

# Degree distribution in random Apollonian networks structures

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joint work with Michèle Soria

26 February 2007

*Systèmes*  *Implantations et*  
*Polynomiaux* **SPIRAL** *Résolutions Algébriques*

# Plan

- 1 Introduction
- 2 Properties of real-life graphs
  - Distinctive properties
  - Existing models
- 3 Random Apollonian networks
  - A bijection with ternary increasing trees
  - Random Apollonian network structures
- 4 Boltzmann sampling
  - The model
  - Generating ternary trees
- 5 Properties
  - Number of edges and connectivity
  - Degree distribution
  - Clustering and mean distance
- 6 Variants

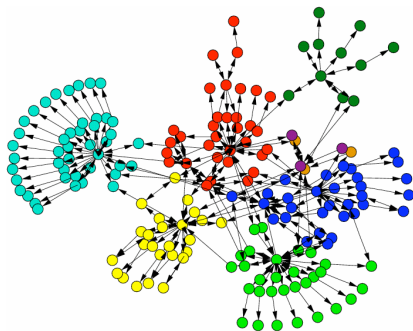
# Real-life Networks

## Application domains

- Computer Science
- Biology
- Sociology
- ...

## Models

- Needed to simulate real-life networks
- Simple classes of random graphs not a good model



Web site

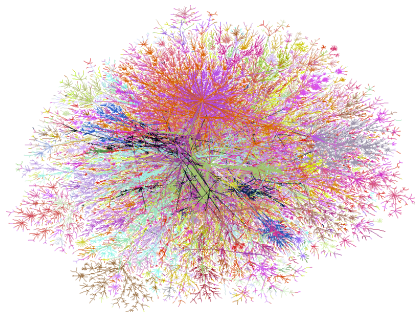
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Internet

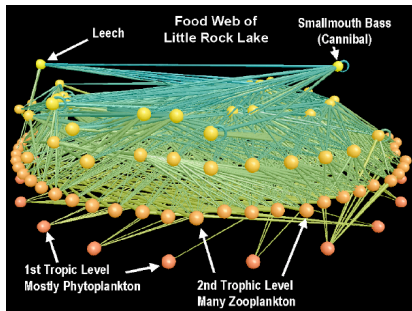
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Food web

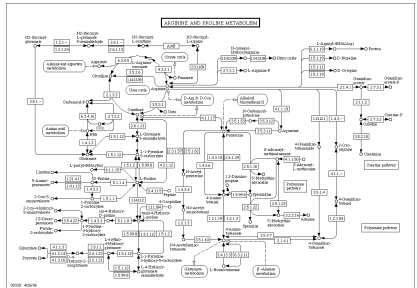
# Real-life Networks

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Metabolism

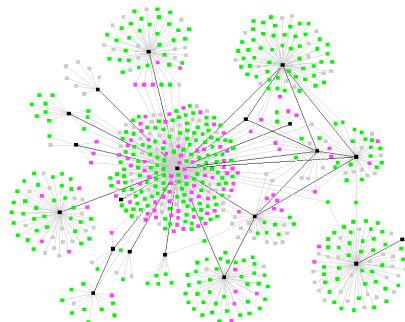
# Real-life Networks

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Contagion of diseases

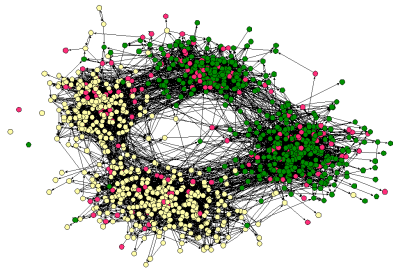
# Real-life Networks

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Friendship

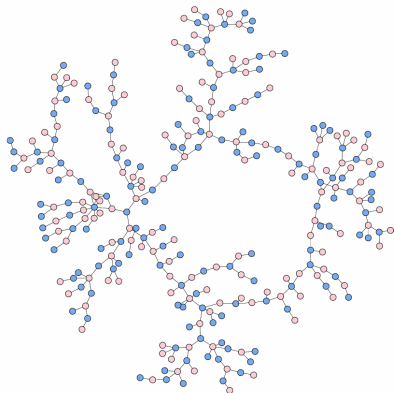
# Real-life Networks

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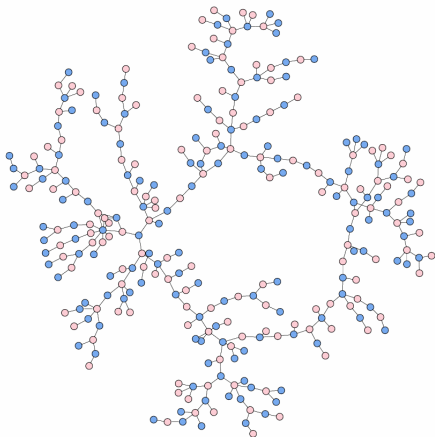
- Needed to simulate real-life networks
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Dating

