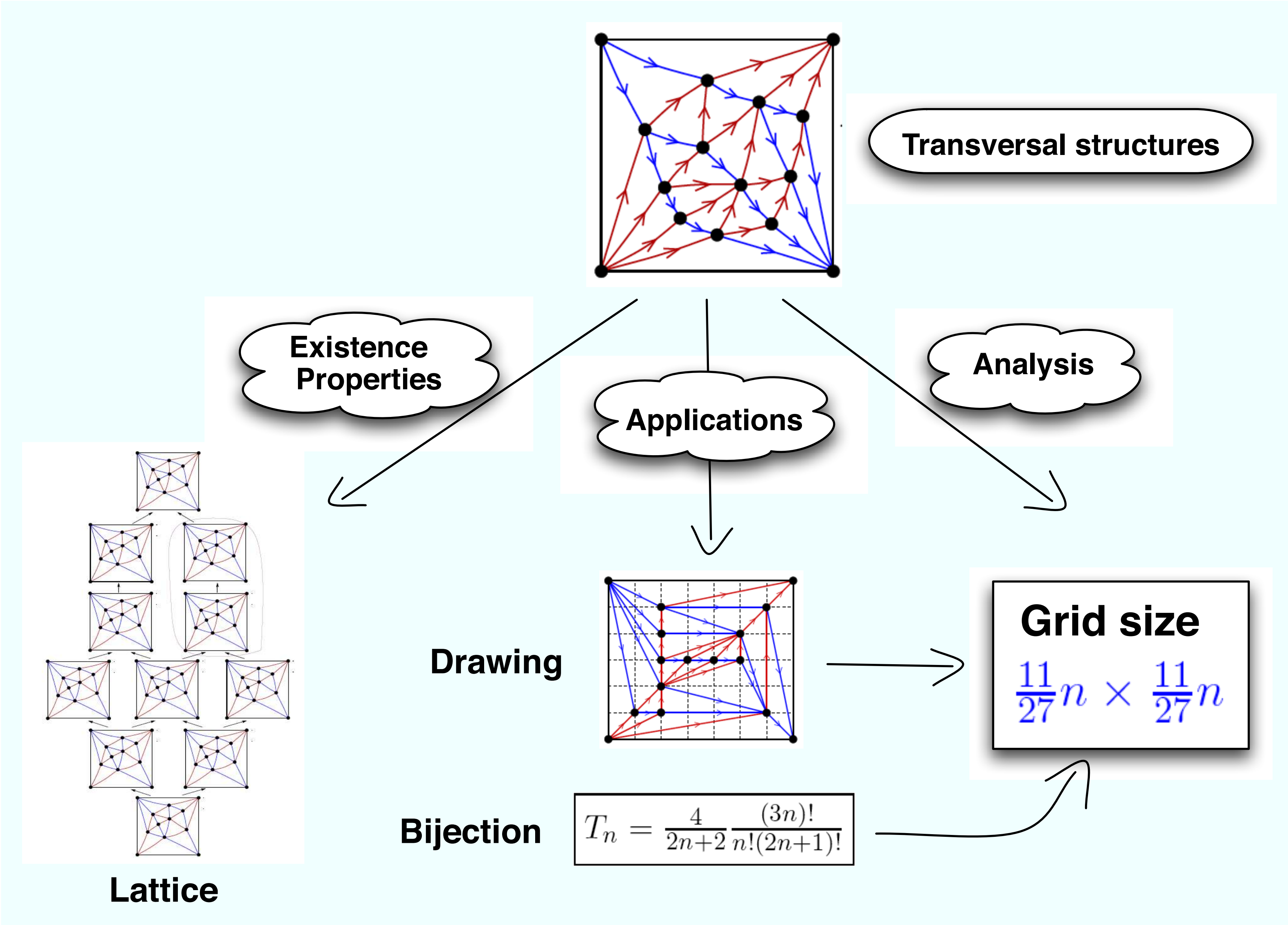


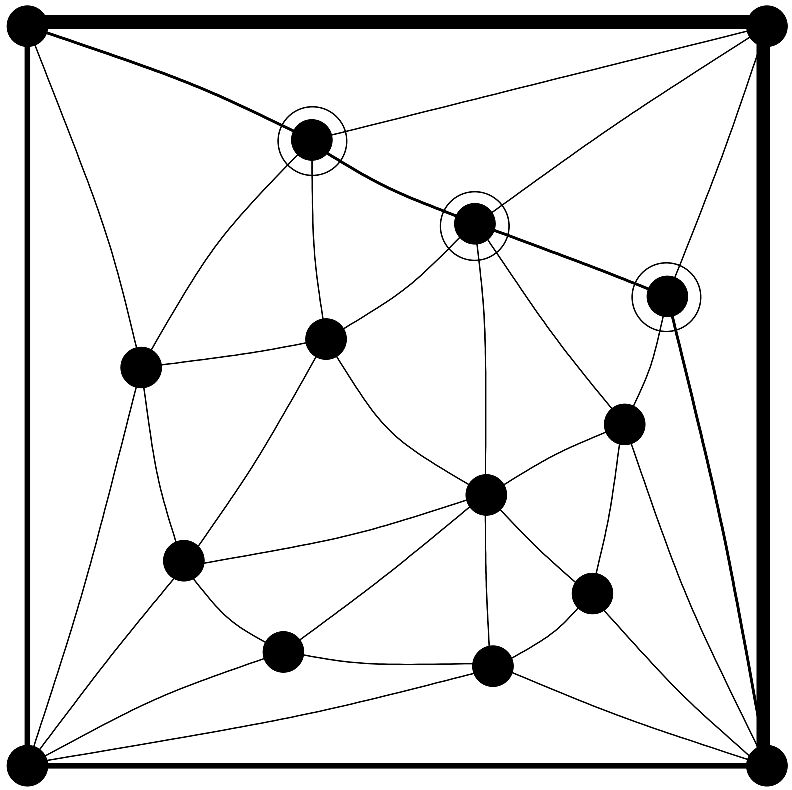
Overview


Transversal structures
Existence
Properties
Applications
Analysis
Lattice
Drawing
Grid size
$\frac{11}{27}n \times \frac{11}{27}n$
Bijection
$T_n = \frac{4}{2n+2} \frac{(3n)!}{n!(2n+1)!}$

Definition and properties of transversal structures on triangulations

Existence of transversal structures

- For each triangulation, there exists a transversal structure (Kant, He 1997)
- There exists linear time iterative algorithms to compute transversal structures



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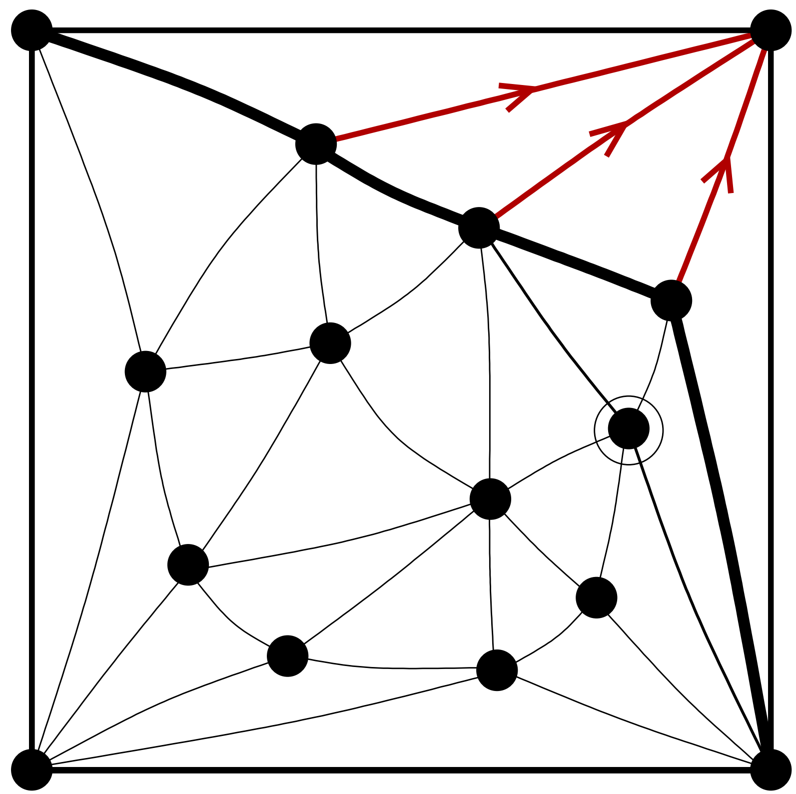

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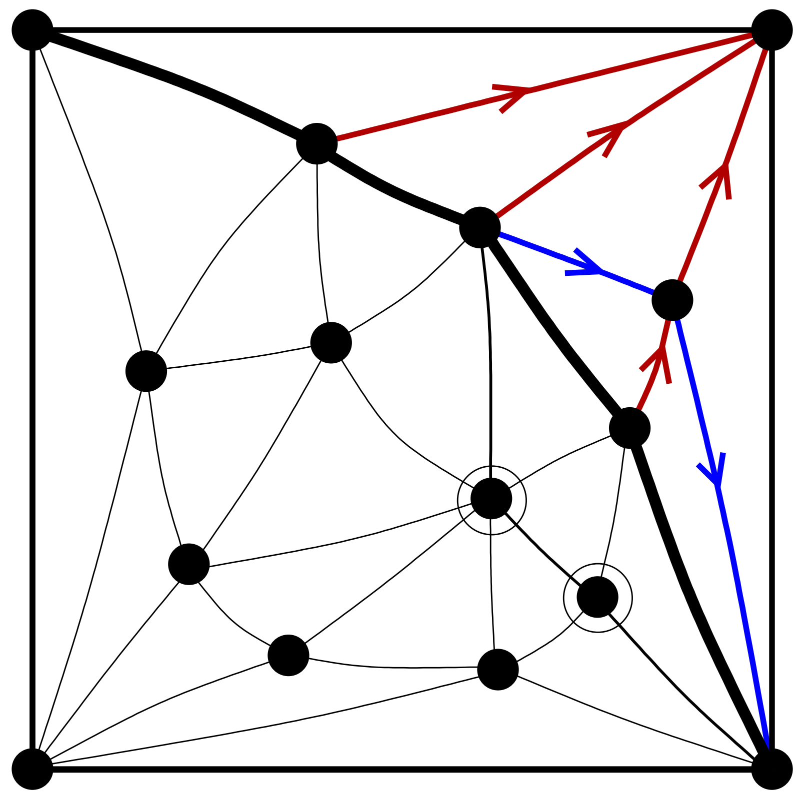

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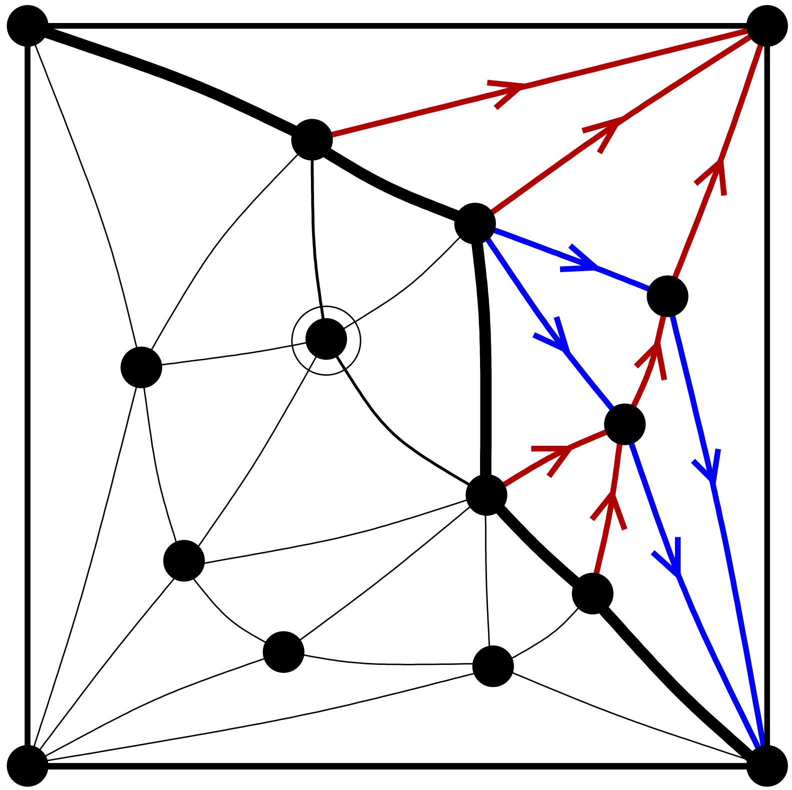

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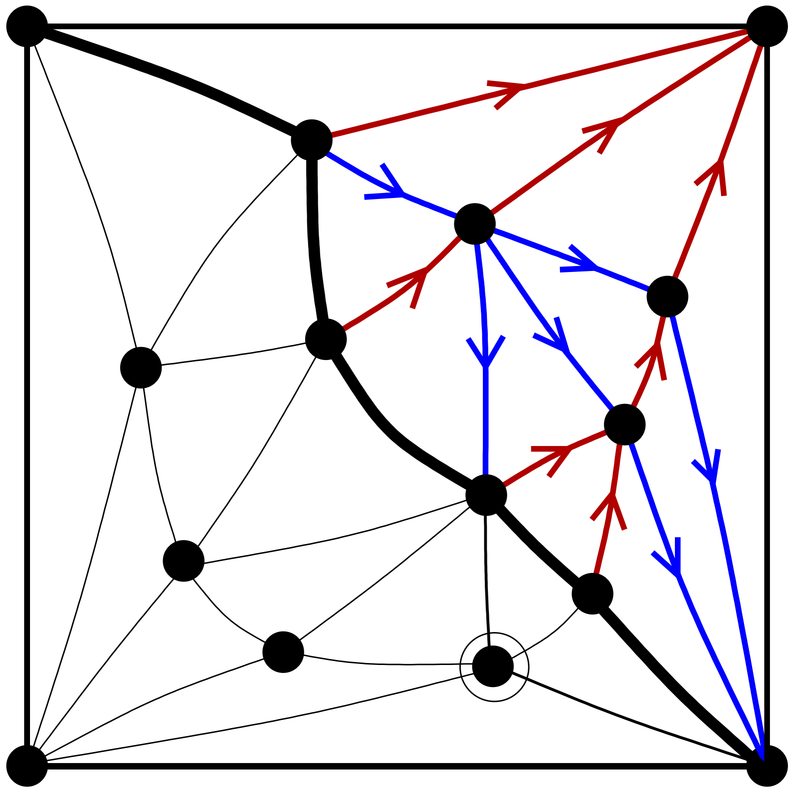

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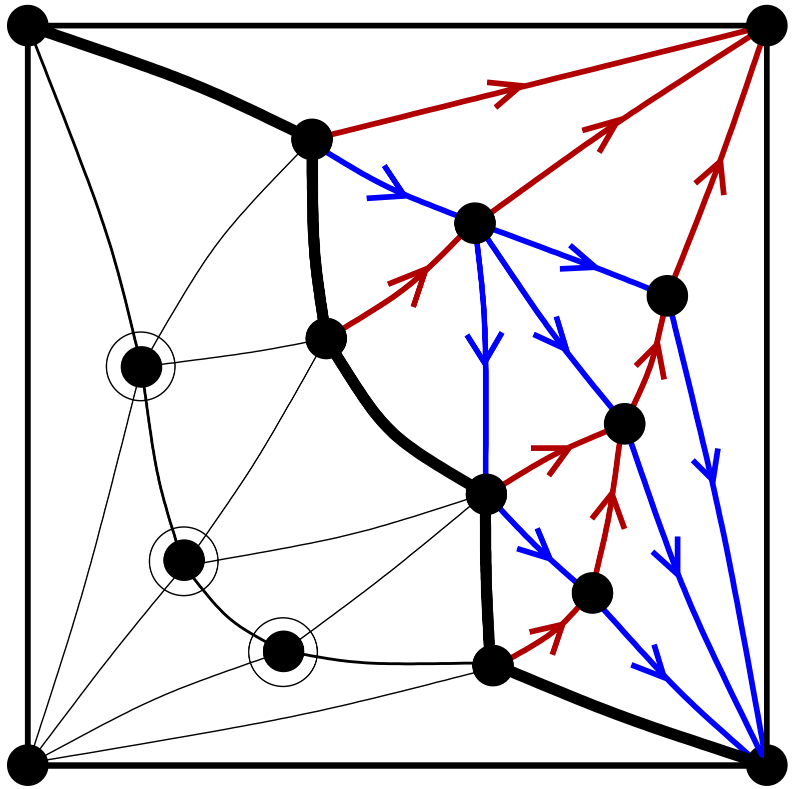

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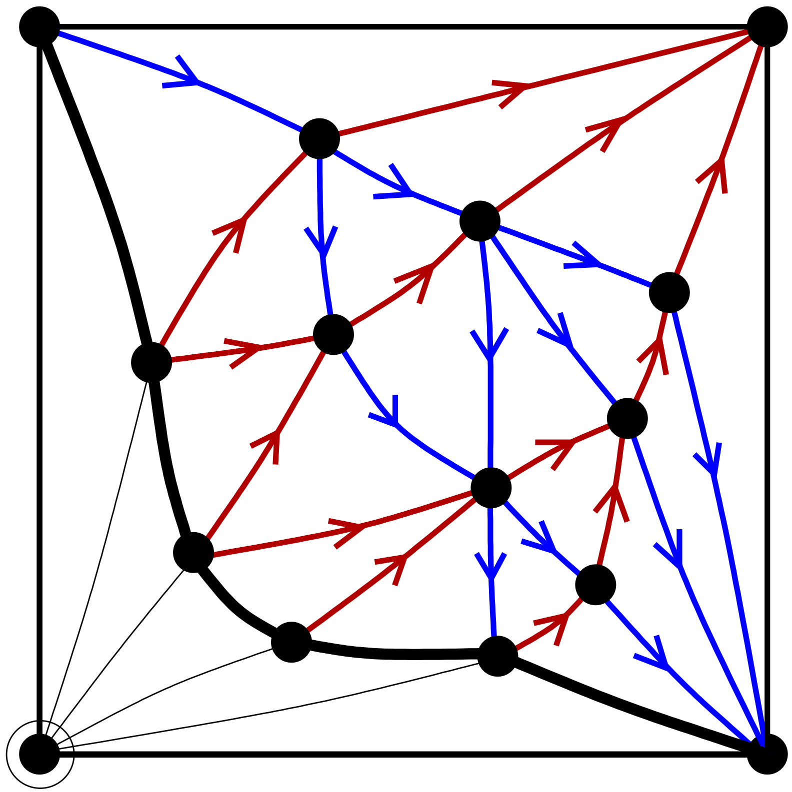

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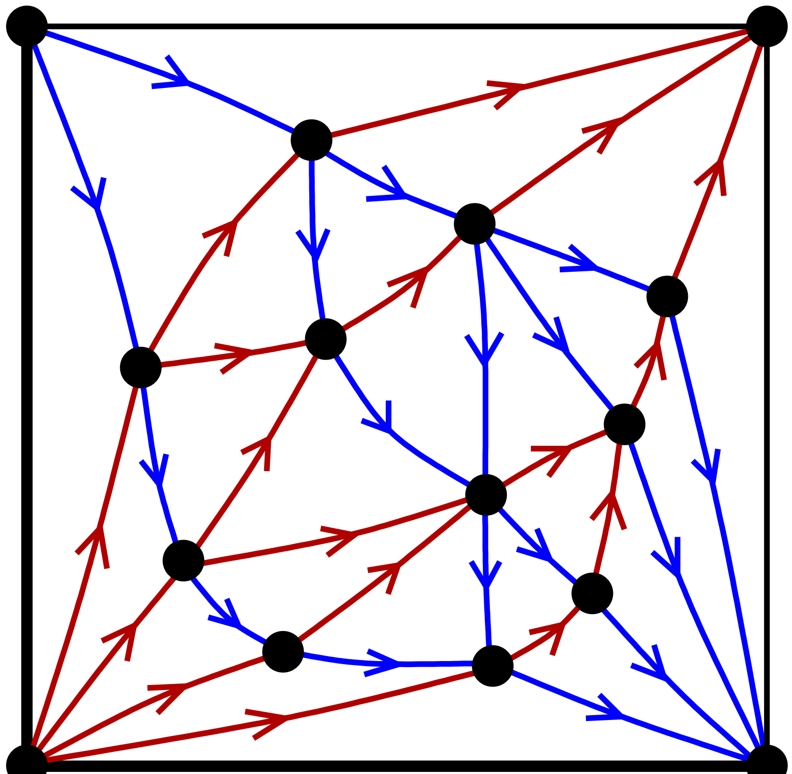
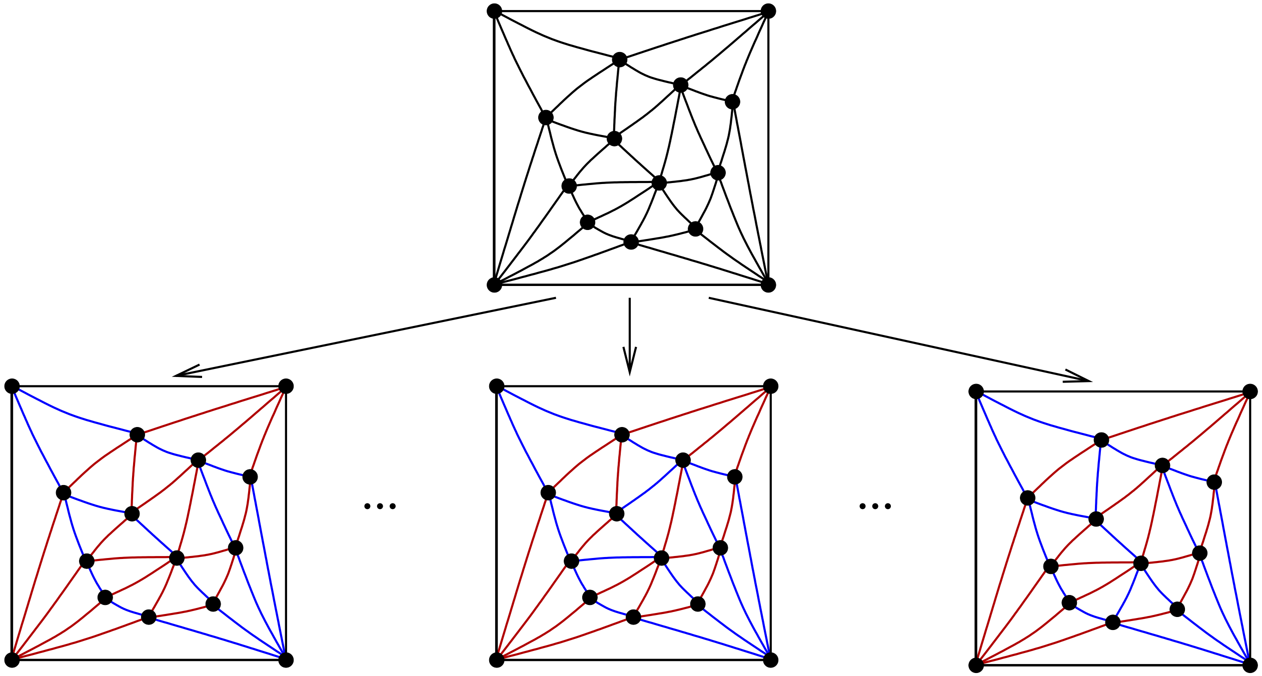


The structure of transversal structures

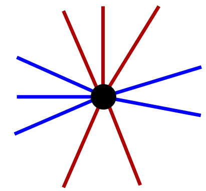
- For each triangulation T, such transversal structures are not unique
- Let X_T be the set of transversal bicolorations of T
- What is the structure of X_T ?



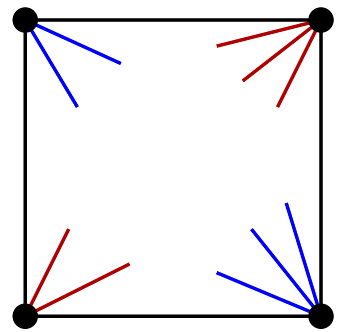
Without orientations

The orientation of edges are not necessary.
The local conditions can be defined just with the bicoloration:

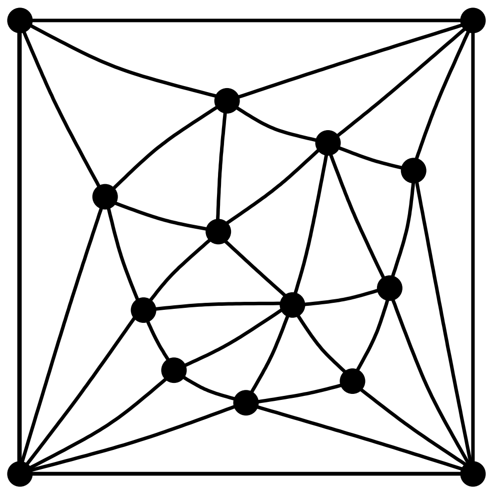
Inner vertex ⇒



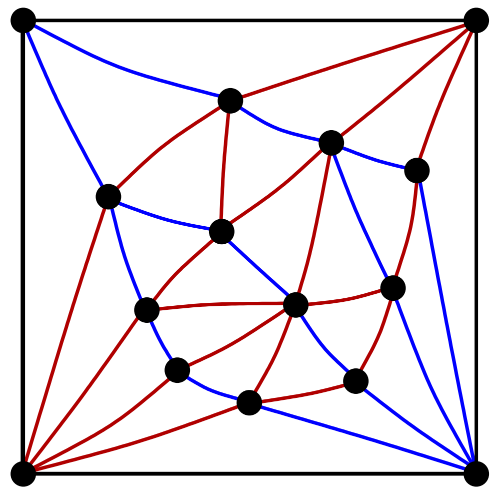
Border vertices ⇒



Example:

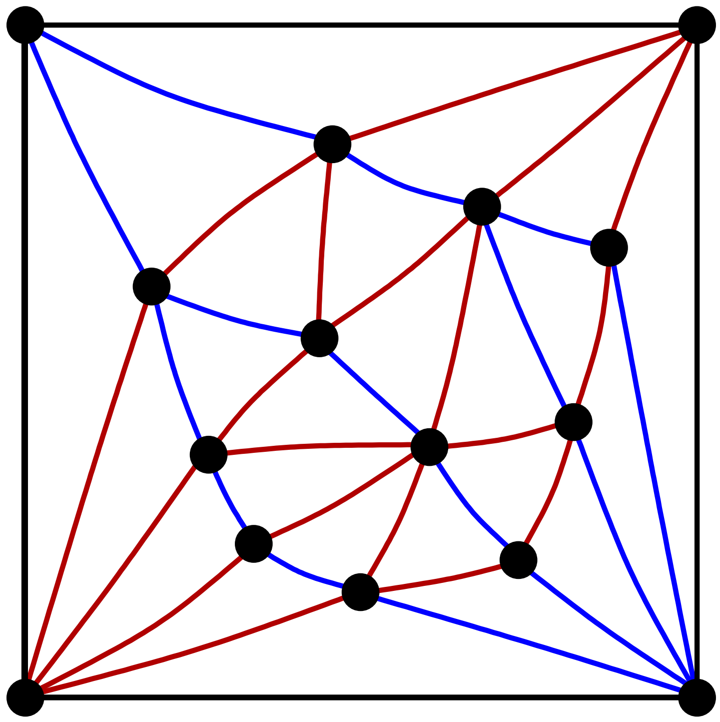


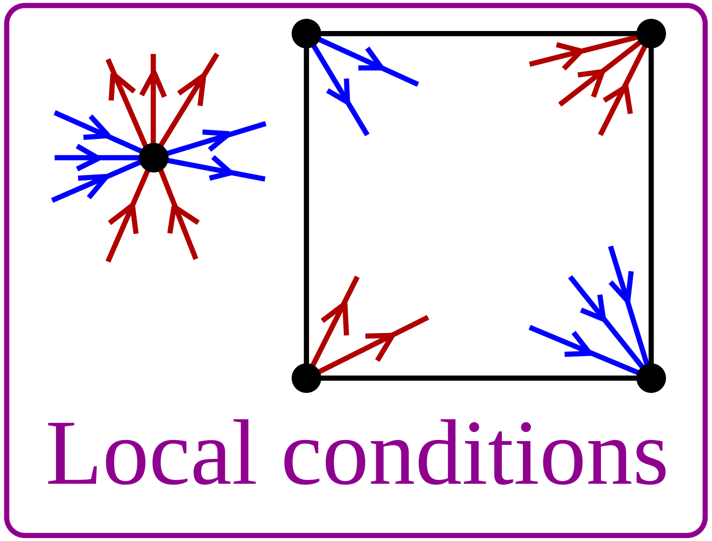
⇒



The two definitions are equivalent

The orientations of edges can be recovered in an unique way
$\Rightarrow$ bijection between the two structures





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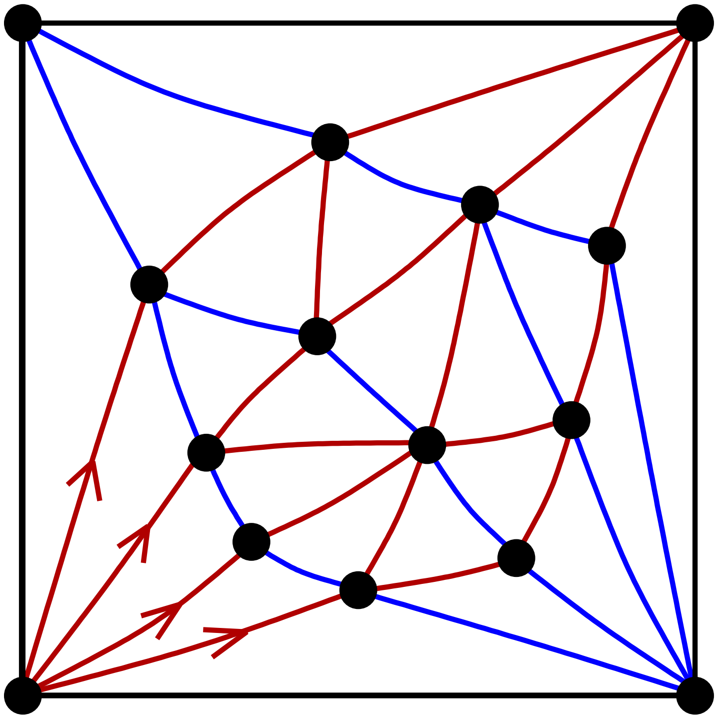

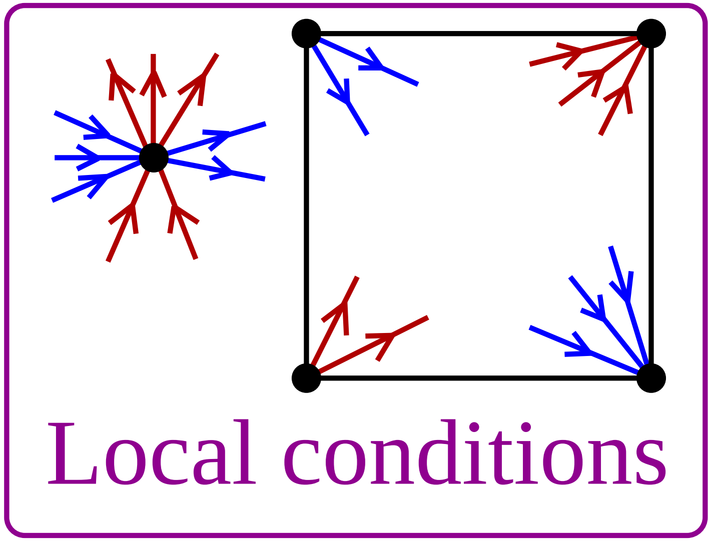

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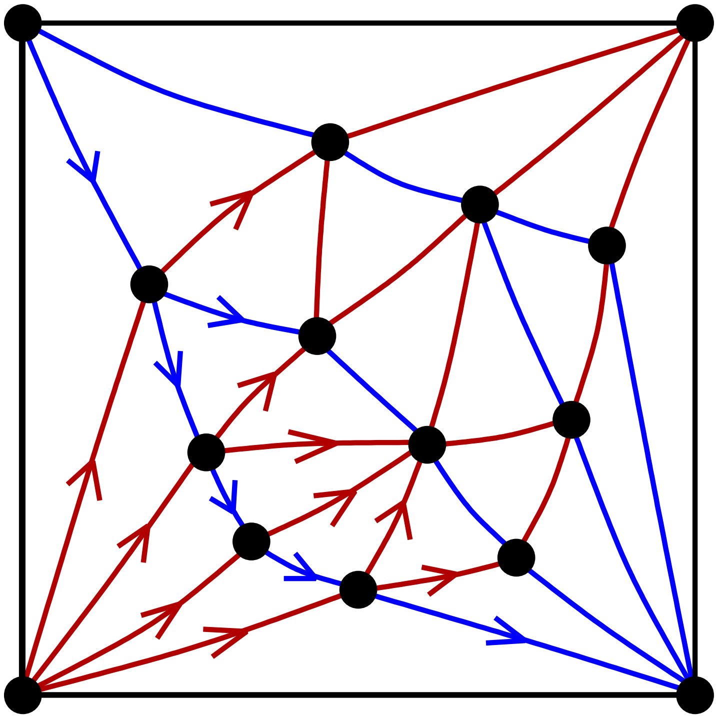

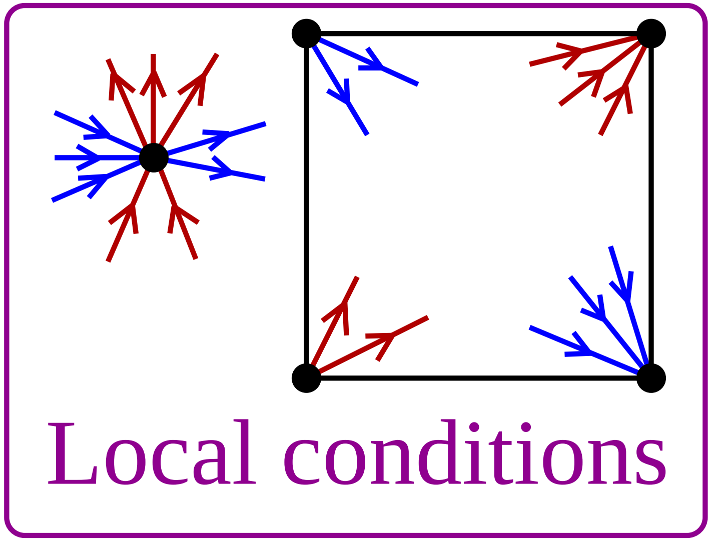

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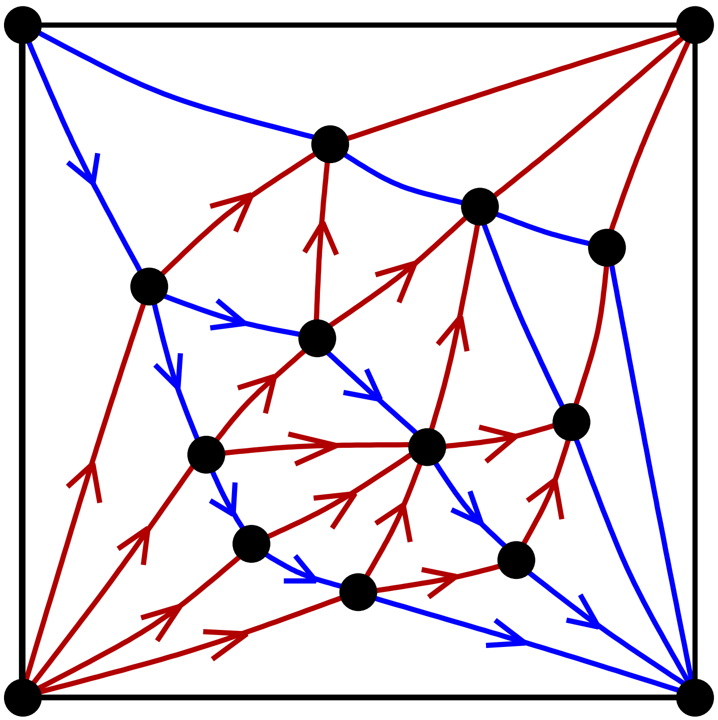

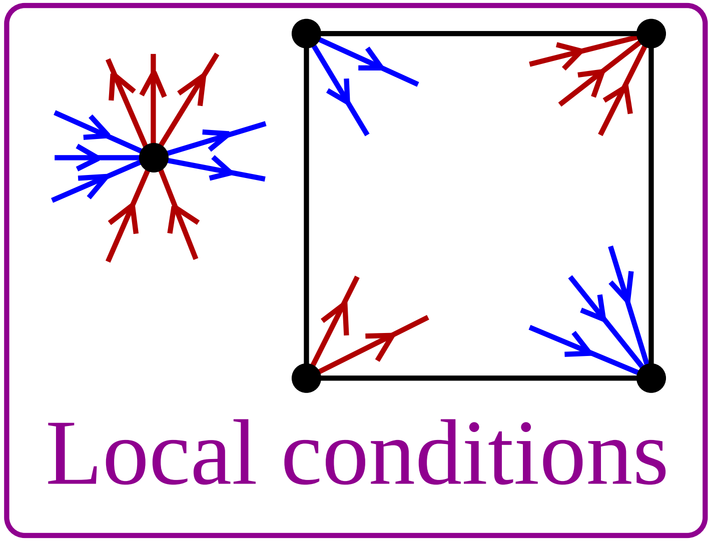

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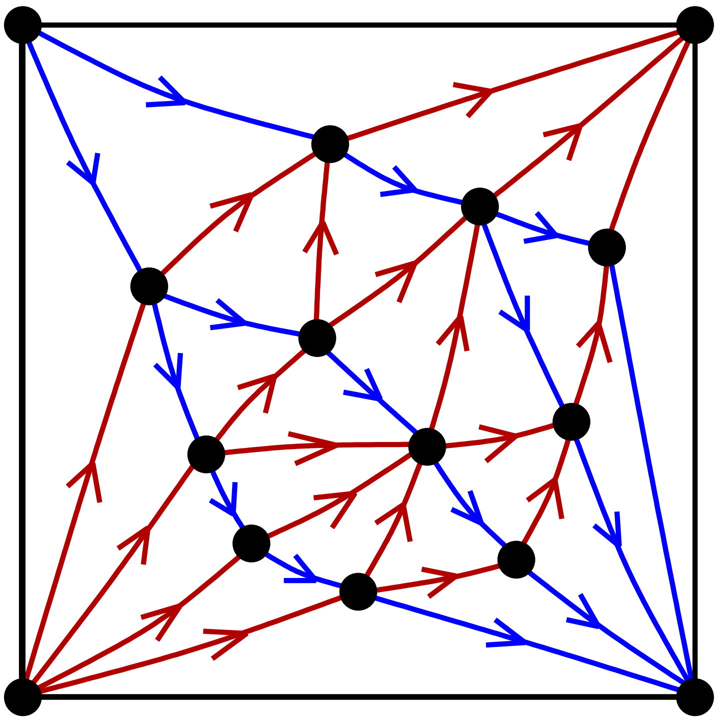

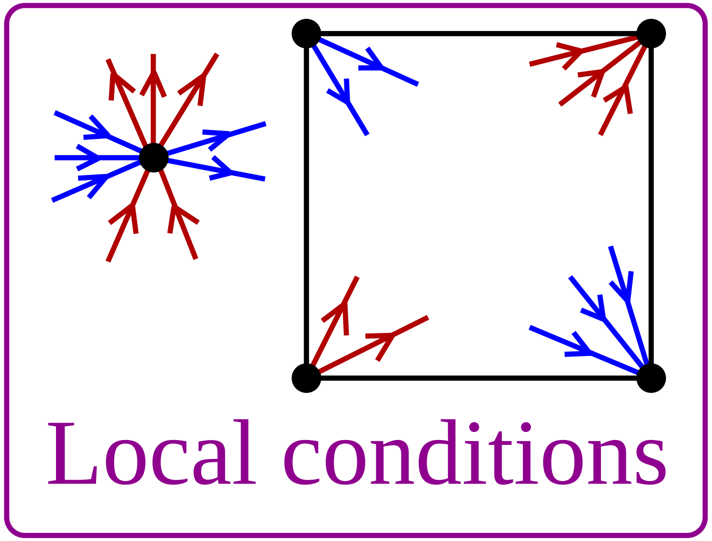

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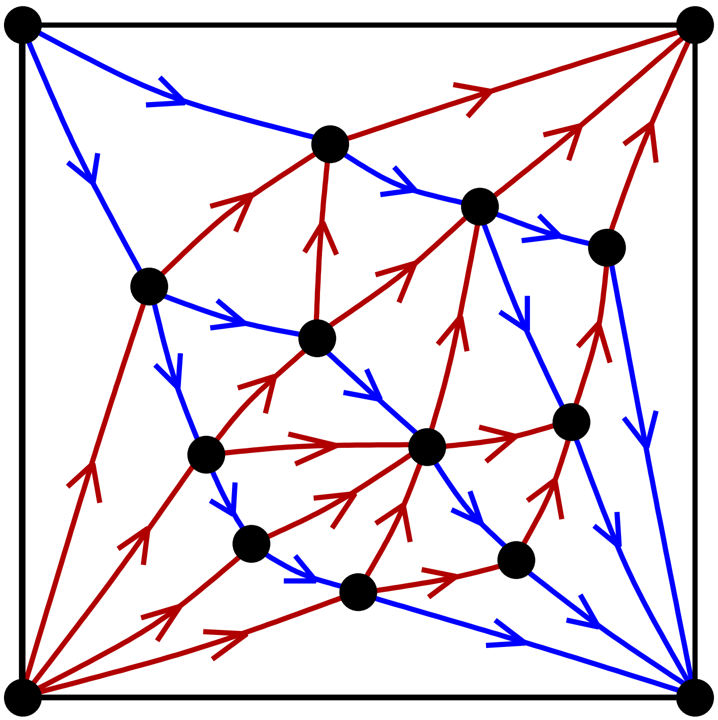

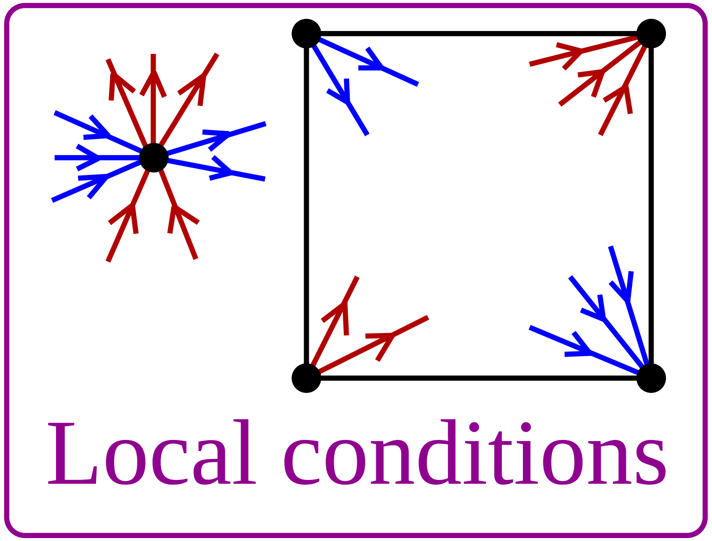

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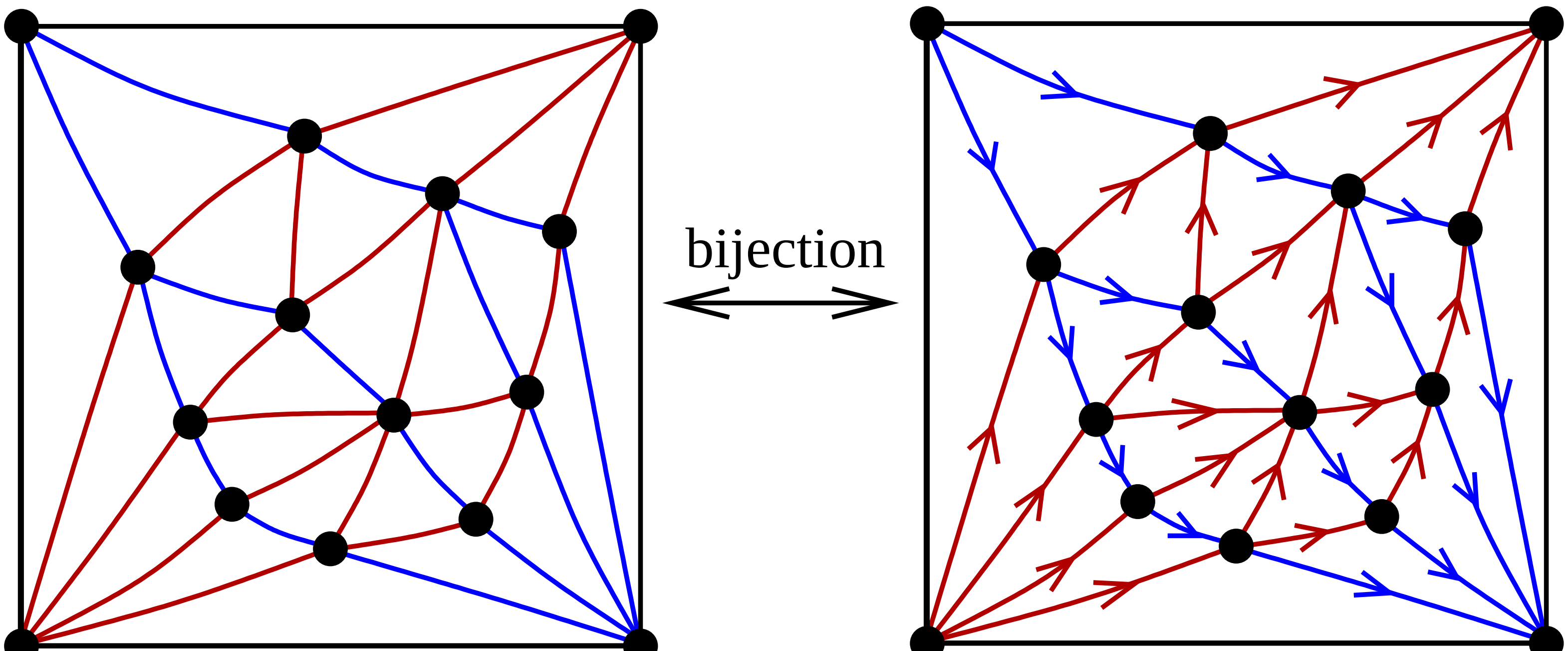
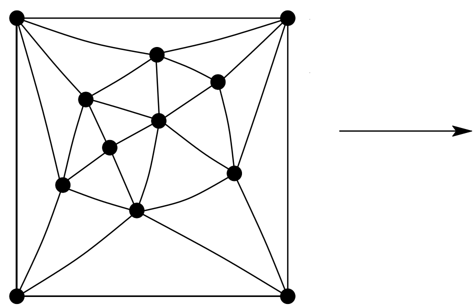
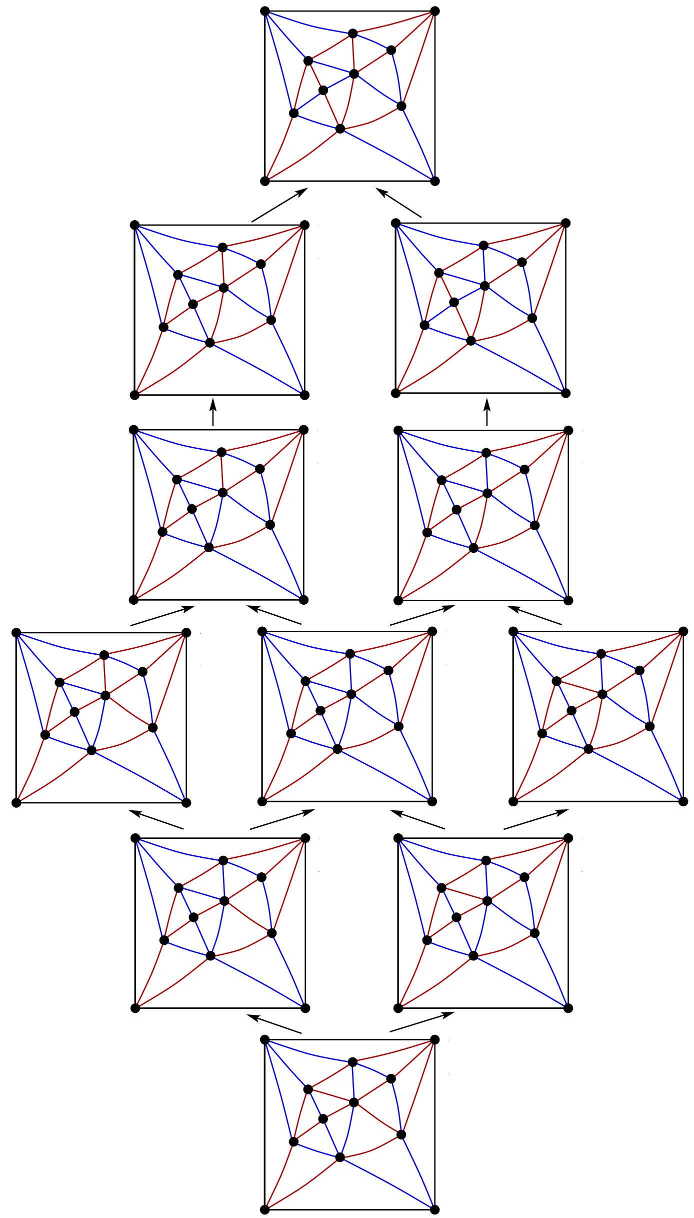
$\Rightarrow$ bijection between the two structures



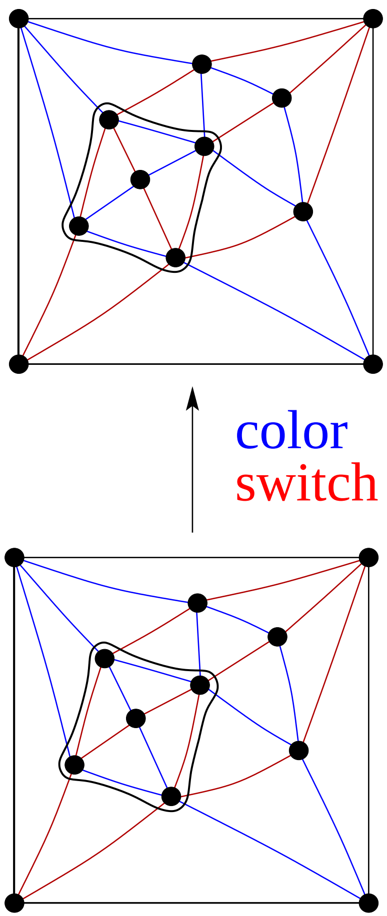
The set X_T is a distributive lattice

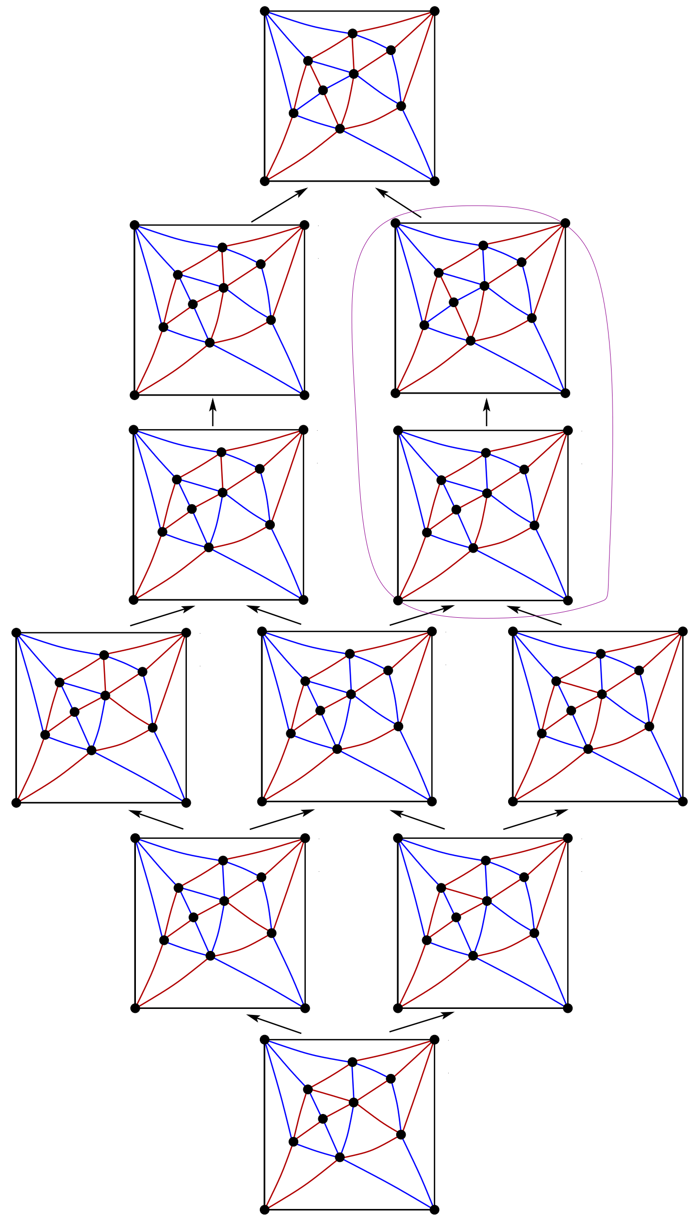
T=





The set X_T is a distributive lattice

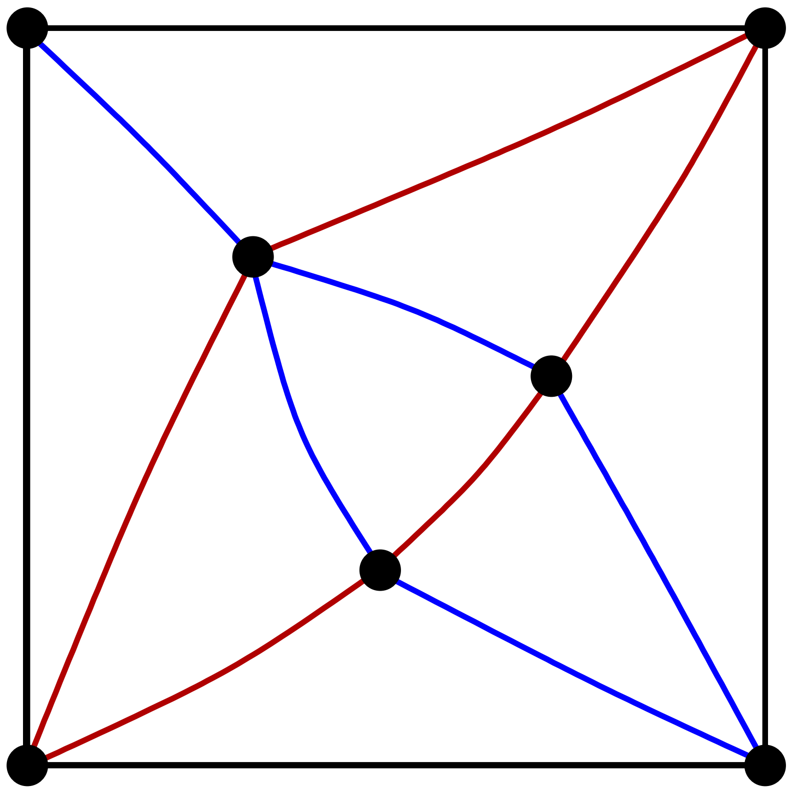




Angular graph of T

We associate to T an **angular graph** $Q(T)$:

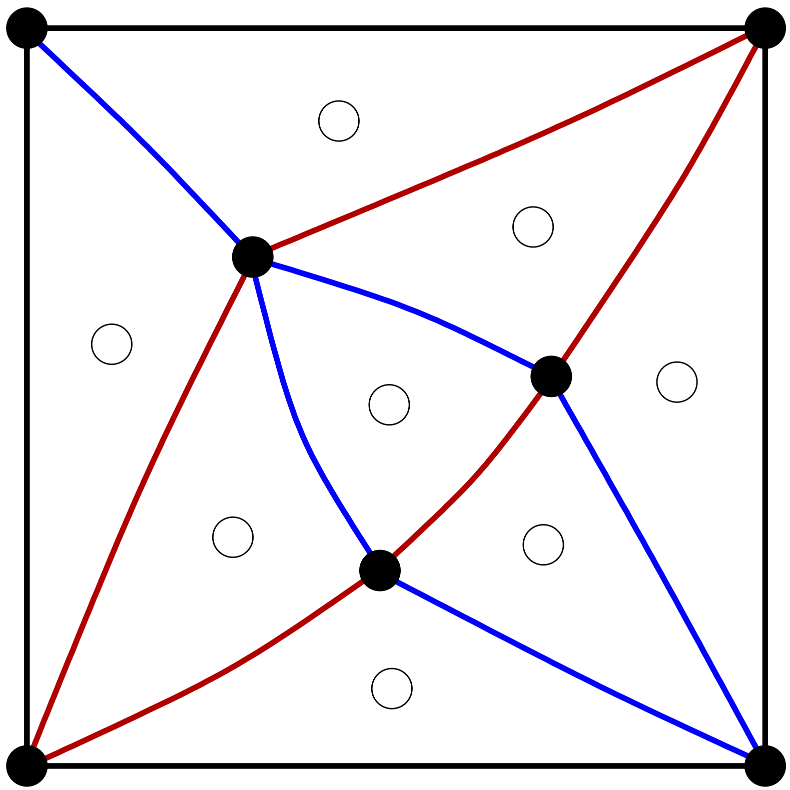
- The **black vertices** of $Q(T)$ are the vertices of T



Angular graph of T

We associate to T an **angular graph** $Q(T)$:

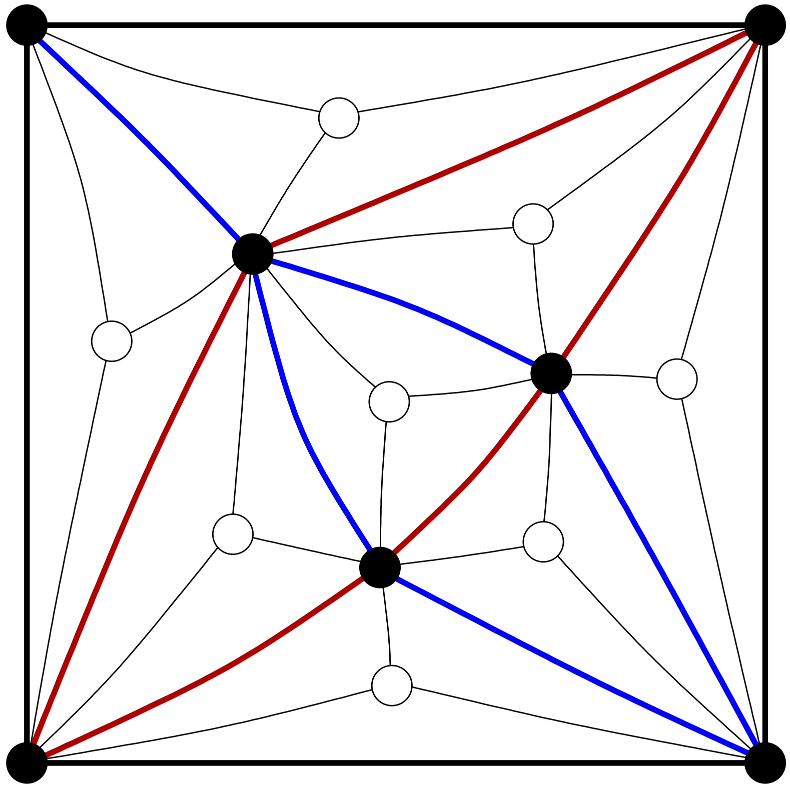
- There is a **white vertex** of $Q(T)$ in each face of T



Angular graph of T

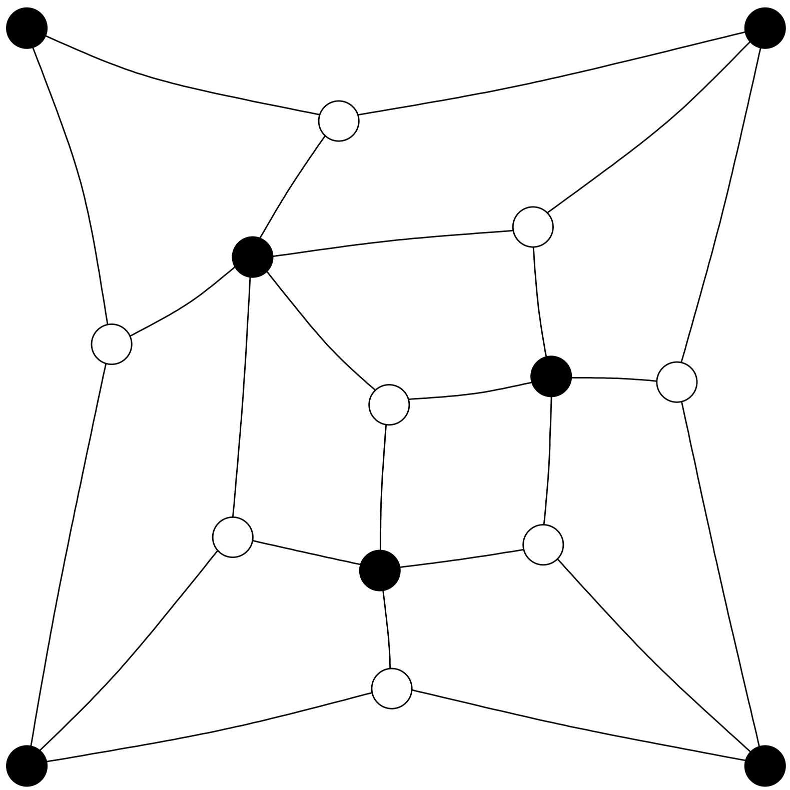
We associate to T an **angular graph** $Q(T)$:

- To each angle of T corresponds an **edge** of $Q(T)$



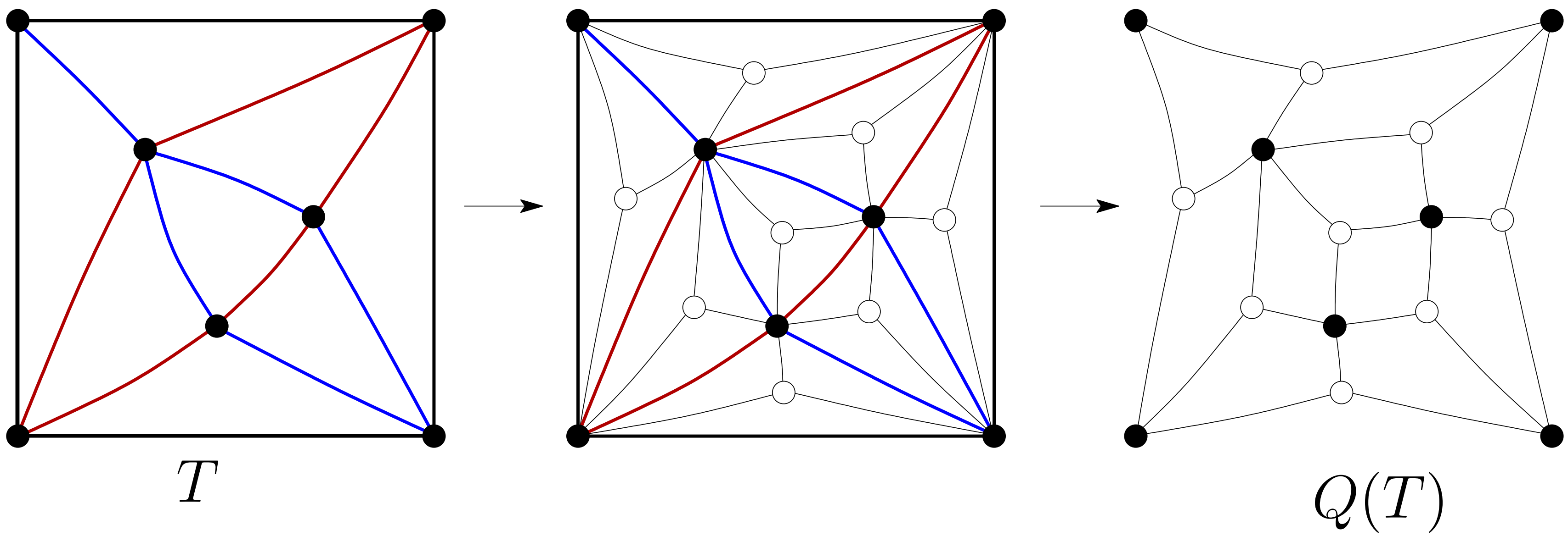
Angular graph of T

We associate to T an **angular graph** $Q(T)$:

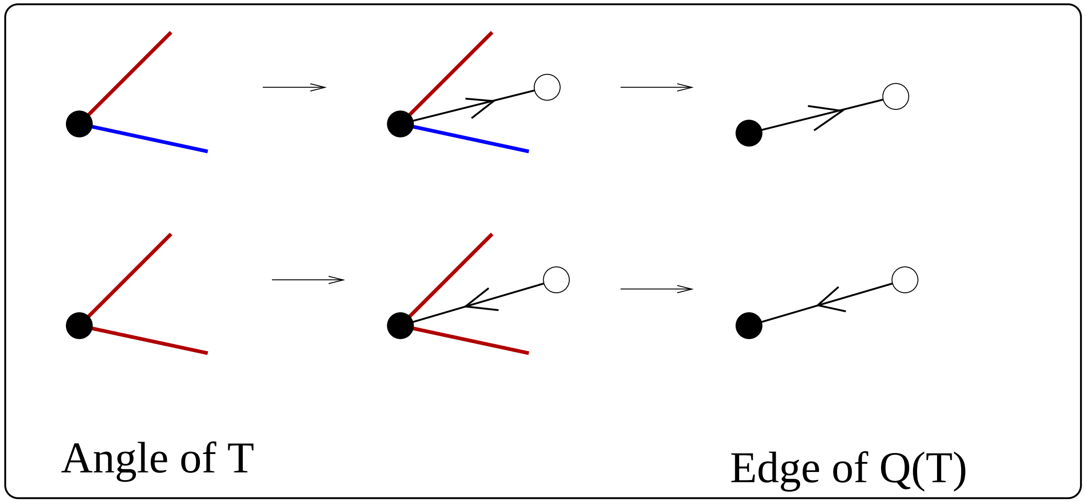


Angular graph of T

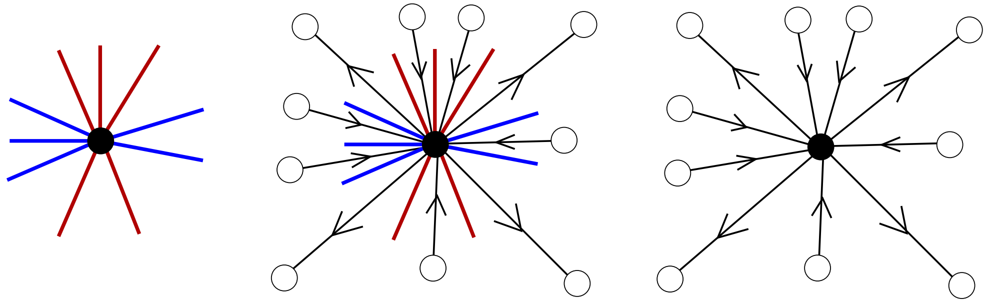
We associate to T an **angular graph** $Q(T)$:



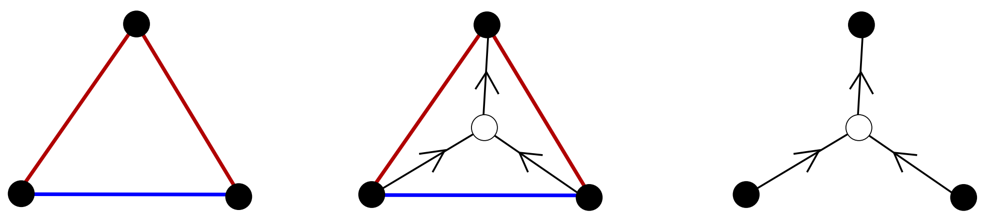
Induced orientation on $Q(T)$



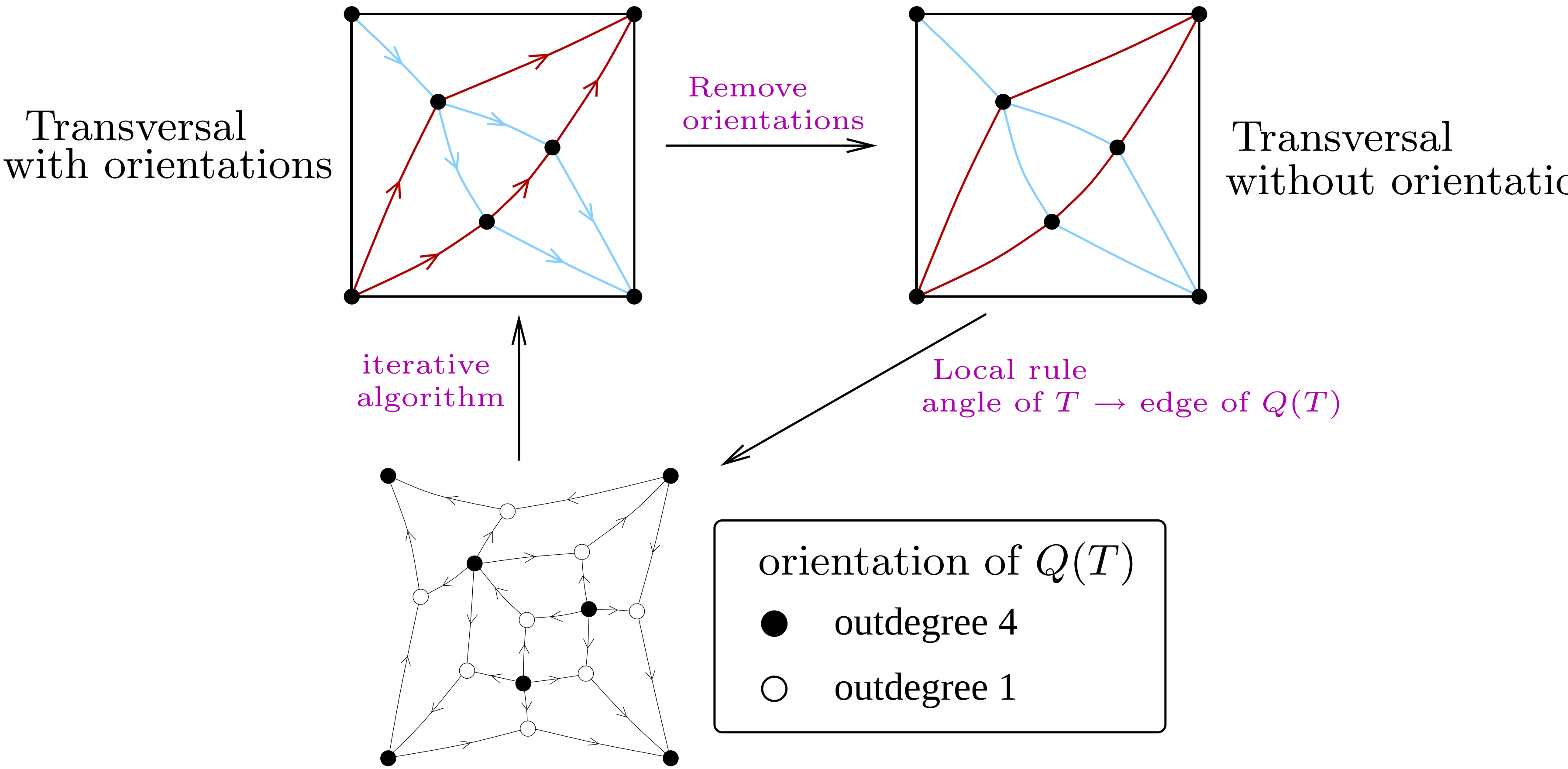
Vertex−condition:



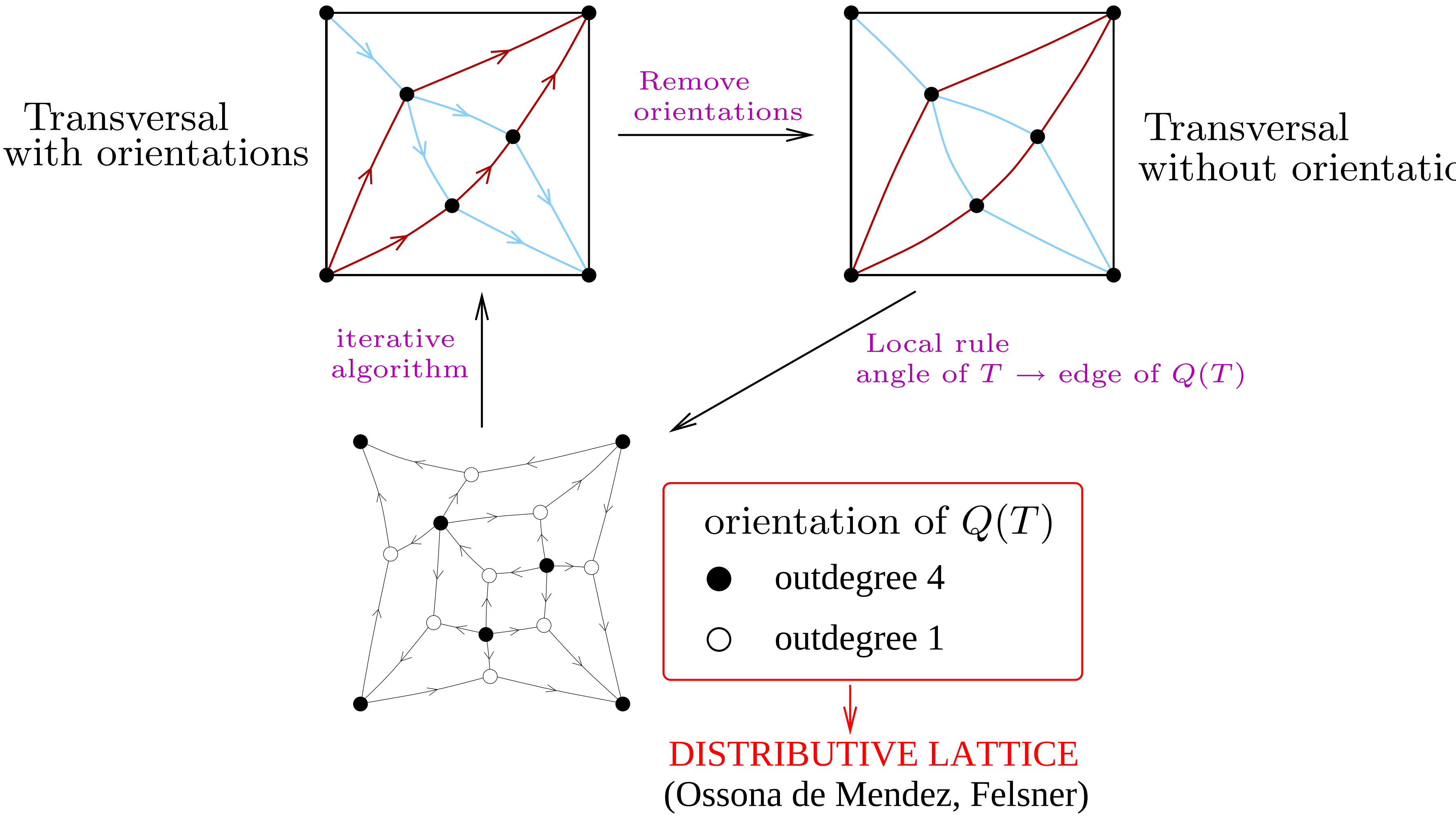
Face−condition:



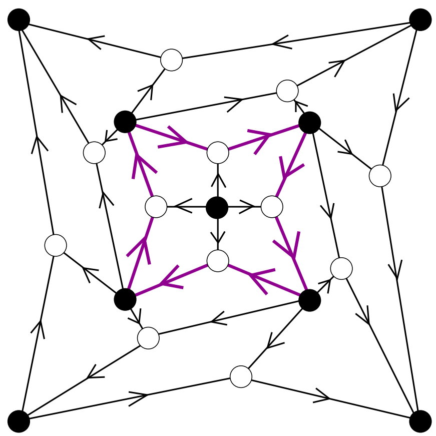
Bijective correspondences



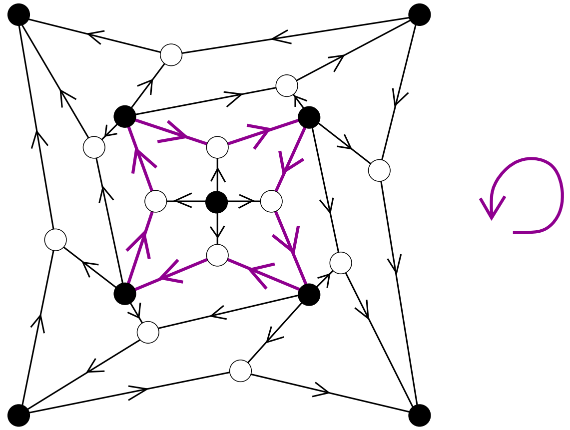
Bijective correspondences



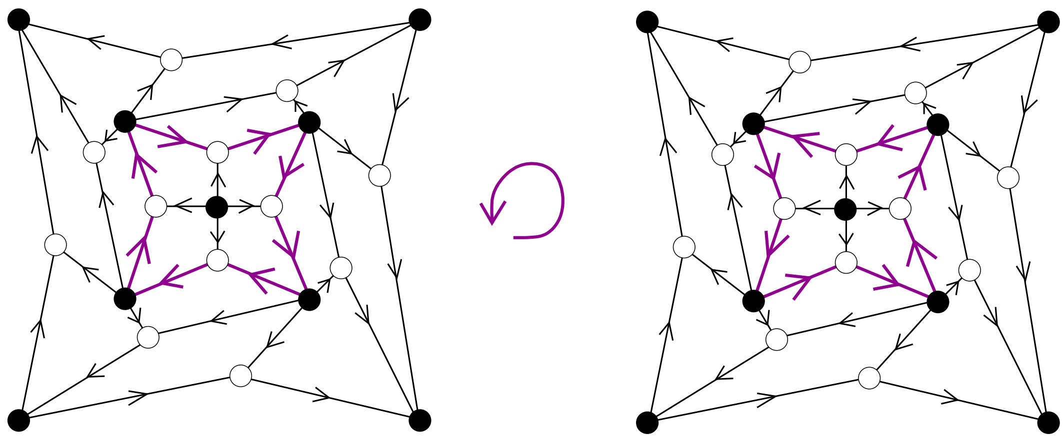
Flip on $Q(T)$ and then on T



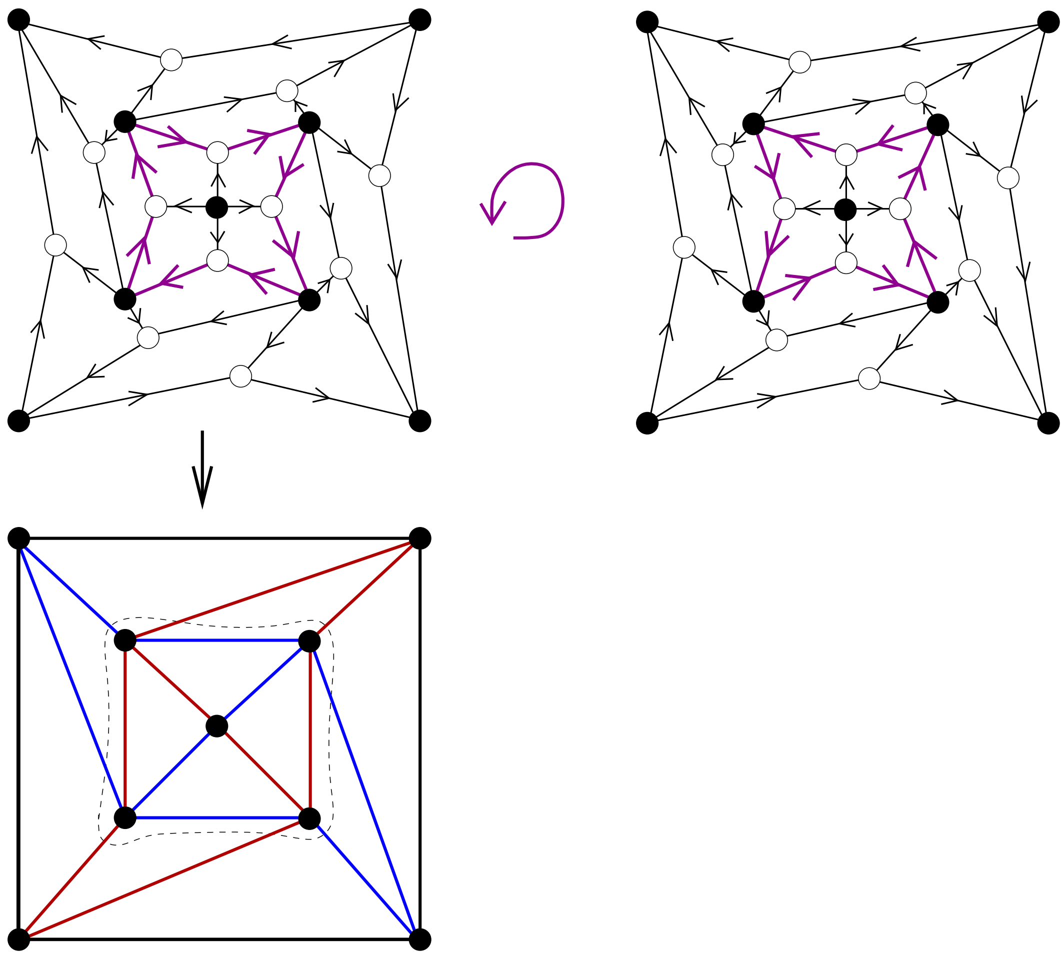
Flip on $Q(T)$ and then on T



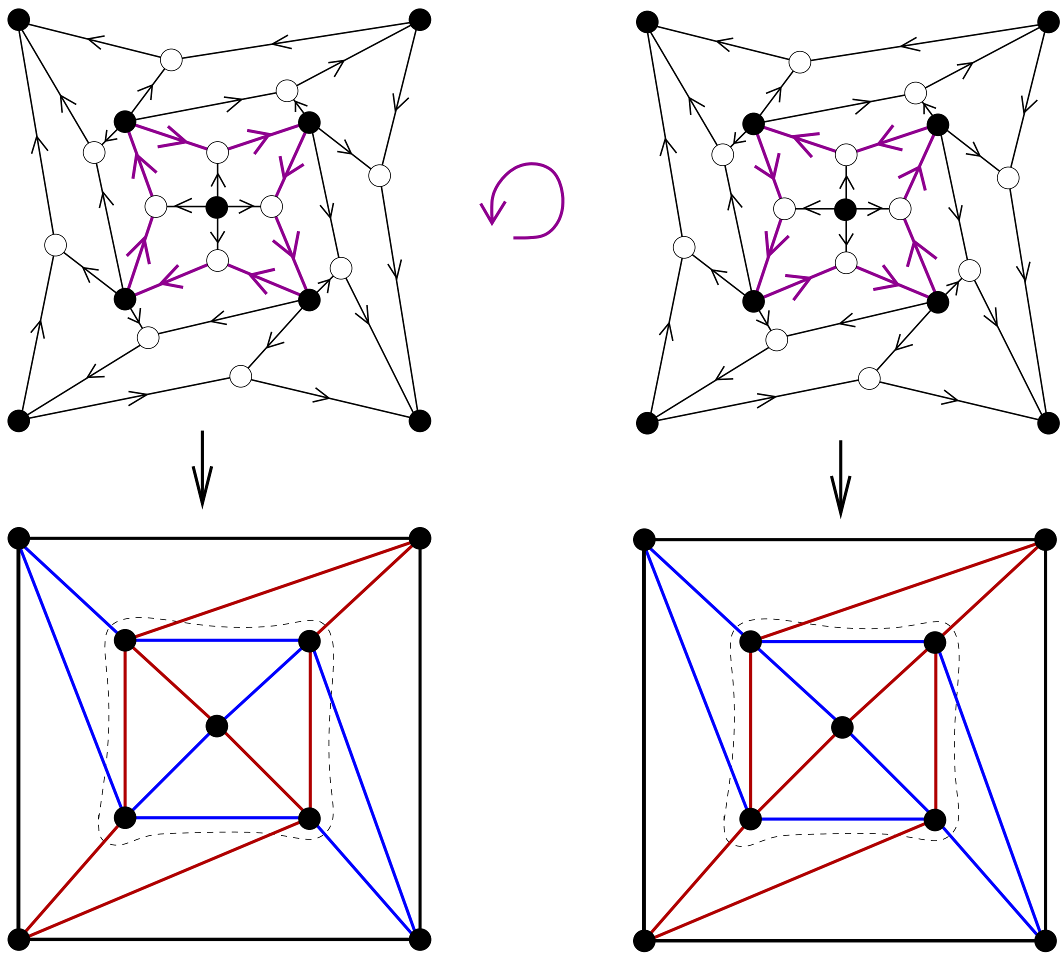
Flip on $Q(T)$ and then on T



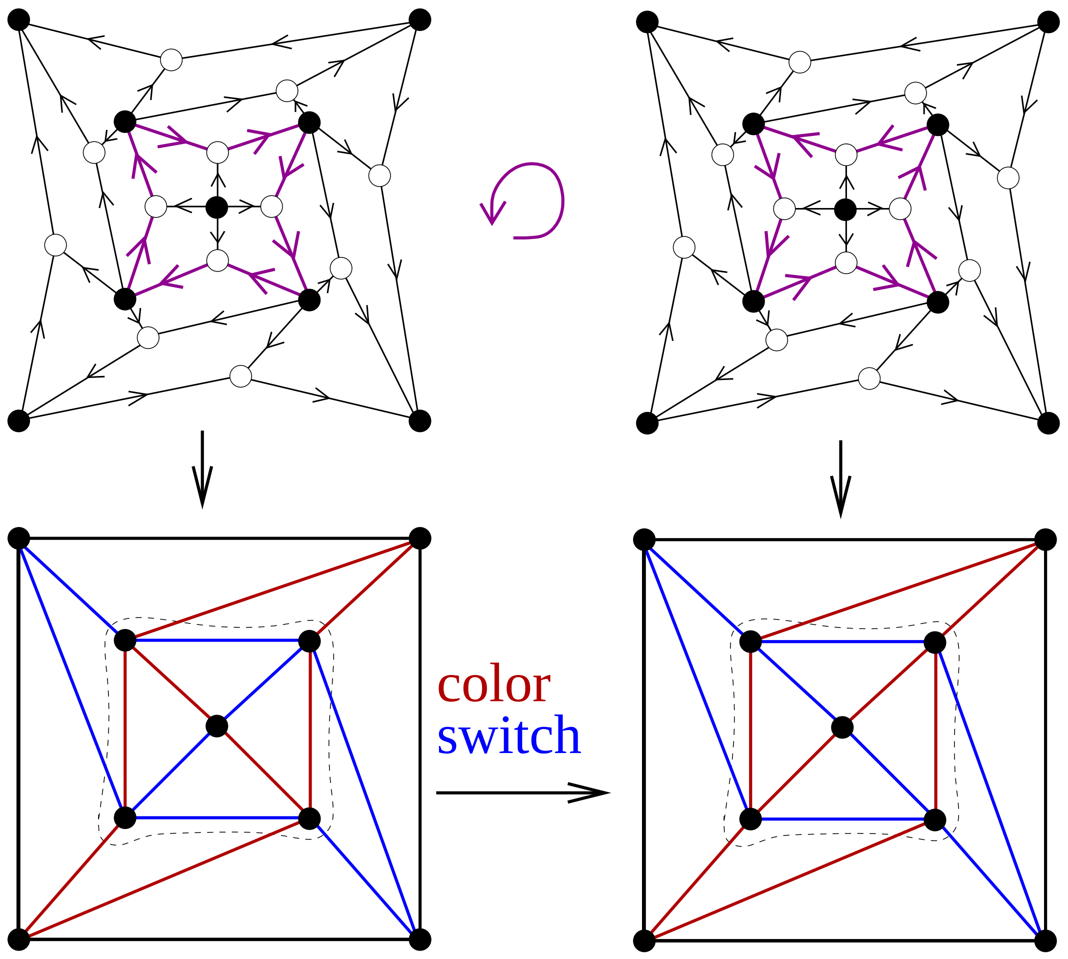
Flip on $Q(T)$ and then on T



Flip on $Q(T)$ and then on T

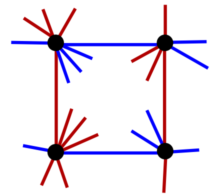


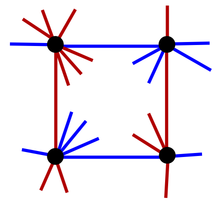
Flip on $Q(T)$ and then on T



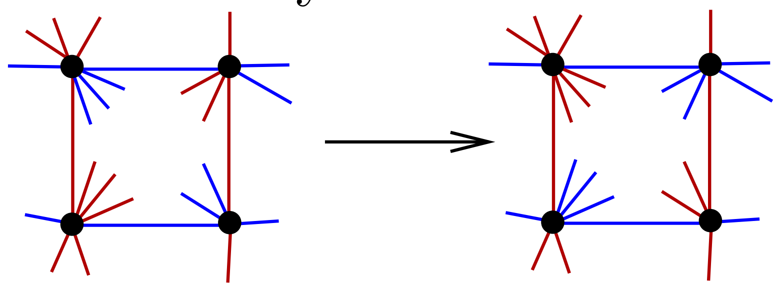
The set X_T is a distributive lattice

We distinguish:

left alternating 4-cycles 

right alternating 4-cycles 

Flip operation: switch colors inside a right alternatin
4-cycle

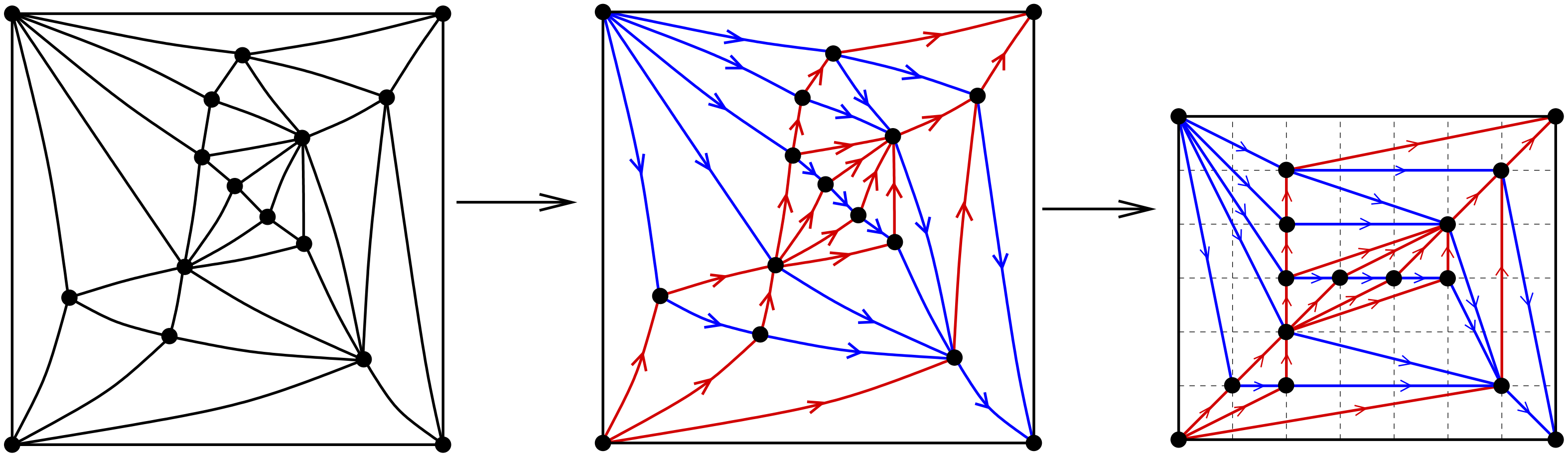


The unique transversal bicoloration of T without right alternating 4-cycle is said minimal

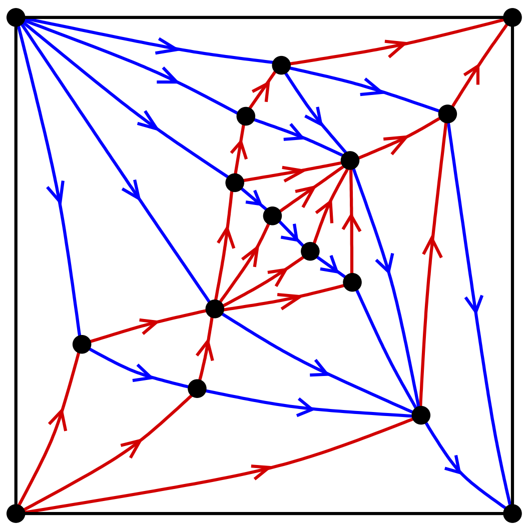
Straight-line drawing algorithm from the transversal structures

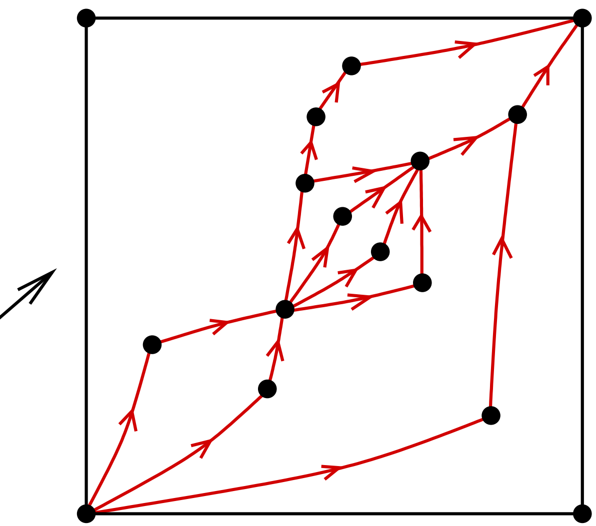
Application to graph drawing

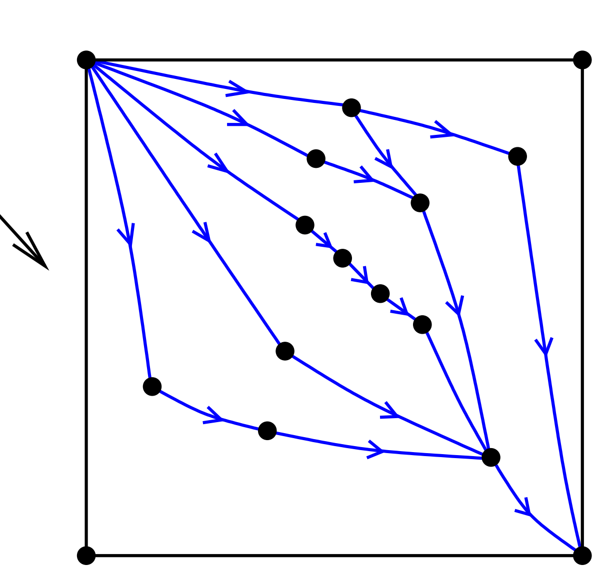
The transversal structure can be used to produce a planar drawing on a regular grid



The red map and the blue map of T





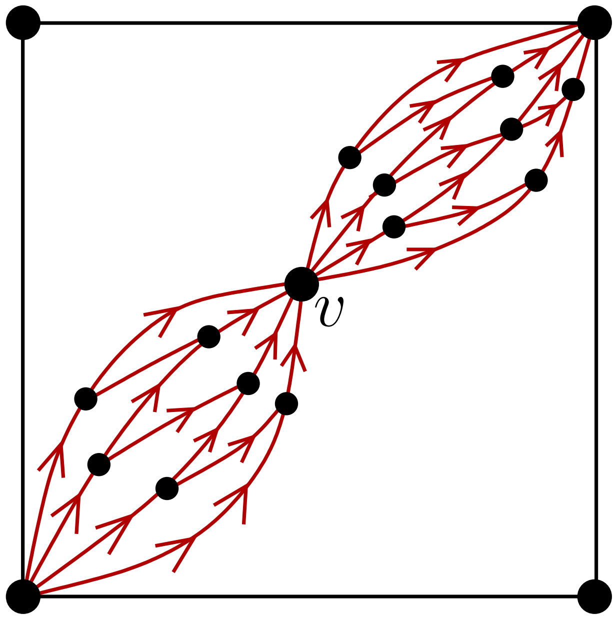


The red map gives abscissas (1)

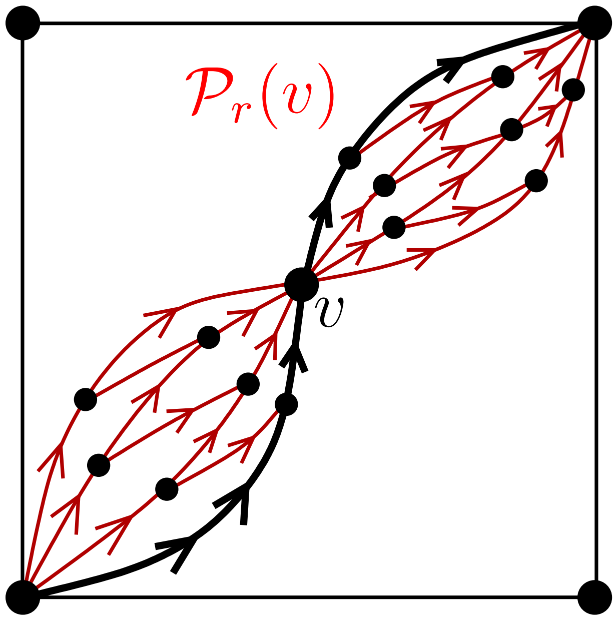
Let v be an inner vertex of T

Let $\mathcal{P}_r(v)$ be the unique path passing by v which is:

- the rightmost one before arriving at v
- the leftmost one after leaving v

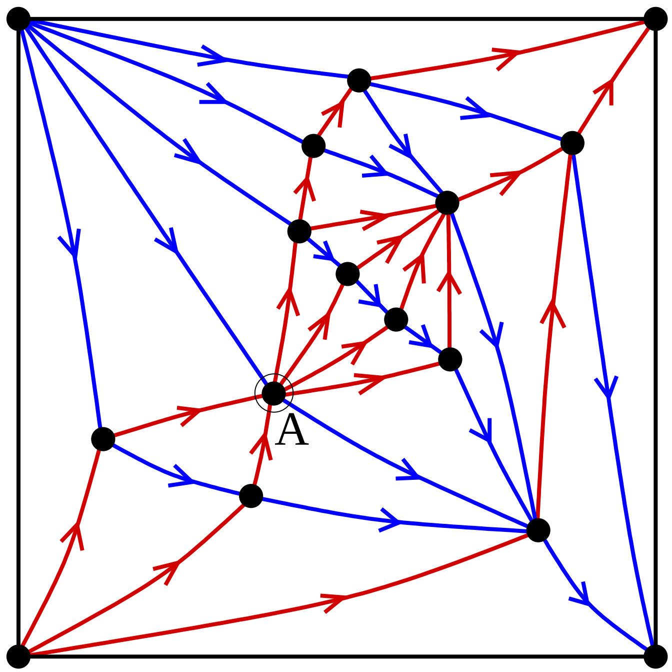


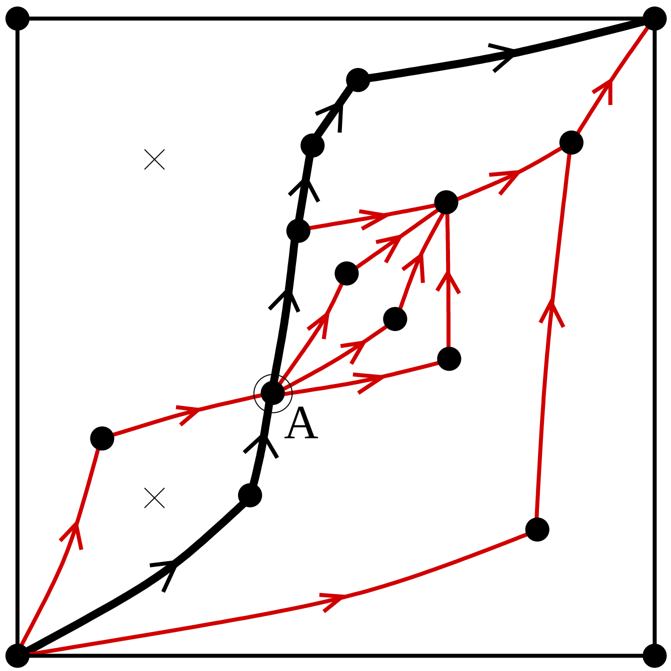
$\Rightarrow$



The red map gives abscissas (2)

The absciss of v is the number of faces of the red map on the left of $\mathcal{P}_r(v)$



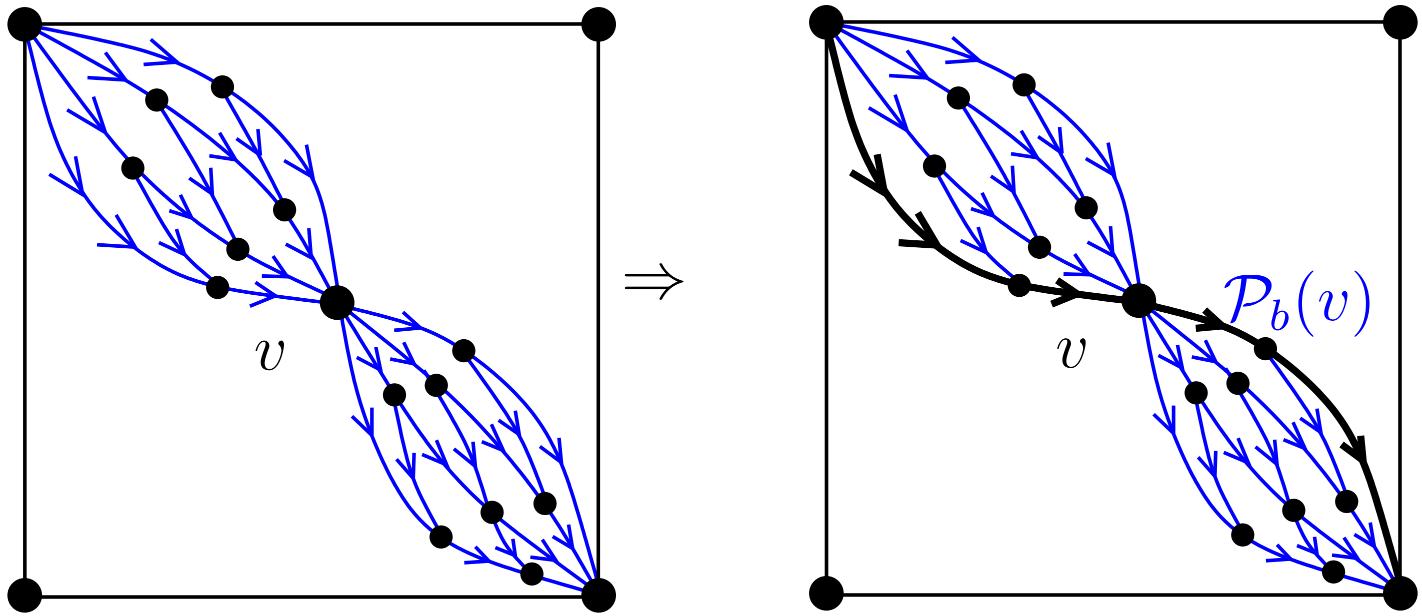


$\Rightarrow$ A has absciss 2

The blue map gives ordinates (1)

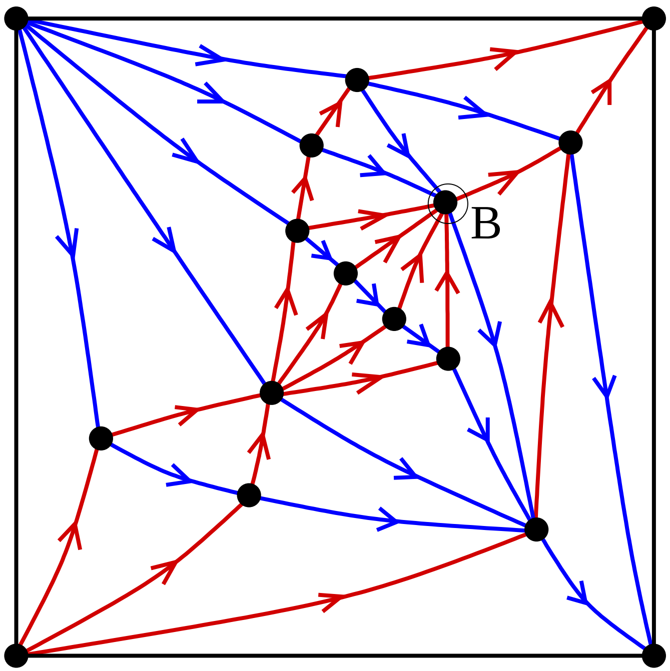
Similarly we define $\mathcal{P}_b(v)$ the unique blue path which is:

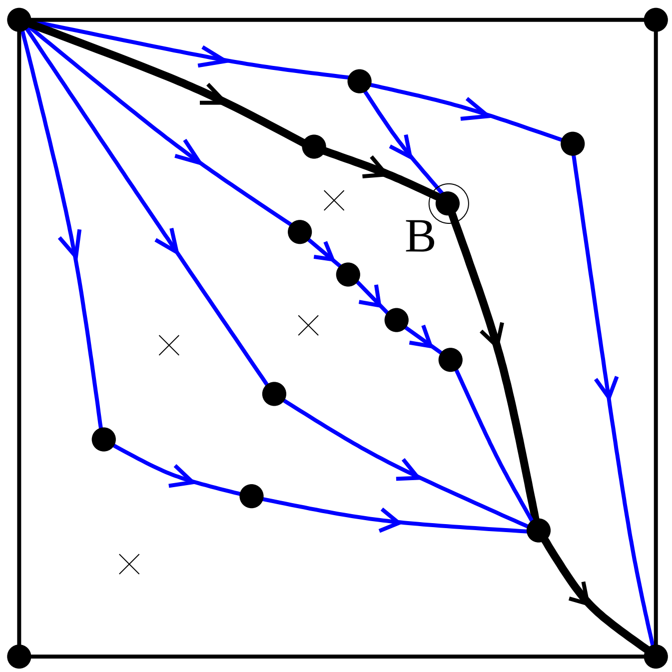
- the rightmost one before arriving at v
- the leftmost one after leaving v



The blue map gives ordinates (2)

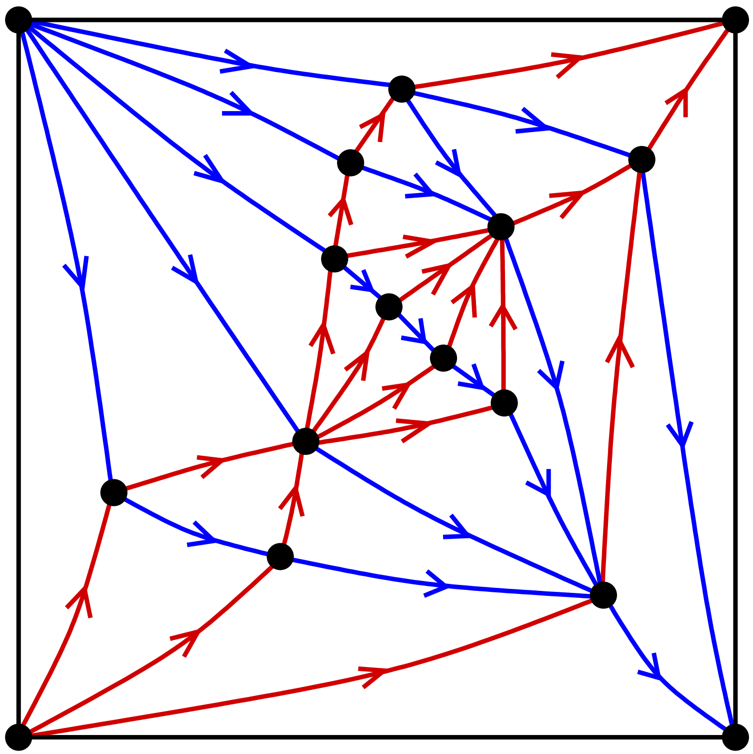
The ordinate of v is the number of faces of the blue map below $\mathcal{P}_b(v)$





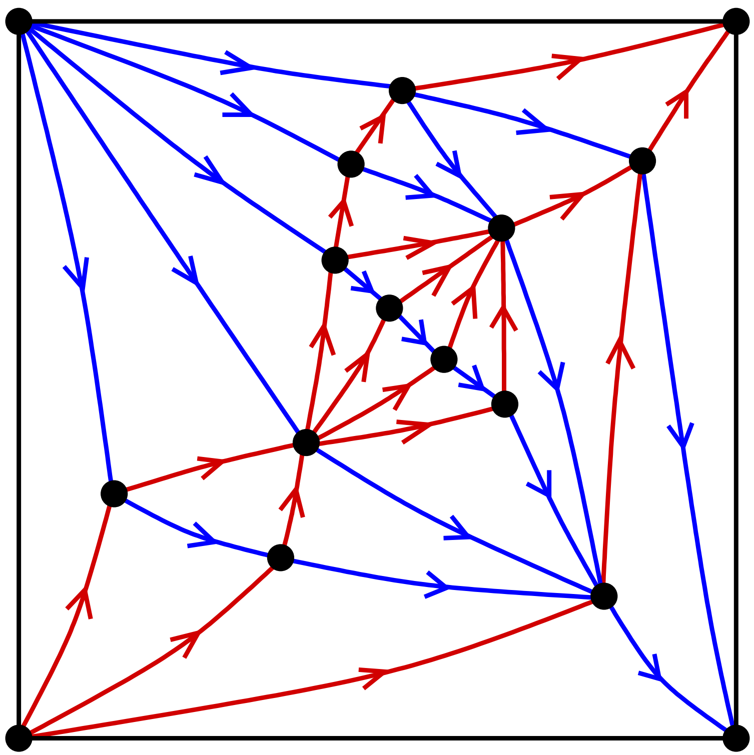
$\Rightarrow B$ has ordinate 4

Execution of the algorithm

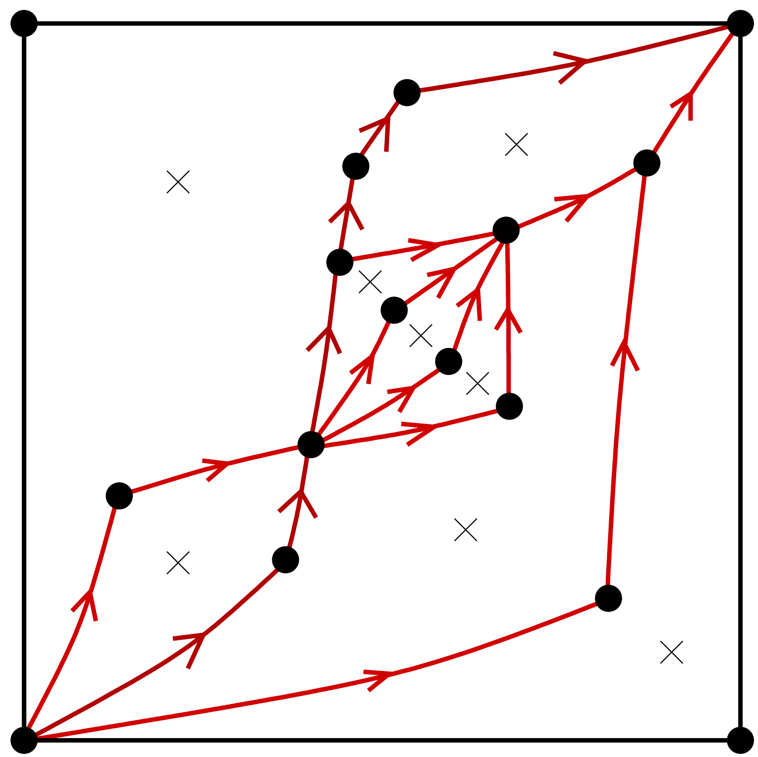


Execution of the algorithm

Let f_r be the number of faces of the red map

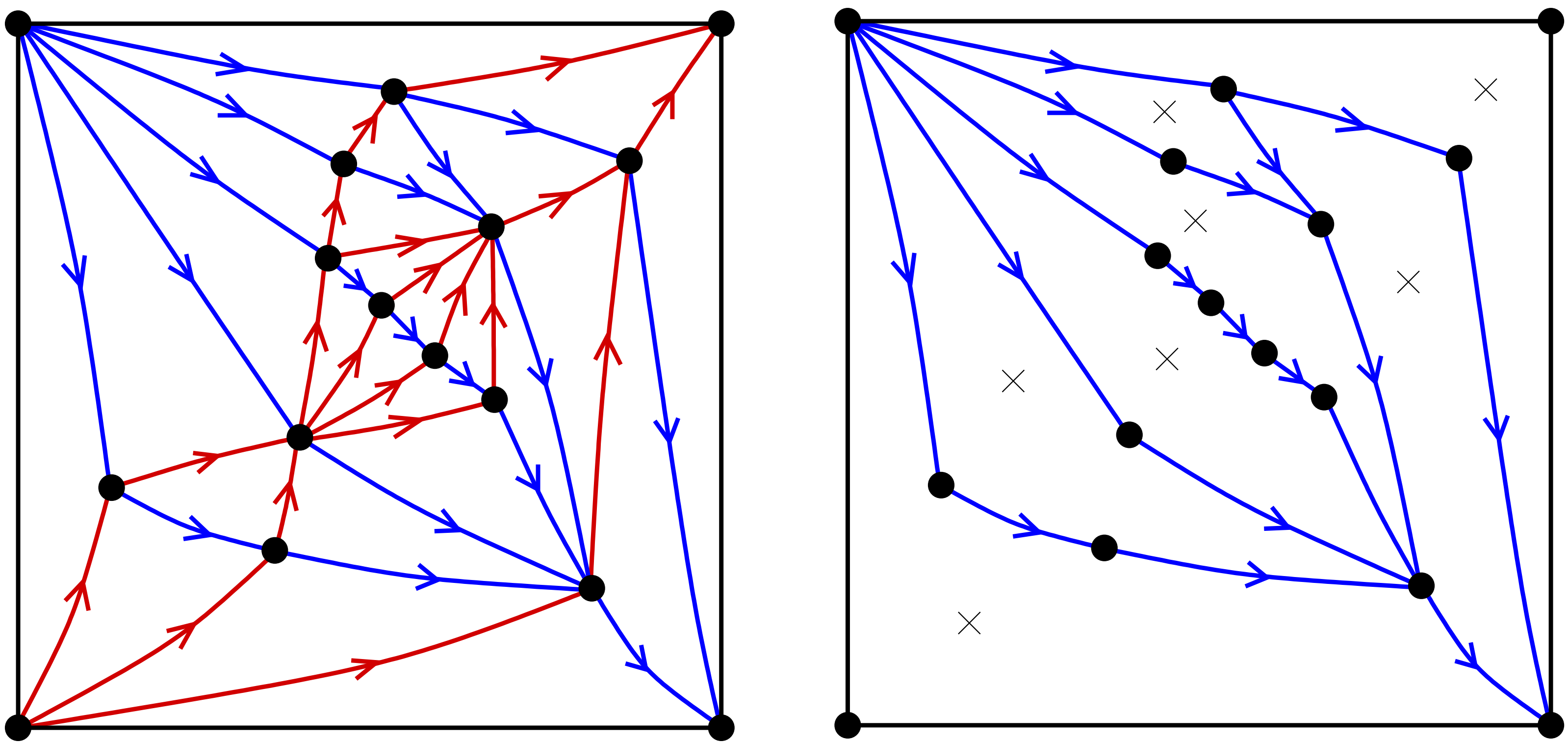


$f_r = 8$



Execution of the algorithm

Let f_b be the number of faces of the blue map

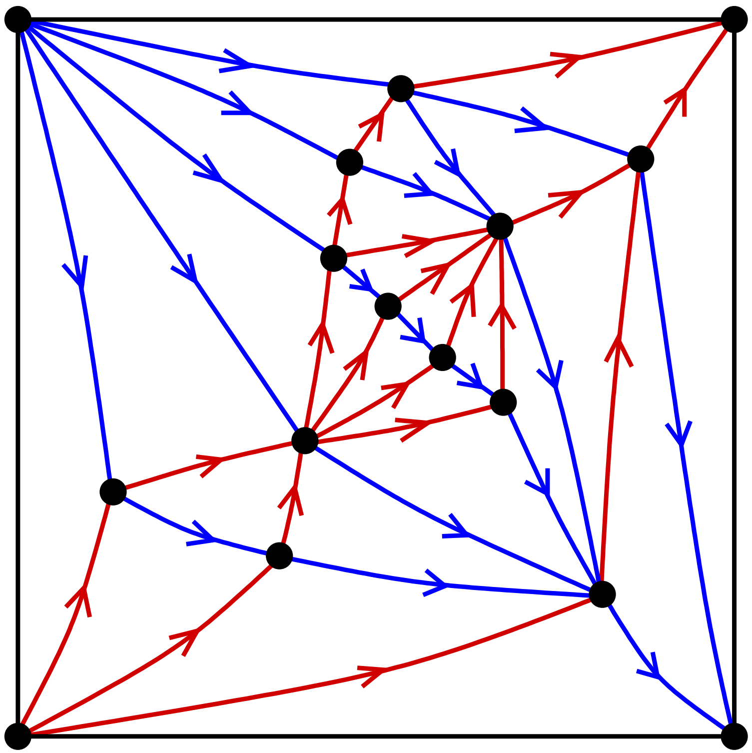


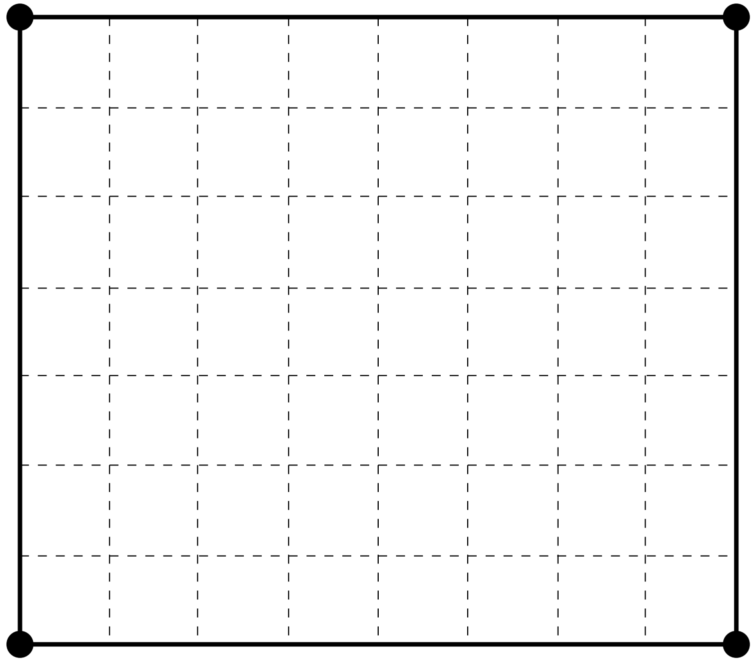
$f_r = 8$

$f_b = 7$

Execution of the algorithm

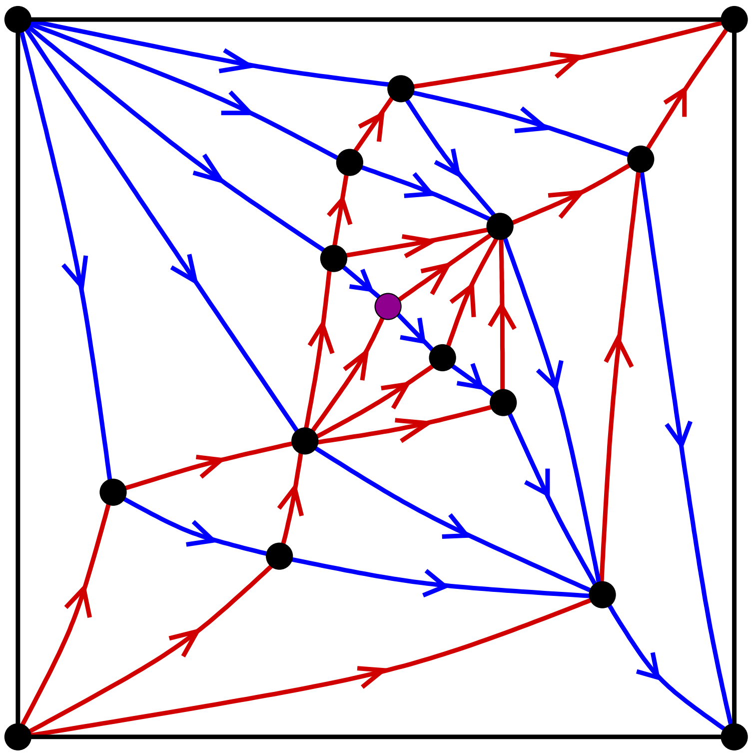
Take a regular grid of width f_r and height f_b and place the 4 border vertices of T at the 4 corners of the grid

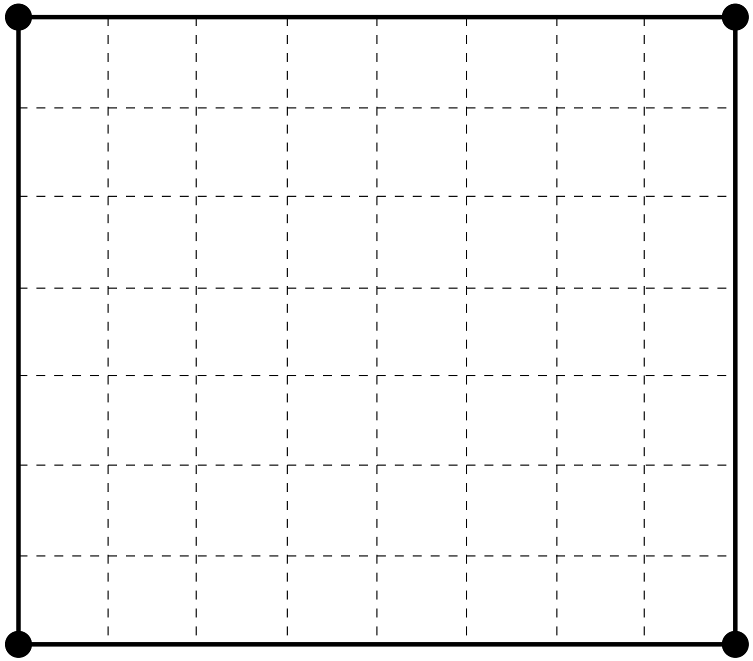




Execution of the algorithm

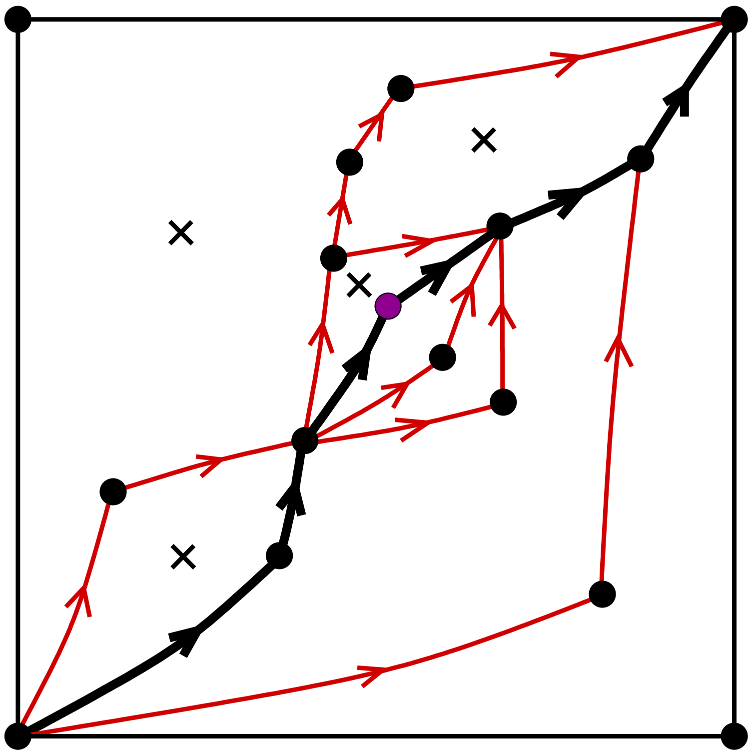
Place all other points using the red path for absciss and the blue path for ordinate



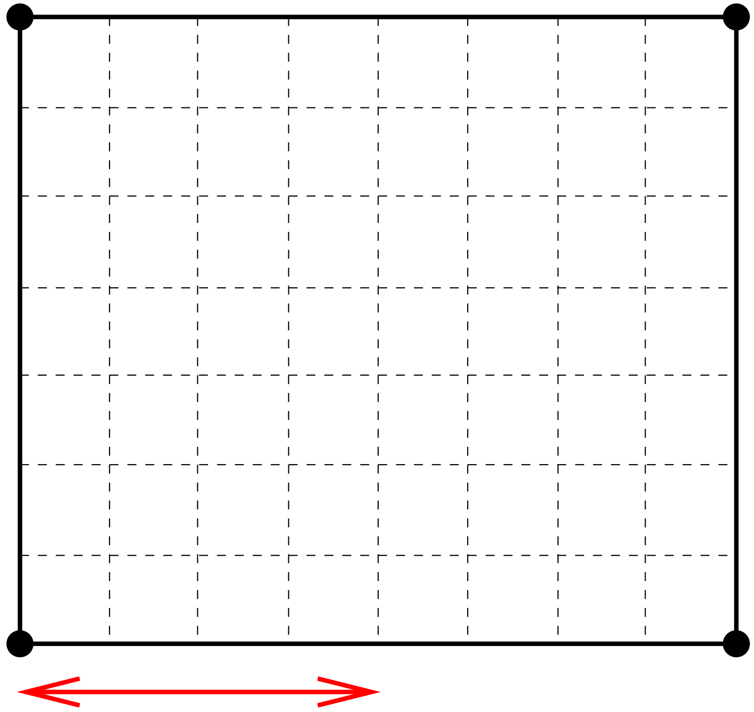


Execution of the algorithm

Place all other points using the red path for absciss and the blue path for ordinate

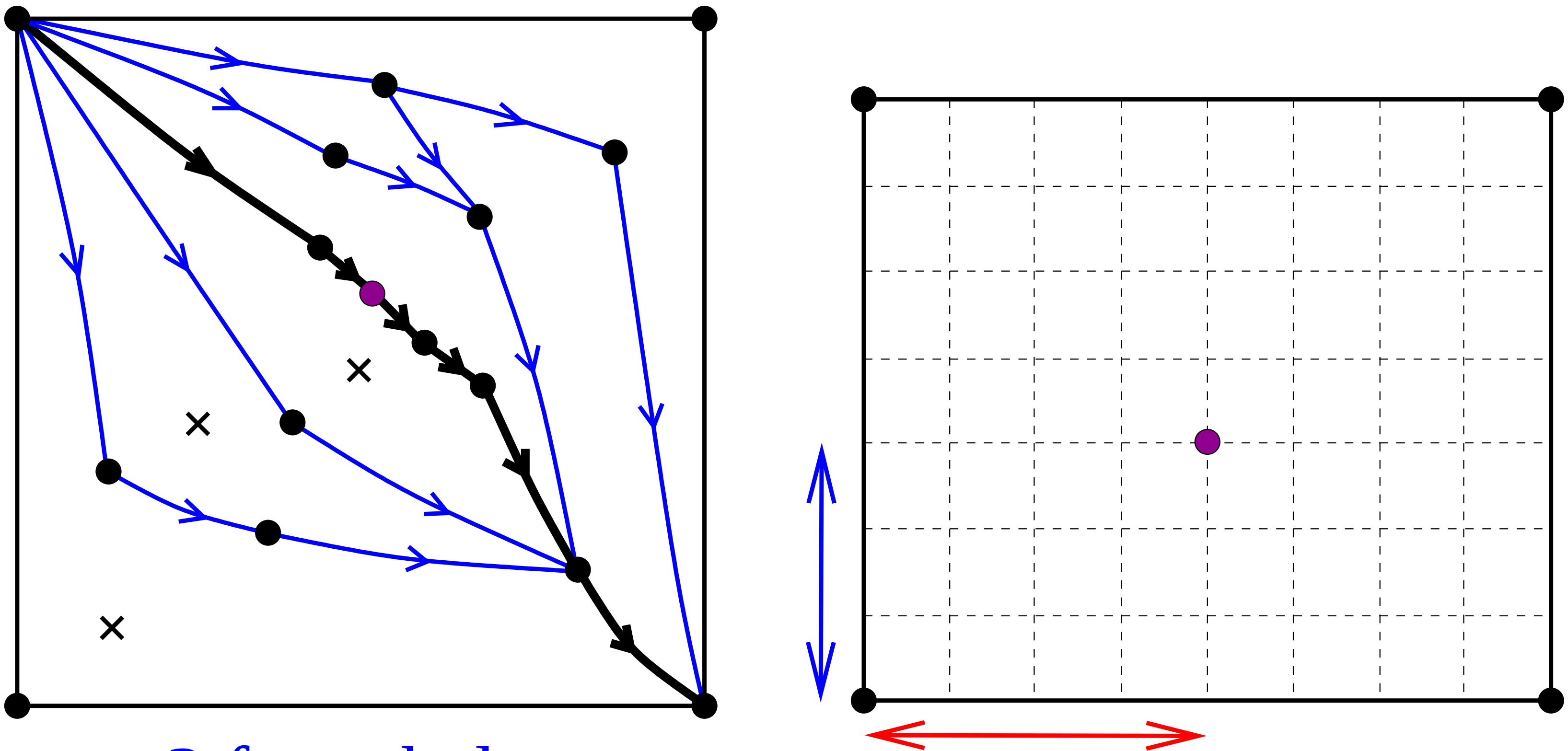


4 faces on the left



Execution of the algorithm

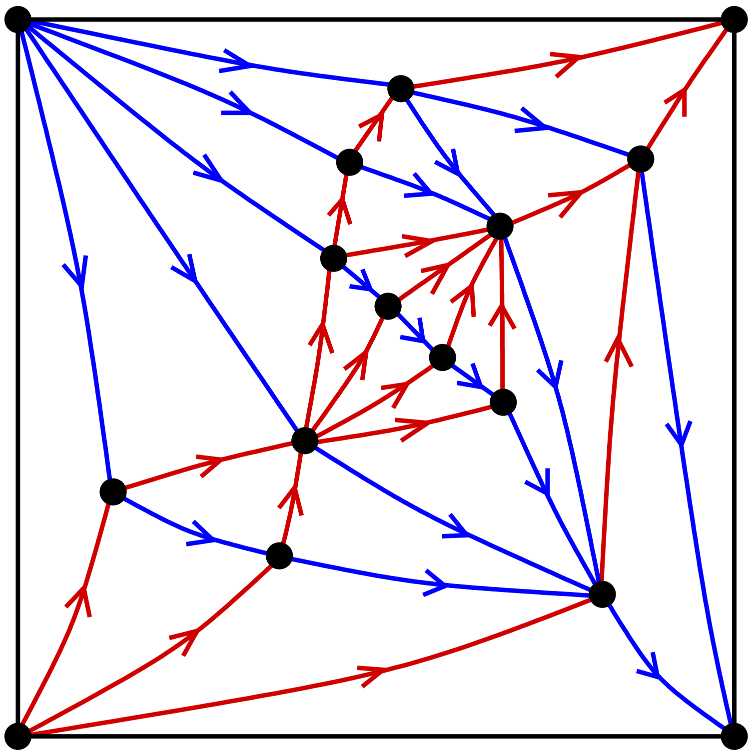
Place all other points using the red path for absciss and the blue path for ordinate

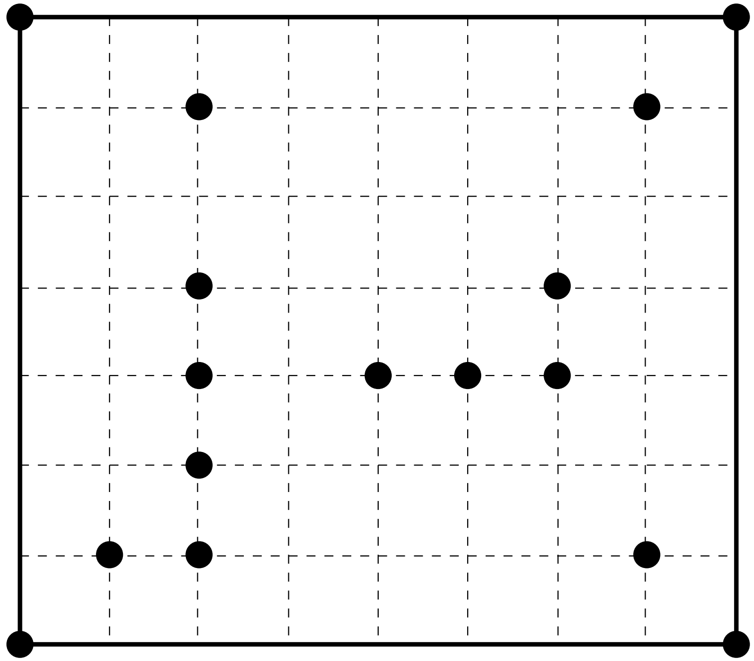


3 faces below

Execution of the algorithm

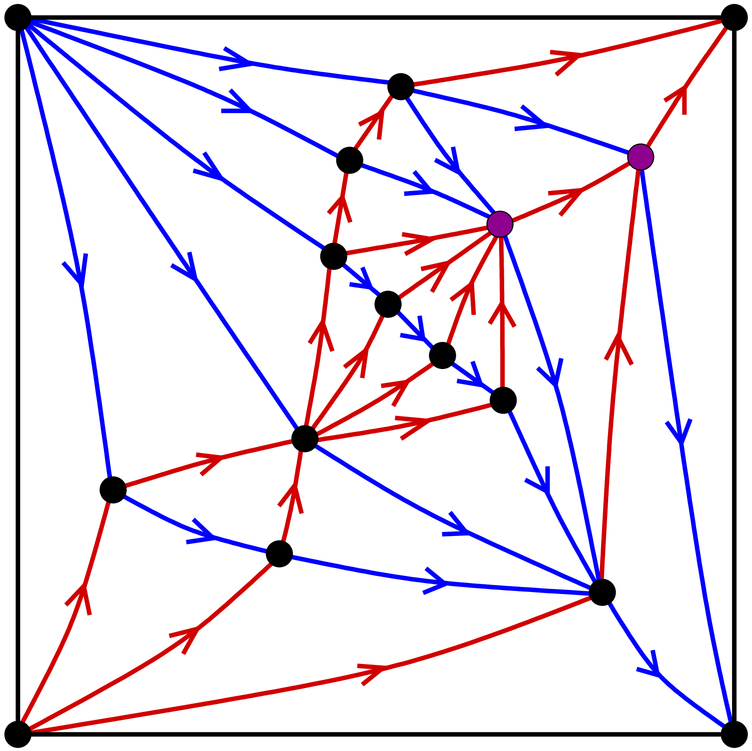
Place all other points using the red path for absciss and the blue path for ordinate

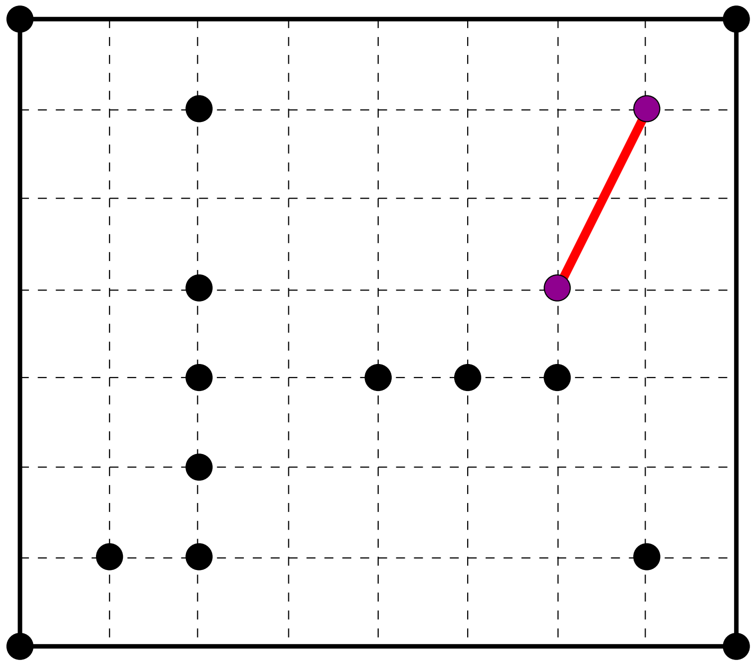




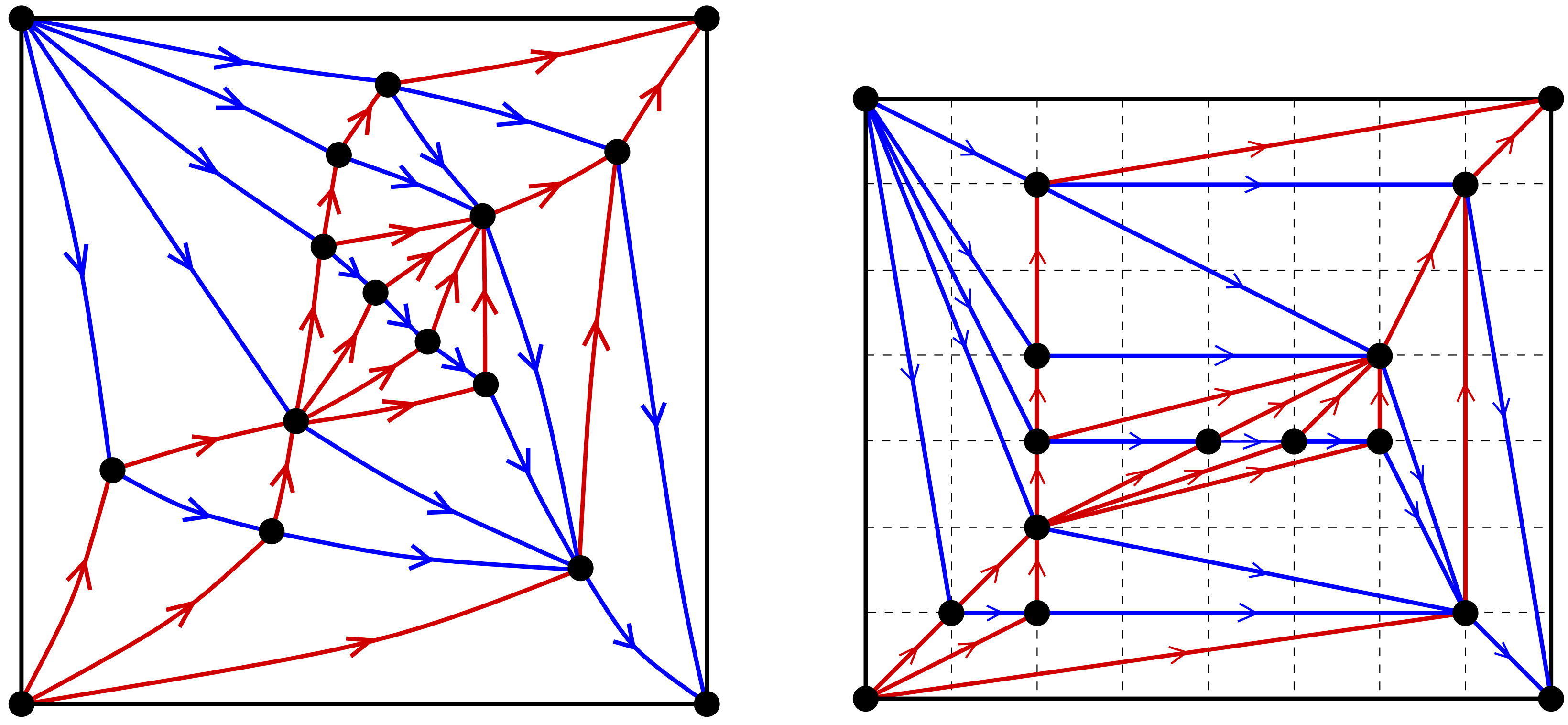
Execution of the algorithm

Link each pair of adjacent vertices by a segment





Execution of the algorithm



Results

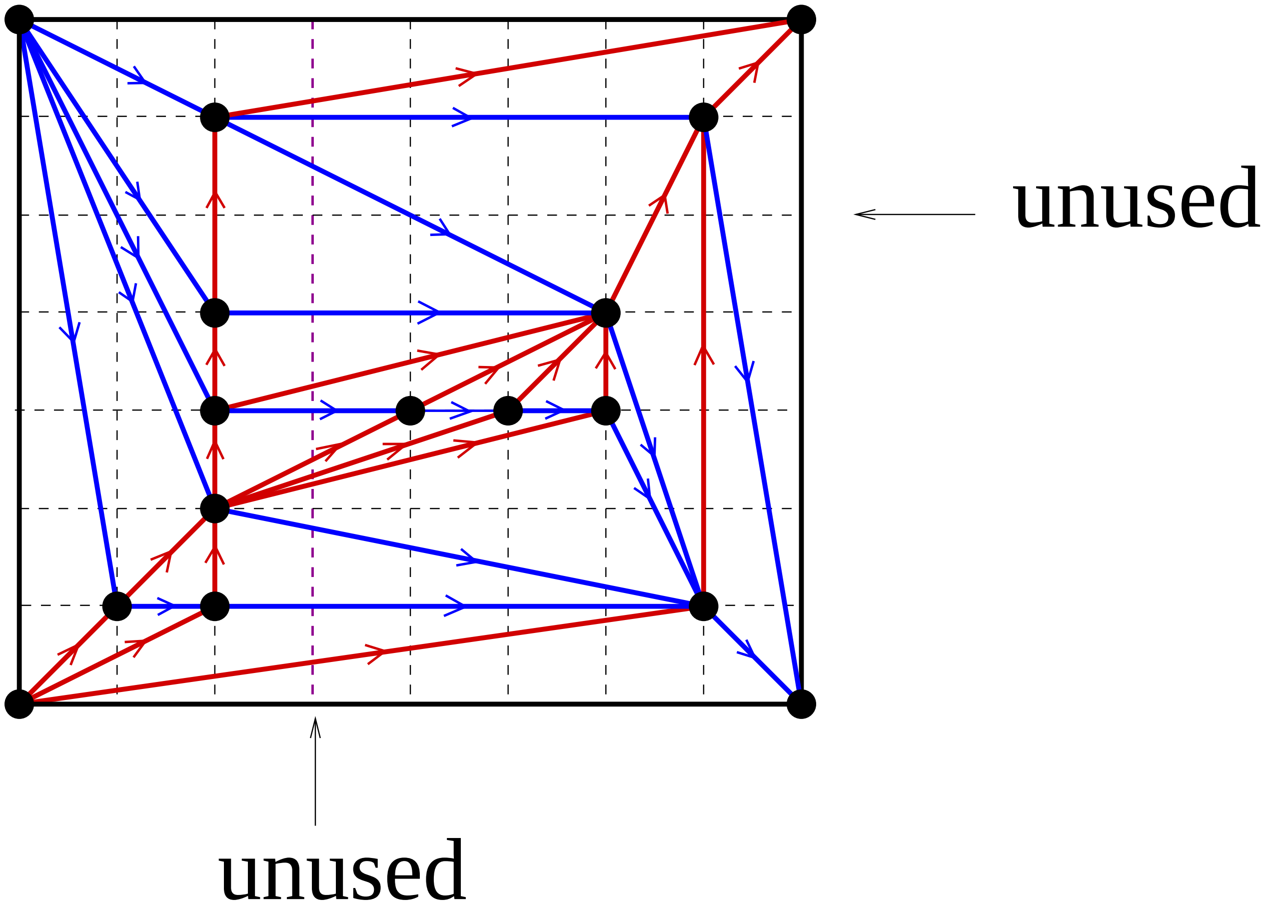
- The obtained drawing is a straight line embedding
- The drawing respects the transversal structure:
  - Red edges are oriented from bottom-left to top-right
  - Blue edges are oriented from top-left to bottom-right
- If T has n vertices, the width W and height H verify

  $$\boxed{W + H = n - 1}$$

  similar grid size as He (1996) and Miura et al (2001)

Compaction step

- Some abscissas and ordinates are not used
- The deletion of these unused coordinates keeps the drawing planar



Compaction step

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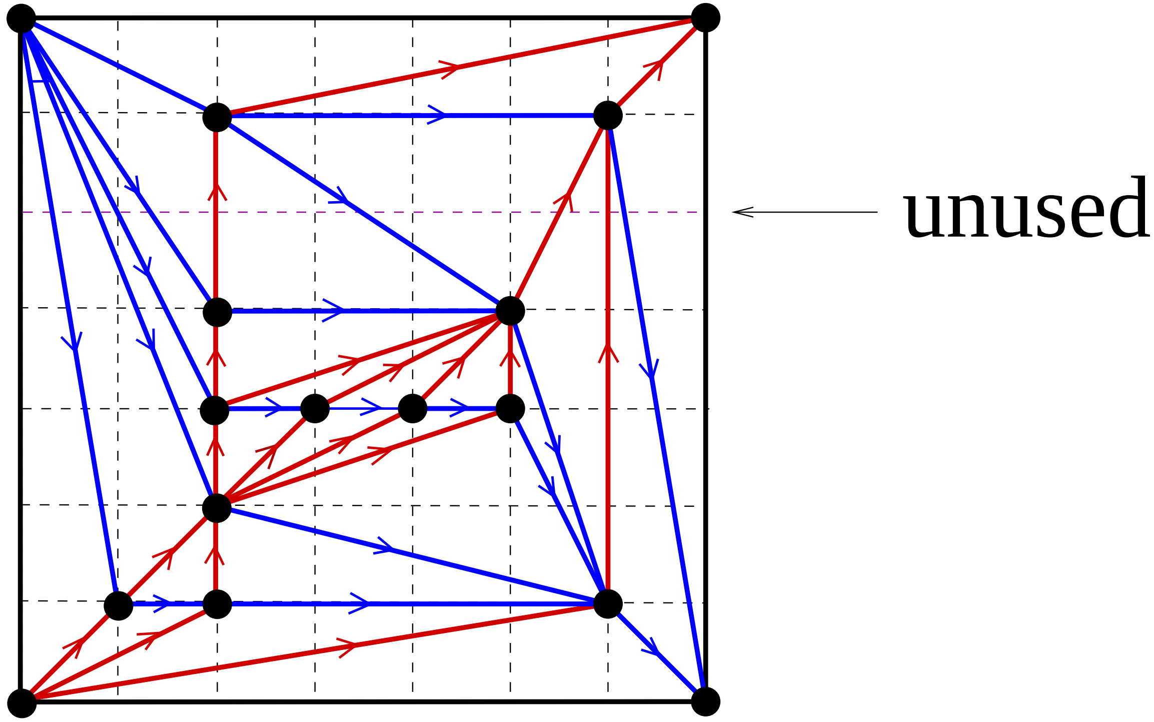

Compaction step

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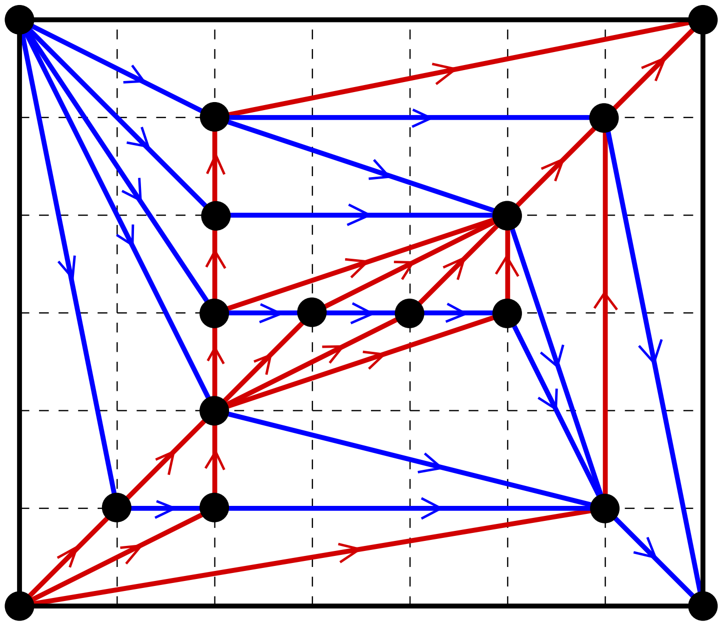
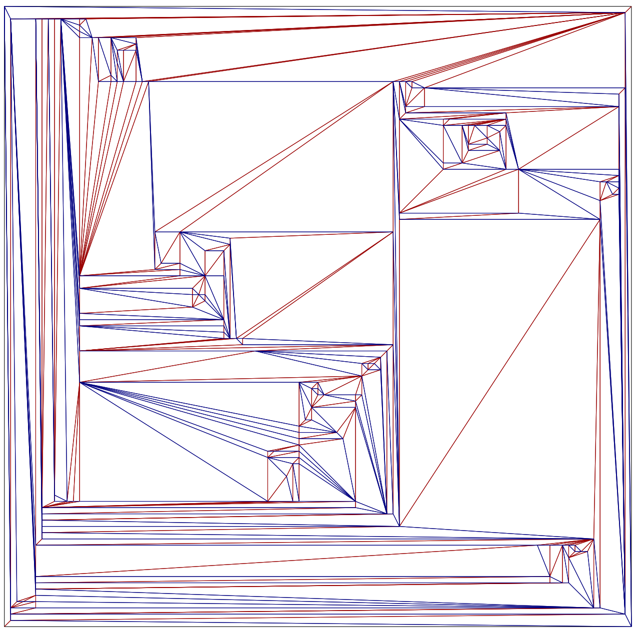
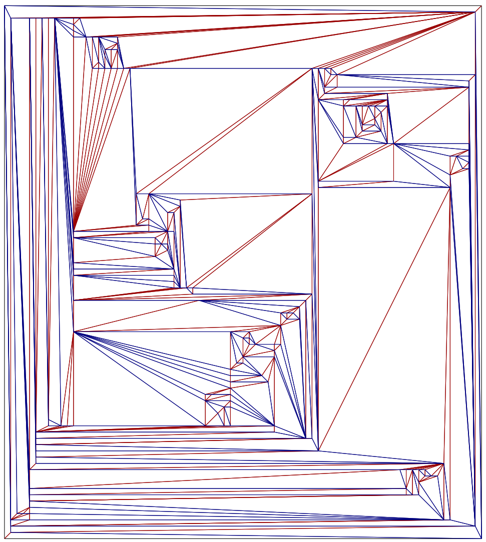


Size of the grid after deletion

- If the transversal structure is the minimal one, the number of deleted coordinates can be analyzed:
- After deletion, the grid has size $\frac{11}{27}n \times \frac{11}{27}n$ "almost surely"
- Reduction of $\frac{5}{27} \approx 18\%$ compared to He and Miura et al





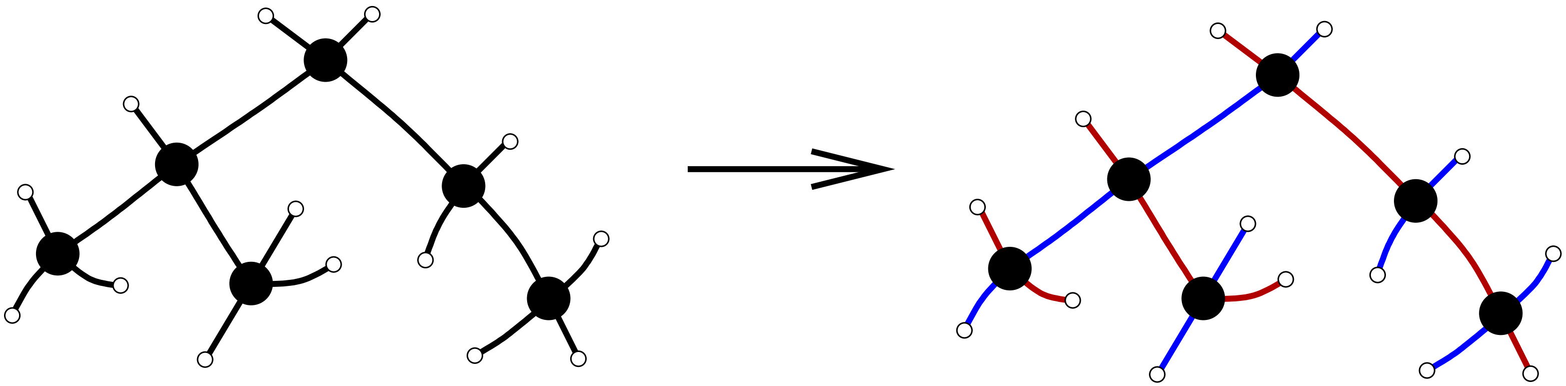
Bijection between triangulations and ternary trees

Ternary trees

A ternary tree is a plane tree with:

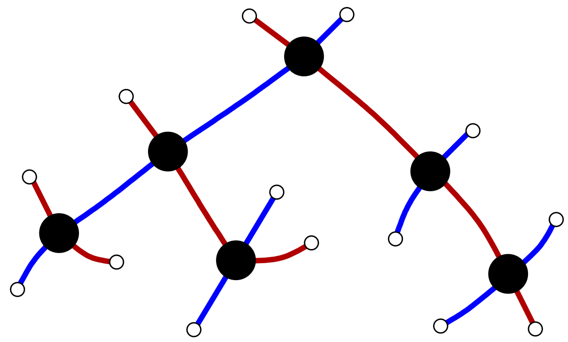
- Vertices of degree 4 called inner nodes
- Vertices of degree 1 called leaves
- An edge connected two inner nodes is called inner edge
- An edge incident to a leaf is called a stem

A ternary tree can be endowed with a transversal structure



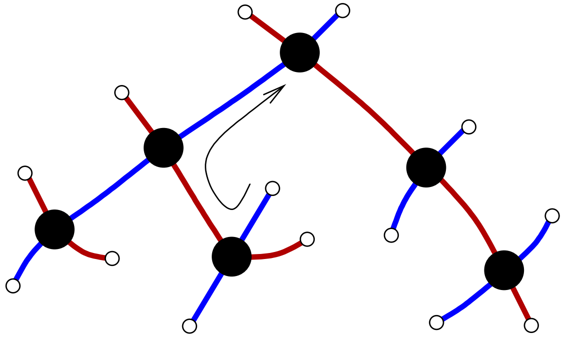
From a ternary tree to a triangulation

Local operations to “close” triangular faces



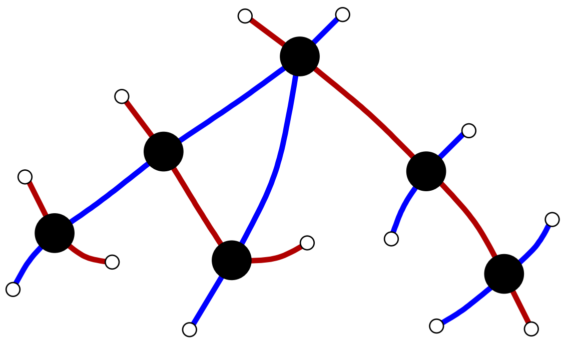
From a ternary tree to a triangulation

Local operations to "close" triangular faces



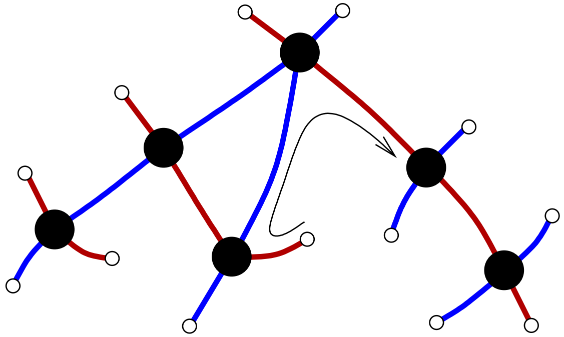
From a ternary tree to a triangulation

Local operations to “close” triangular faces



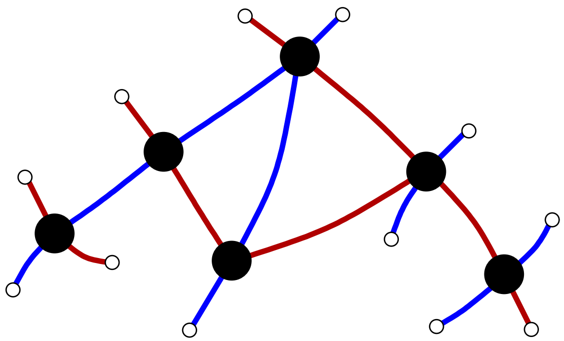
From a ternary tree to a triangulation

Local operations to "close" triangular faces



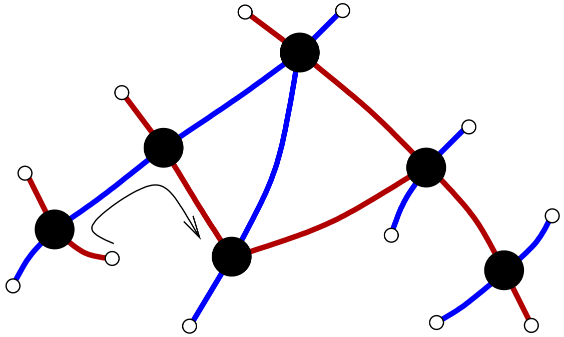
From a ternary tree to a triangulation

Local operations to "close" triangular faces



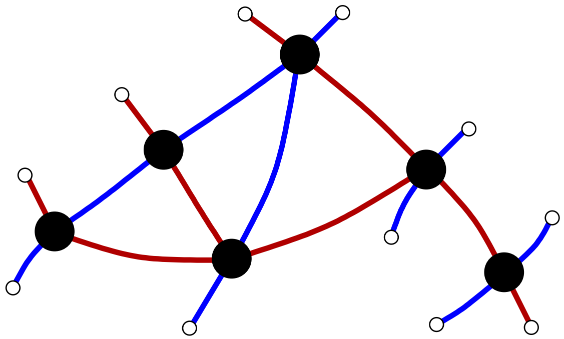
From a ternary tree to a triangulation

Local operations to “close” triangular faces



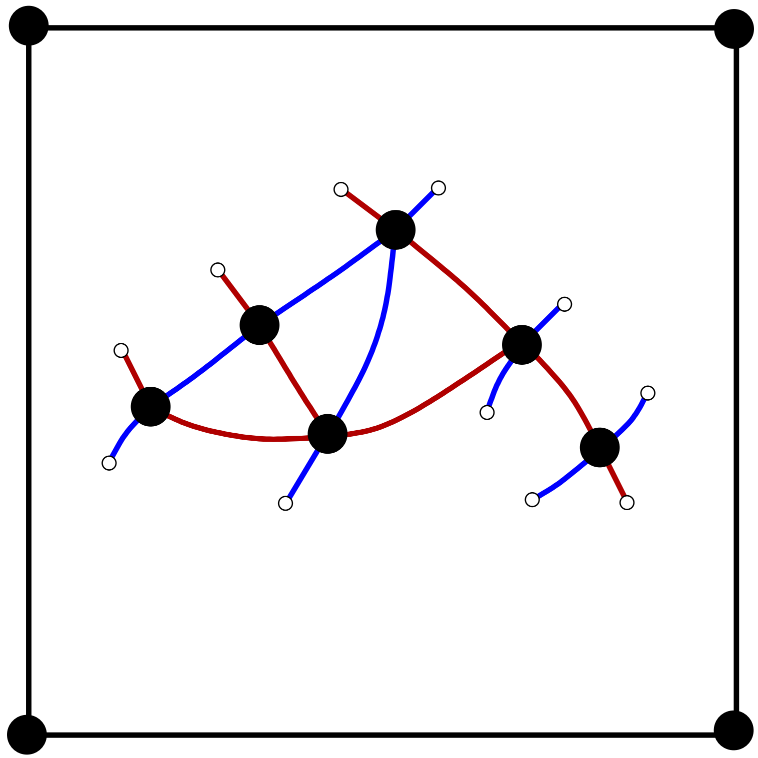
From a ternary tree to a triangulation

Local operations to "close" triangular faces



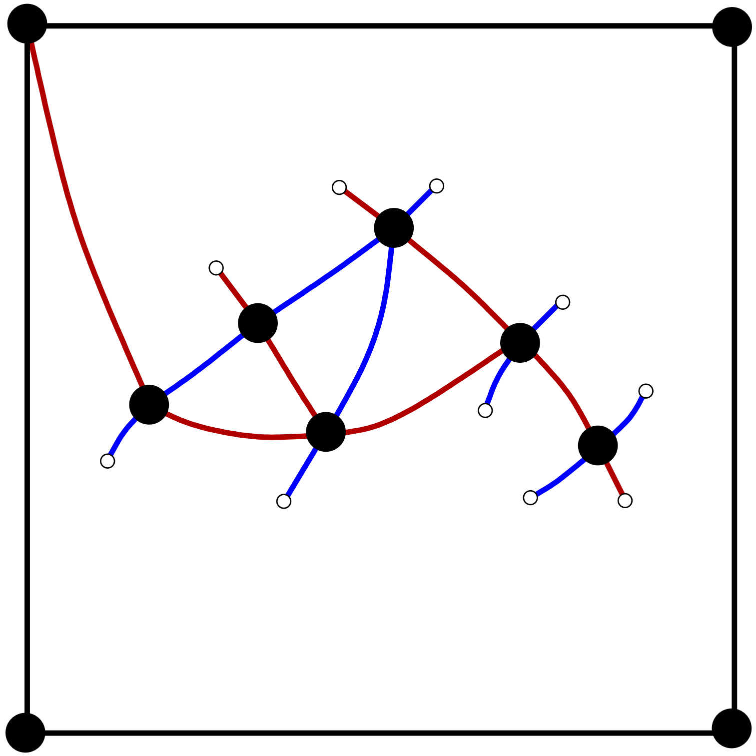
From a ternary tree to a triangulation

Draw a quadrangle outside of the figure



From a ternary tree to a triangulation

Merge remaining stems to form triangles



From a ternary tree to a triangulation

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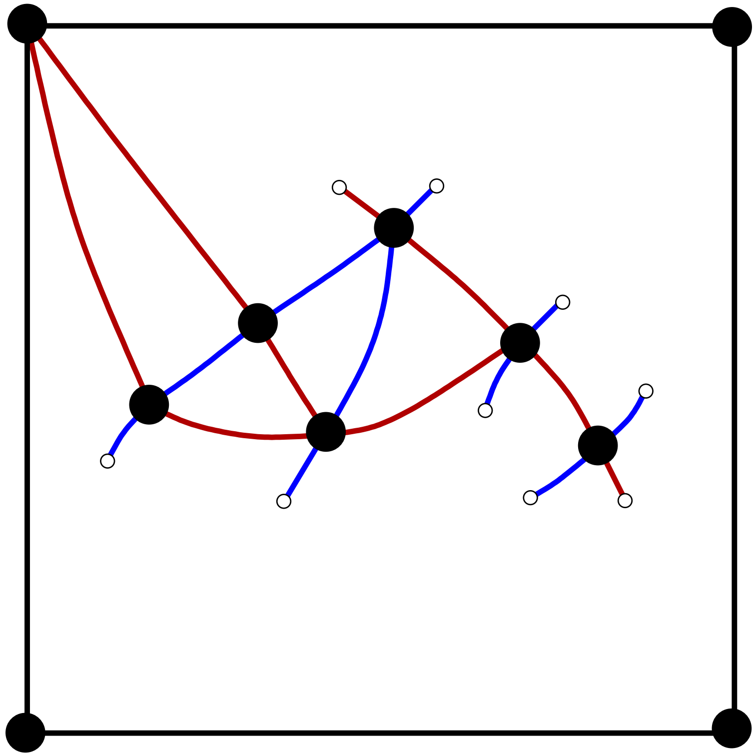
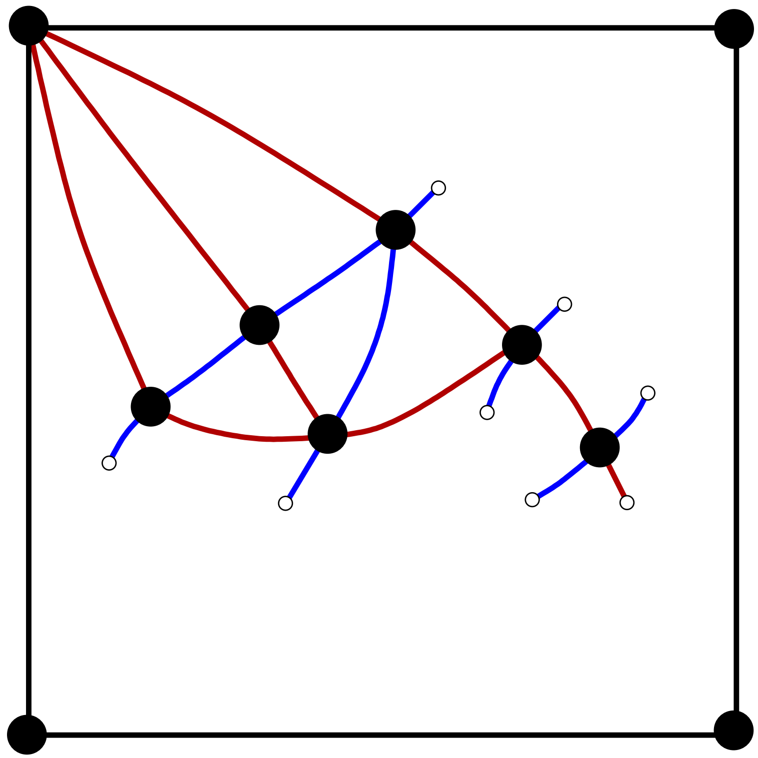


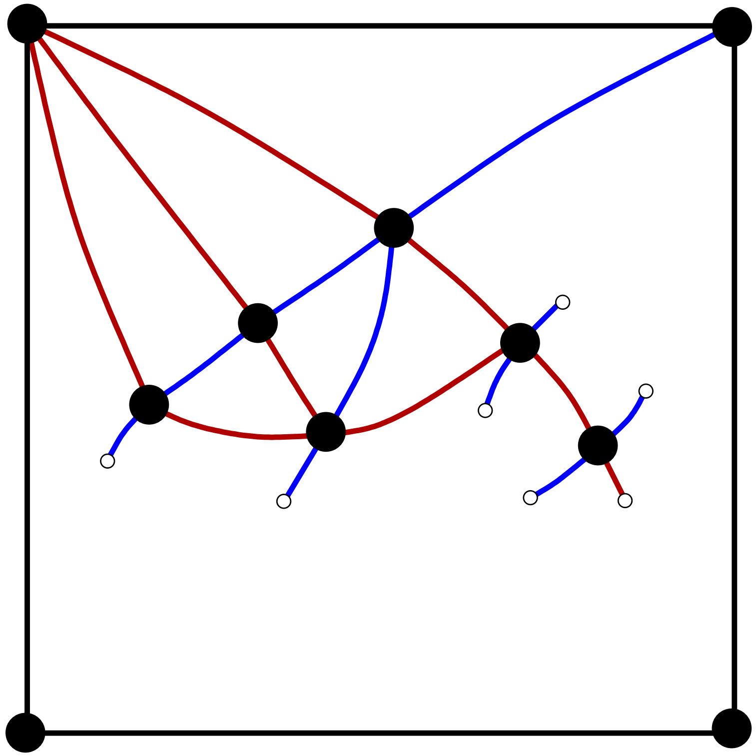
From a ternary tree to a triangulation

Merge remaining stems to form triangles



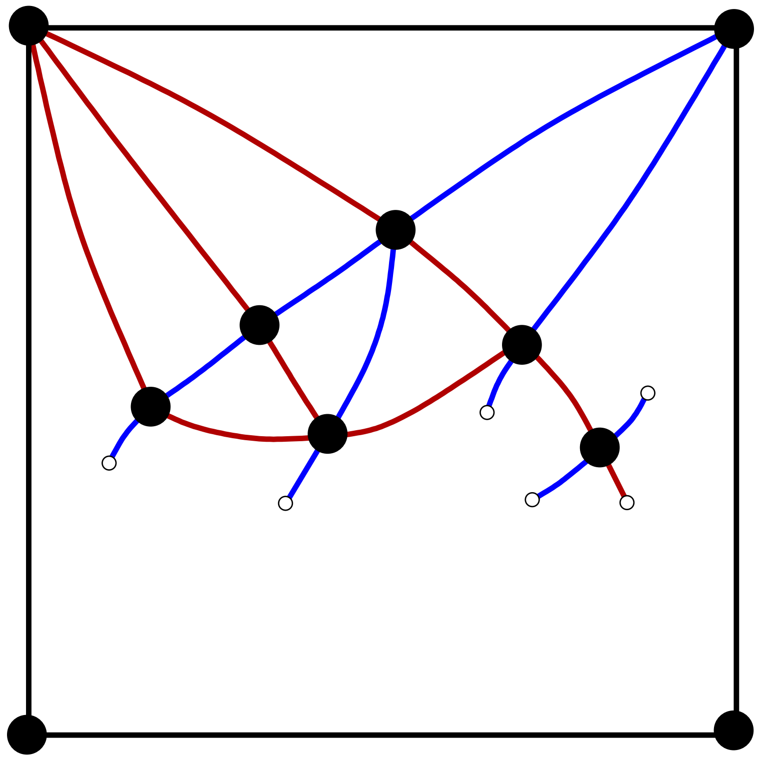
From a ternary tree to a triangulation

Merge remaining stems to form triangles



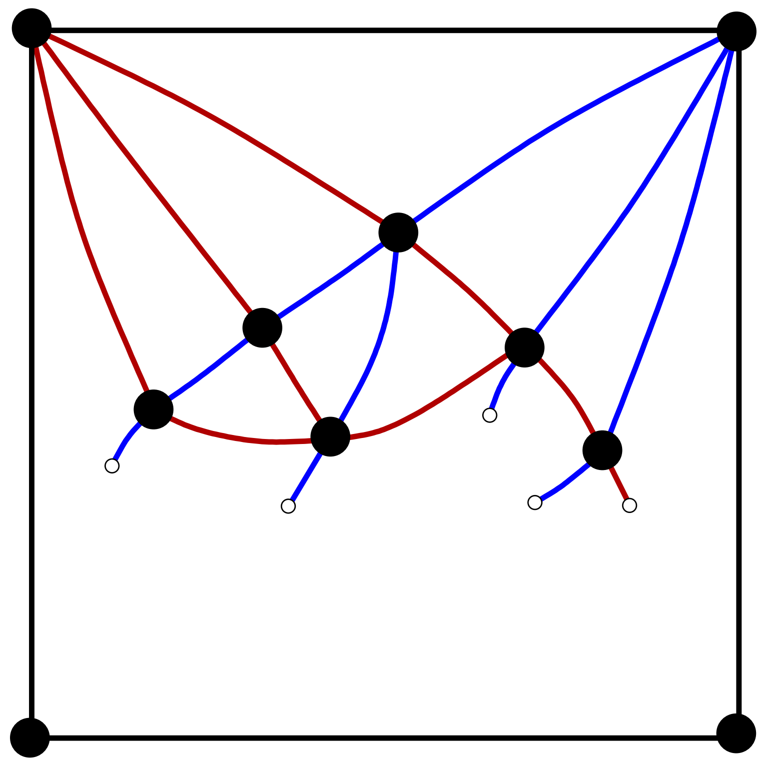
From a ternary tree to a triangulation

Merge remaining stems to form triangles



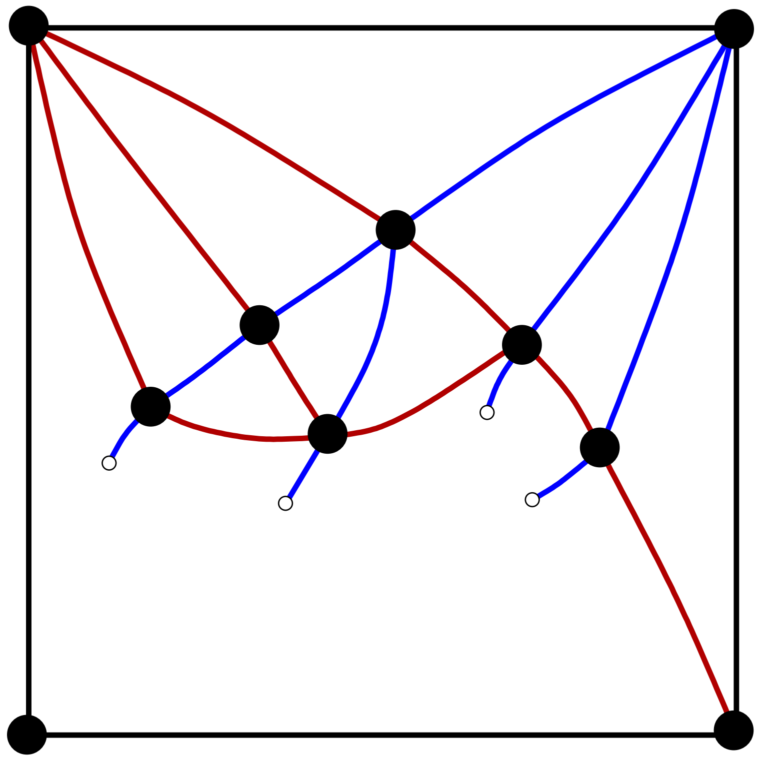
From a ternary tree to a triangulation

Merge remaining stems to form triangles



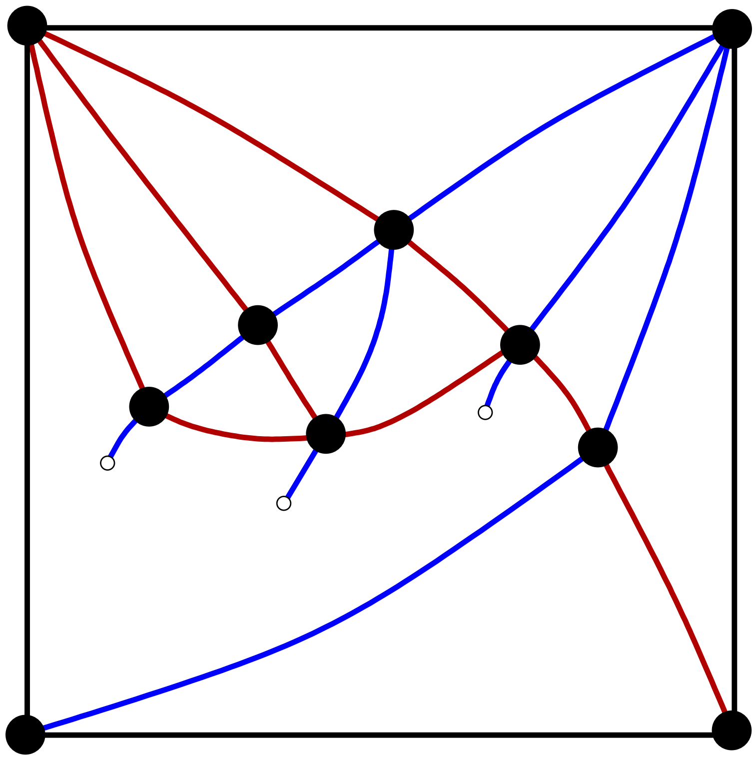
From a ternary tree to a triangulation

Merge remaining stems to form triangles



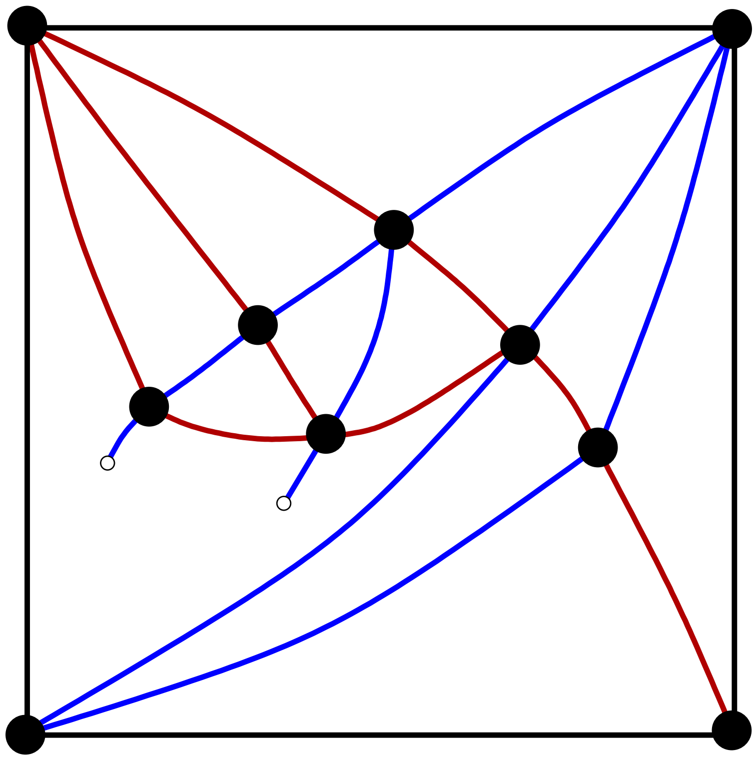
From a ternary tree to a triangulation

Merge remaining stems to form triangles



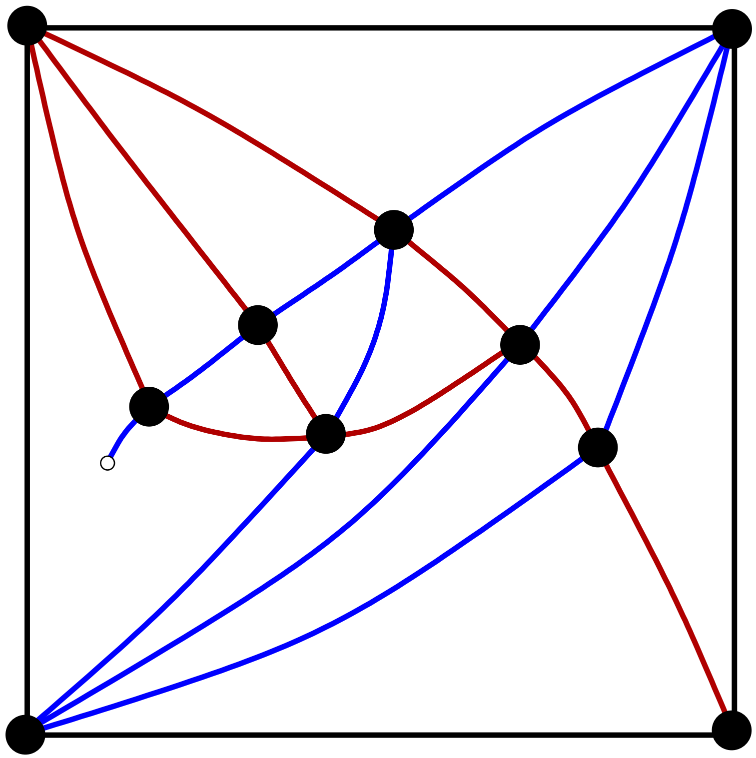
From a ternary tree to a triangulation

Merge remaining stems to form triangles



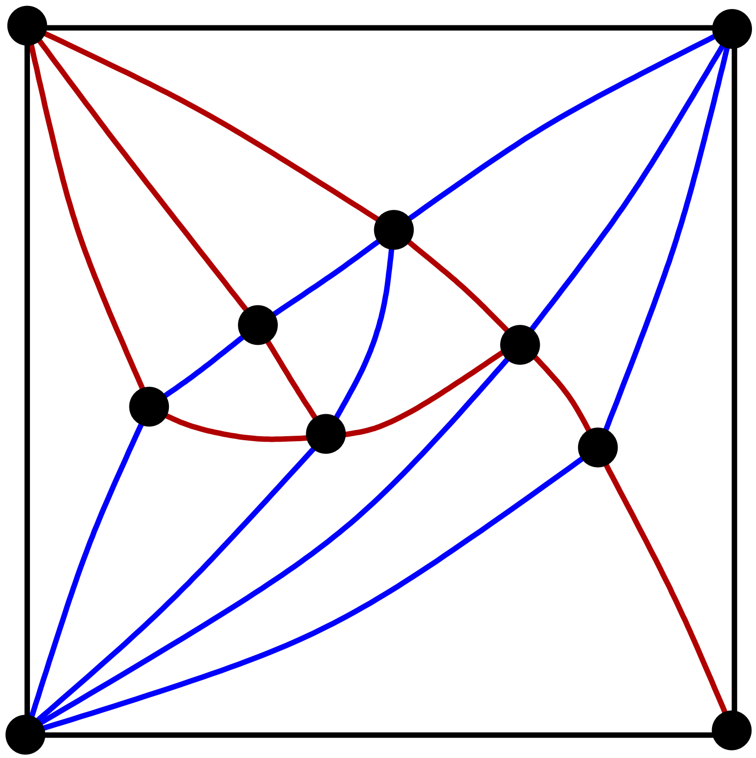
From a ternary tree to a triangulation

Merge remaining stems to form triangles



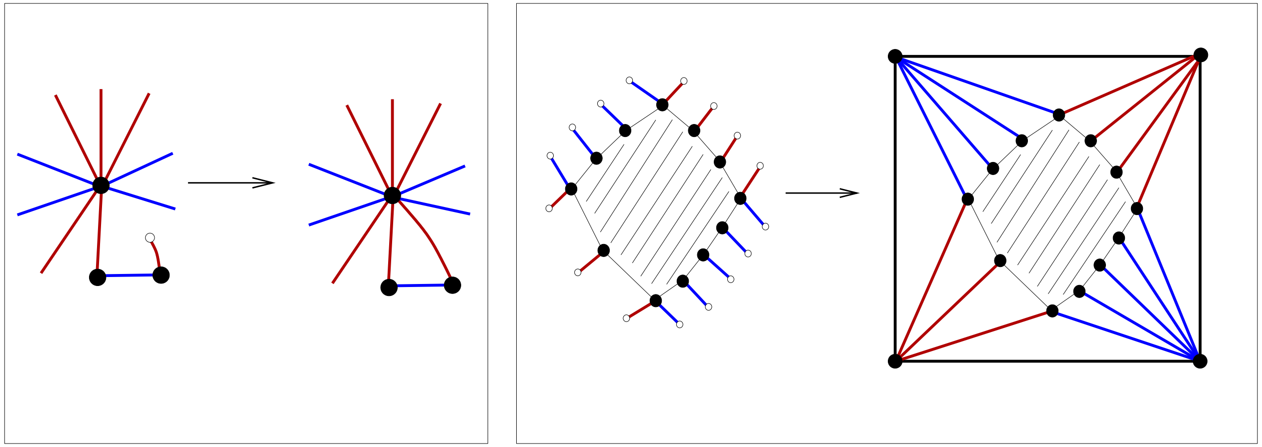
From a ternary tree to a triangulation

Merge remaining stems to form triangles

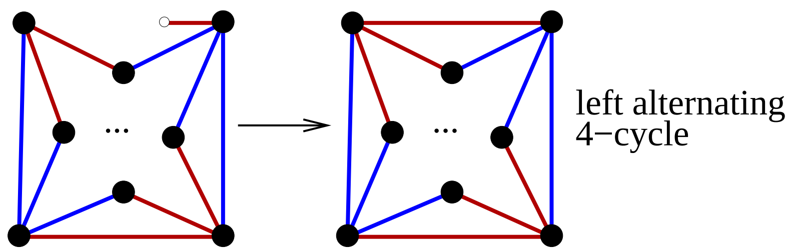


Properties of the closure-mapping

- The closure mapping is a bijection between ternary trees with n inner nodes and triangulations with n inner vertices.
- The closure transports the transversal structure

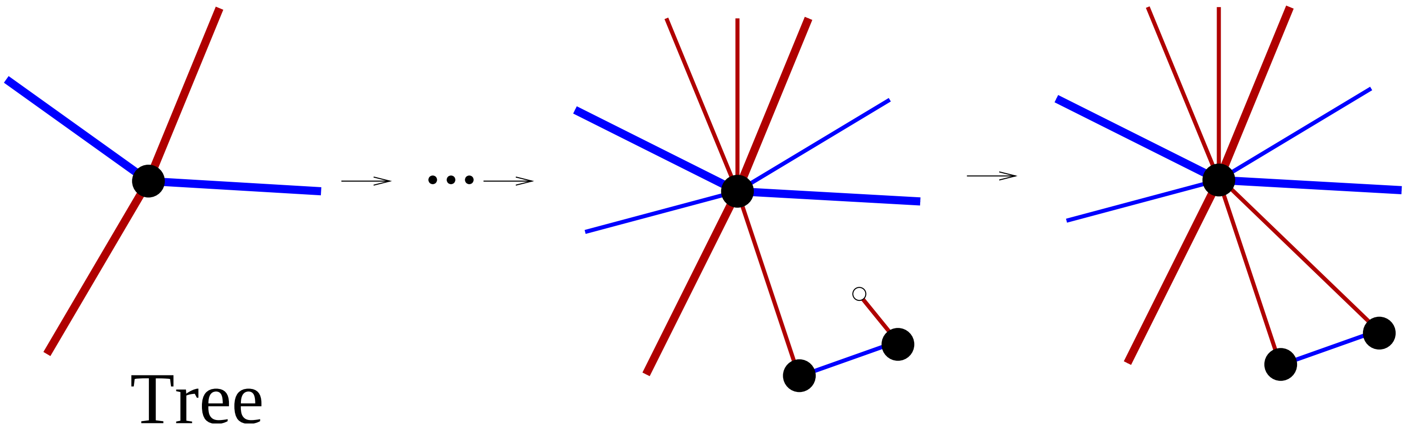


- The obtained transversal structure on T is minimal



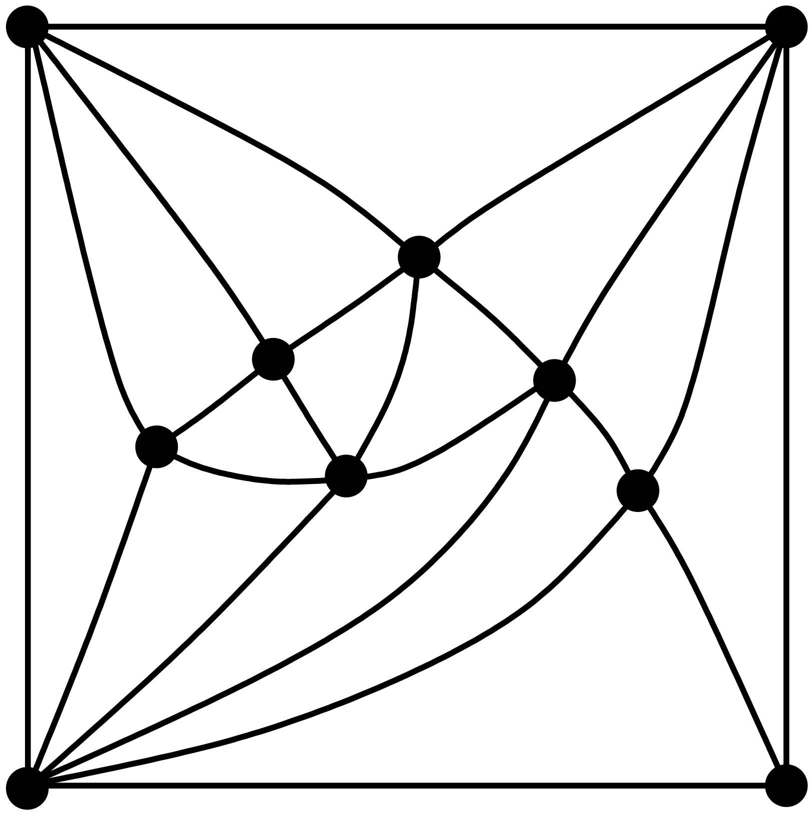
Observation to find the inverse mappin

The original 4 incident edges of each inner vertex of T remain the clockwise-most edge in each bunch

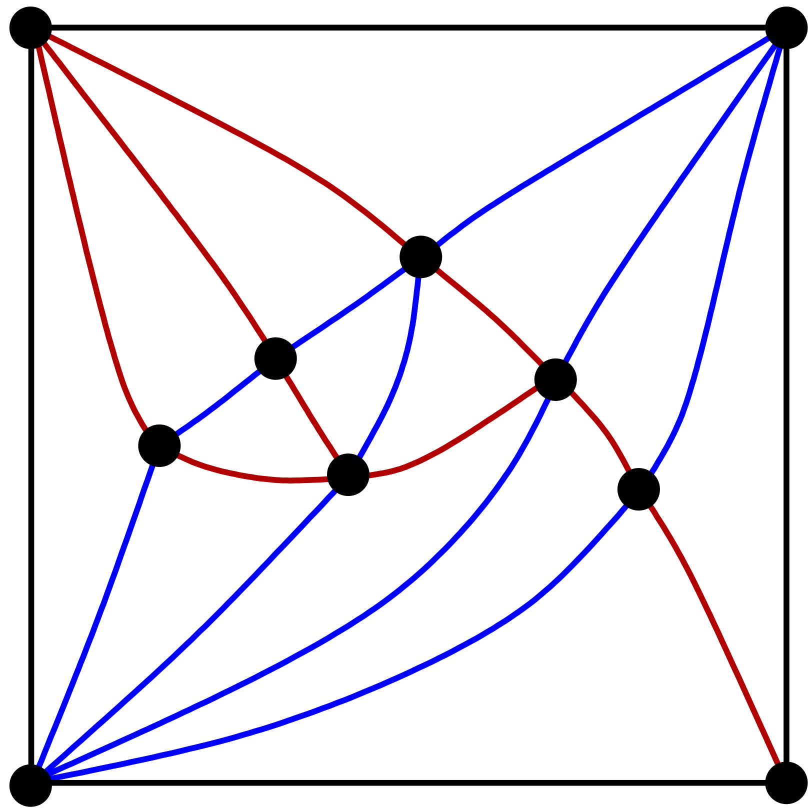


Recover the tree

Compute the minimal transversal structure

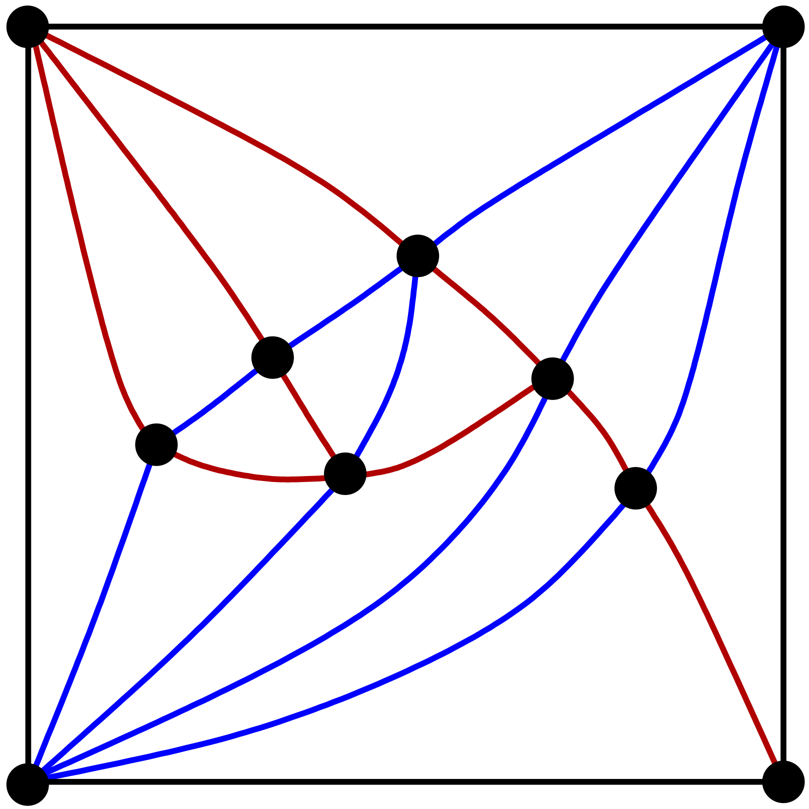


Recover the tree

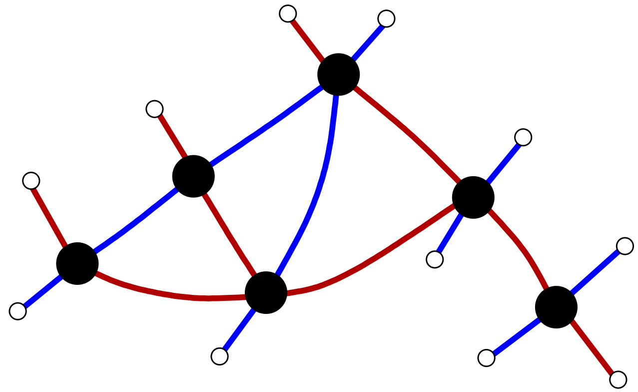


Recover the tree

Remove quadrangle

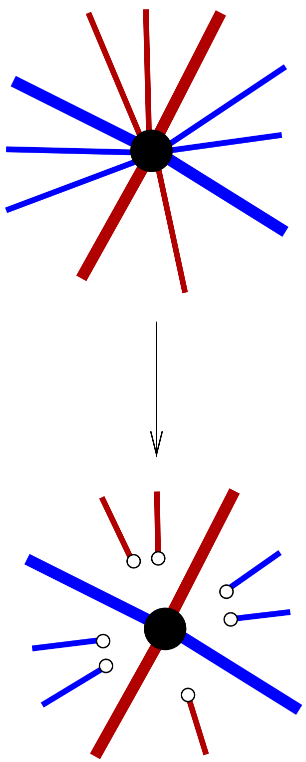


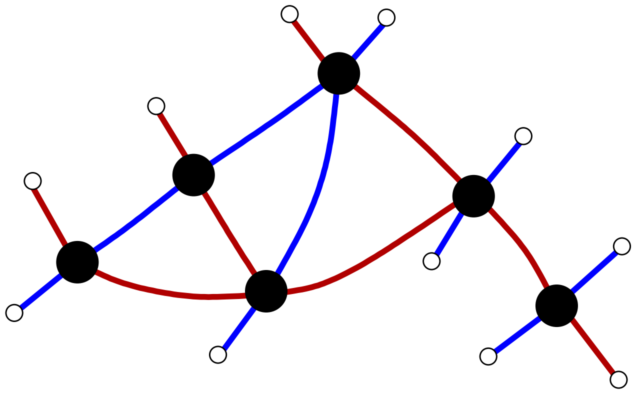
Recover the tree



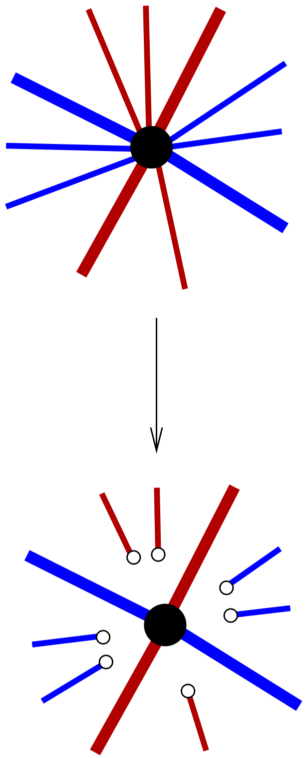
Recover the tree

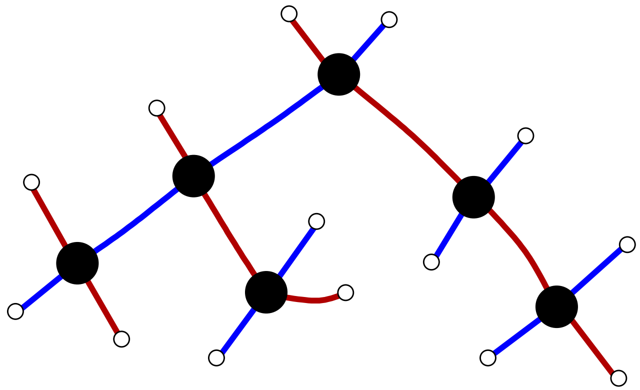
Keep the clockwisemost edge in each bunch





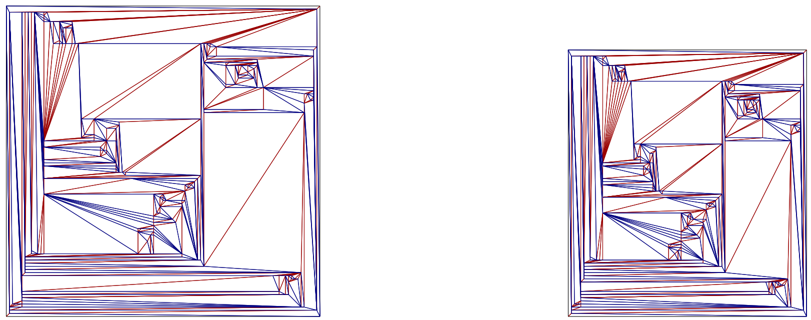
Recover the tree





Applications of the bijection

- Enumeration:$\Rightarrow$ $\boxed{T_n = \frac{4}{2n+2}\frac{(3n)!}{n!(2n+1)!}}$
- Random generation: linear-time uniform random sampler of triangulations with n vertices



- Analysis of the grid size: almost surely $5n/27$ deleted coordinates for a random triangulation with n vertices

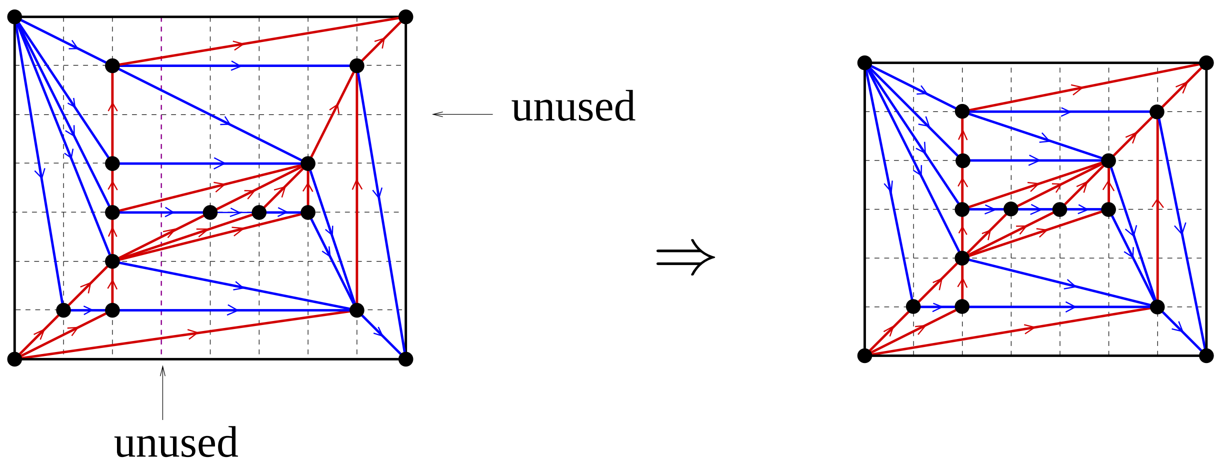
Analysis of the size of the grid using the bijection

Size of the compact drawing ?

Let T be a triangulation with n vertices endowed with its minimal transversal structure

- Unoptimized drawing: $W + H = n - 1$
- Delete unused coordinates$\Rightarrow$Compact drawing:

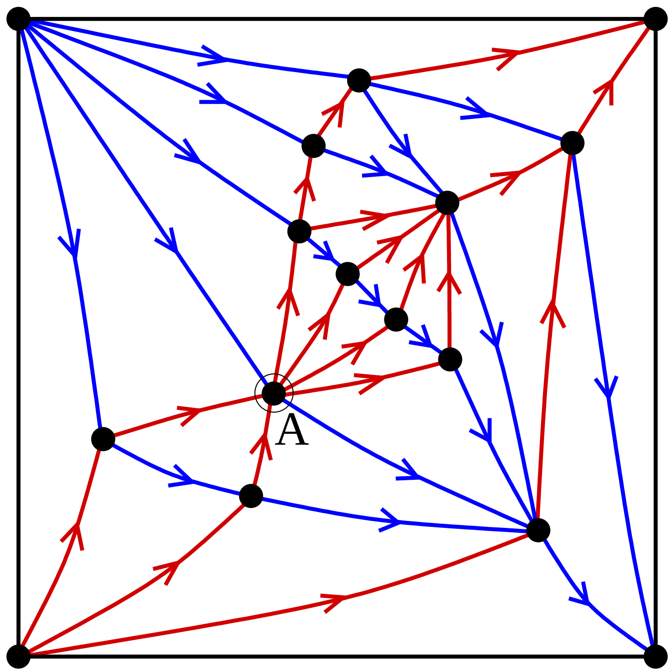
$$W_c + H_c = n - 1 - \#(unused\ coord.)$$

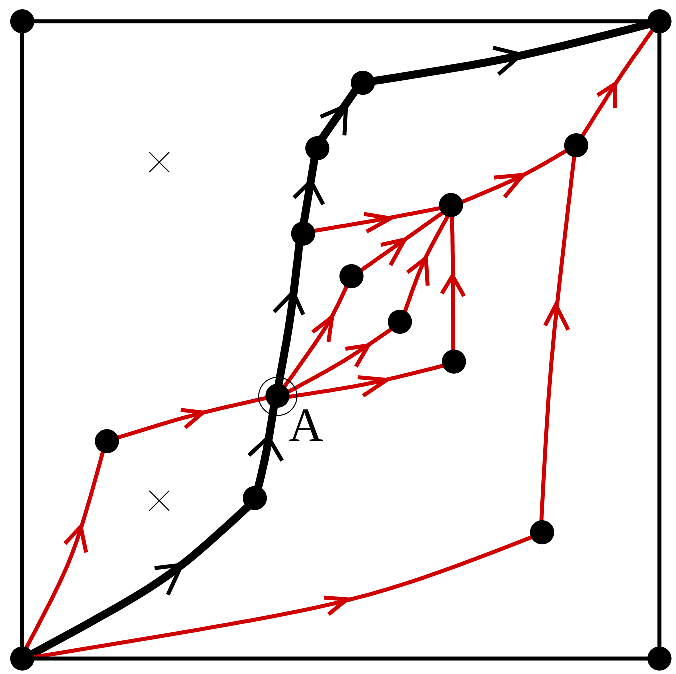


Theorem: $\#(unused\ coord.) \sim \frac{5n}{27}$ almost surely

Rule to give abscissa

The absciss of v is the number of faces of the red map on the left of $\mathcal{P}_r(v)$





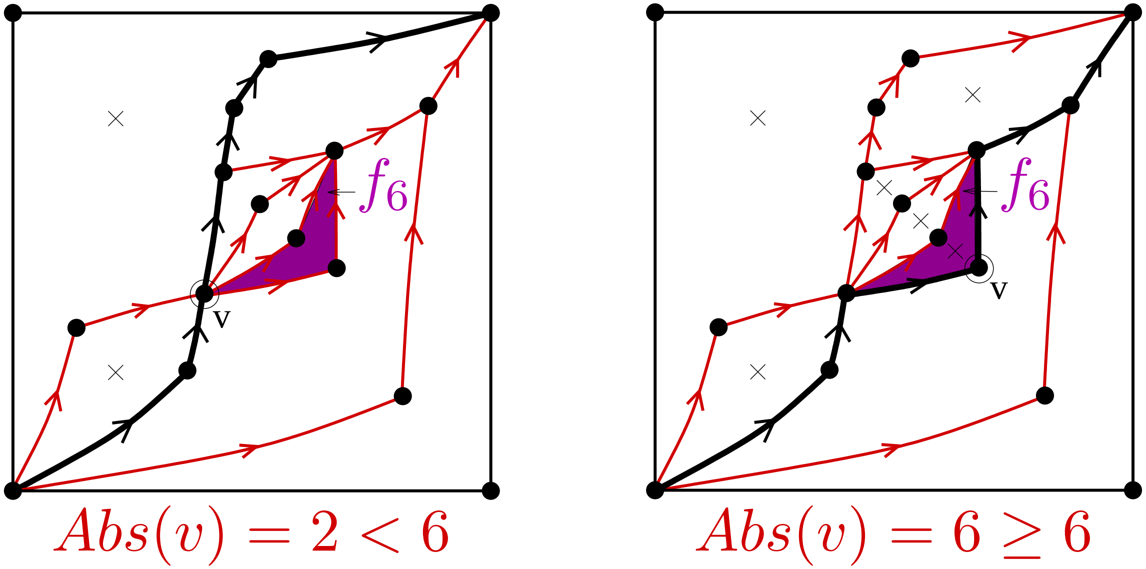
$\Rightarrow$ A has abscissa 2

Absciss $\leftrightarrow$ face of the red-map

- Let f_r be the number of faces of the red-map
- Let $i \in [1, f_r]$ be an absciss-candidate
- There exists a face f_i of the red-map such that:

$$\boxed{Abs(v) \geq i \Leftrightarrow f_i \text{ is on the left of } \mathcal{P}_r(v)}$$

Example: $i = 6$



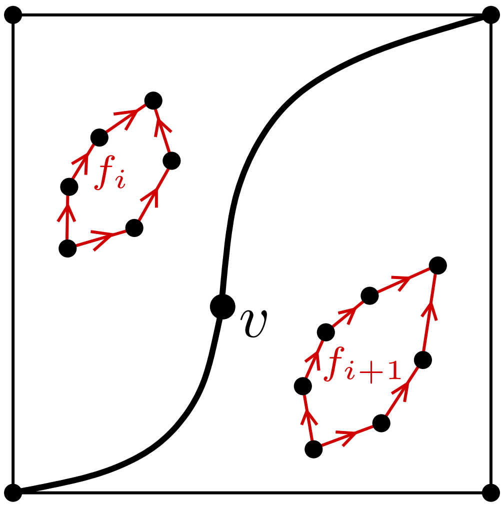
Unused abscissa

An absciss-candidate $i \in [1, f_r]$ is unused iff:

$$\boxed{Abs(v) \geq i \Rightarrow Abs(v) \geq i+1}$$

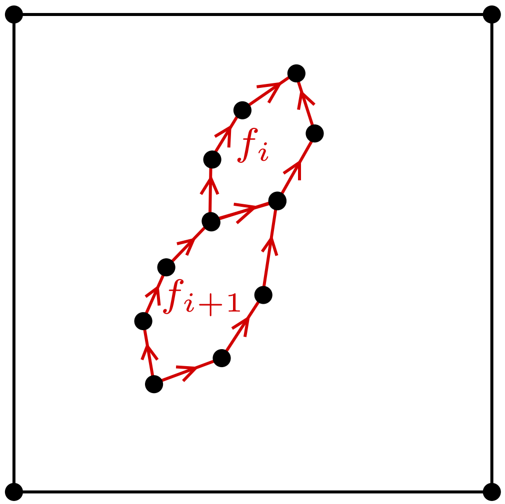
$\Rightarrow$ Faces f_i and f_{i+1} can not be separated by a path $\mathcal{P}_r(v)$

$\Rightarrow$ f_i and f_{i+1} are contiguous



CAN NOT HAPPEN

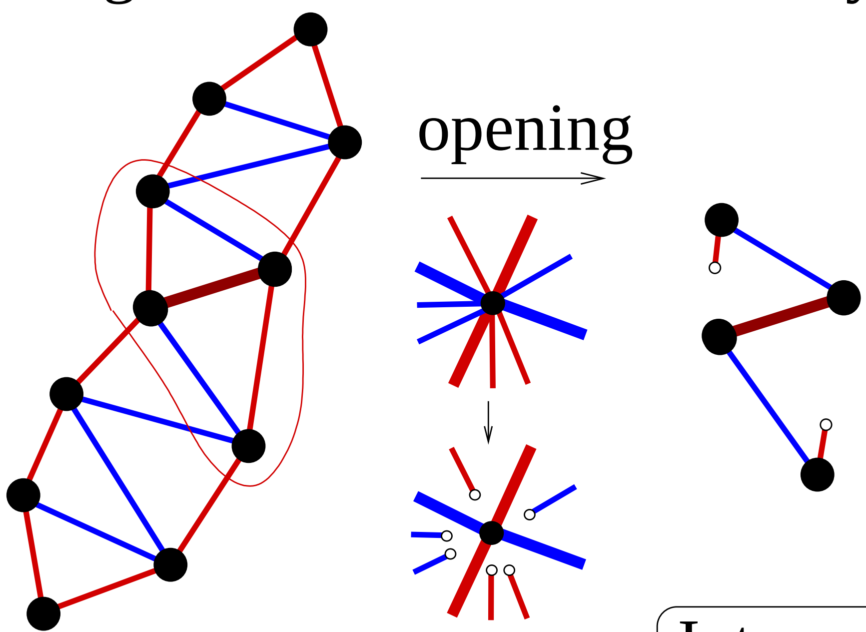
$\Rightarrow$



BottomRight(f_i)=TopLeft(f_{i+1})

Unused abscissa and opening

Triangulation ⟷ Ternary tree



Unused abscissa ⟷ Internal edge such that cw−consecutive edge at each extremity is an internal edge

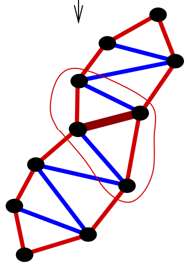
Reduction to a tree-problem

Width of the grid of the compact drawing ?

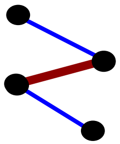
↓

How many unused abscissas in a random triangulation

↓

How many  in a random triangulation

↓

How many  in a random ternary tree

↓

$\Rightarrow \boxed{\sim \frac{1}{2}\frac{5n}{27}}$ (using generating functions)

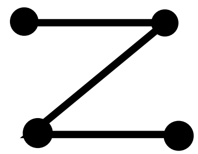
Analysis of the tree parameter

Ternary trees

- One-variable grammar $\Rightarrow$ $T(z) = \sum_n T_n z^n$

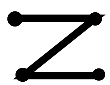
$$\mathcal{T} = \mathcal{Z} \times (1 + \mathcal{T})^3 \quad \Rightarrow \quad T(z) = z(1 + T(z))^3$$

- Two-variables grammar $\Rightarrow$ $T(z,u) = \sum_{n,k} T_{n,k} z^n u^k$

node marked by z  marked by u

- Use quasi-power theorem (Hwang, Flajolet Sedgewick)

$$\rho(u) := \mathrm{Sing}(u \to T(z,u)) \qquad -\frac{\rho'(1)}{\rho(1)} = \frac{5}{27}$$

$\Rightarrow$ The number of  is $\sim \frac{5n}{27}$ almost surely