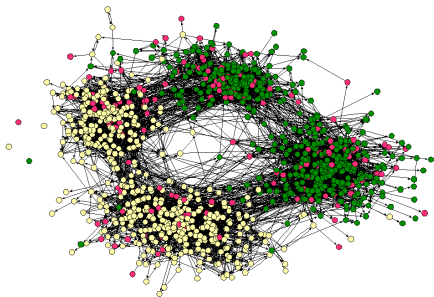
# Distinctive properties of real-life graphs

- **Number of edges**
  - Of the same order as the number of vertices
- Connectivity
- Degree distribution
- Mean distance
- Clustering



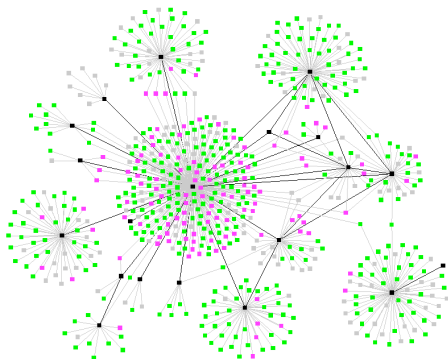
# Distinctive properties of real-life graphs

- Number of edges
- **Connectivity**
  - Strong  
(Giant component)
- Degree distribution
- Mean distance
- Clustering



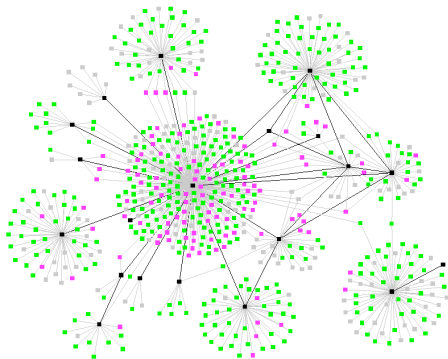
# Distinctive properties of real-life graphs

- Number of edges
- Connectivity
- Degree distribution
  - Heavy tailed  
(Power law,  
Scale-free)
- Mean distance
- Clustering



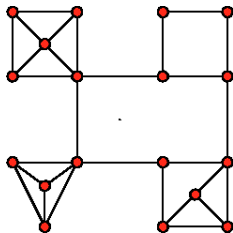
# Distinctive properties of real-life graphs

- Number of edges
- Connectivity
- Degree distribution
- Mean distance
  - Small
- Clustering

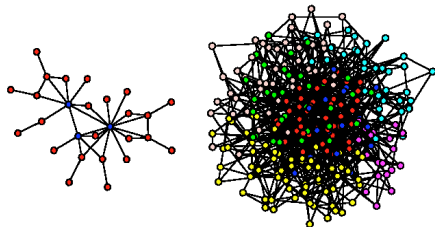


# Distinctive properties of real-life graphs

- Number of edges
- Connectivity
- Degree distribution
- Mean distance
- **Clustering**
  - Strong



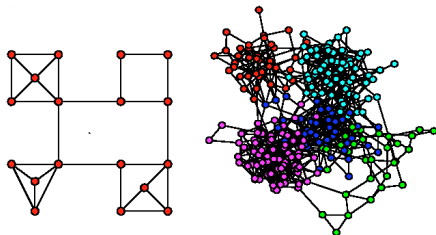
## Scale-free networks



A.-L. Barabási & R. Albert  
Emergence of scaling in random  
networks  
*Science* **286**, 509 (1999)

- Number of edges
- Connectivity
- Degree distribution
- Mean distance
- Clustering

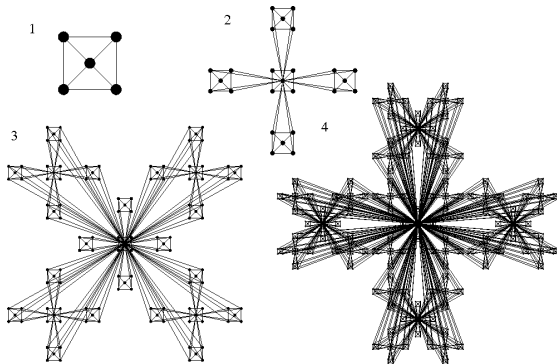
## “Small world” networks



Watts D. J. & Strogatz S. H.  
Collective dynamics of “small-world”  
networks  
Nature **393**, 440 (1998)

- Number of edges
- Connectivity
- Degree distribution
- Mean distance
- Clustering

# Hierarchical networks

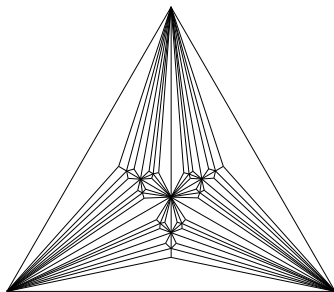


- All properties satisfied
- A new property : hierarchical modularity
- Model is deterministic



E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, A.-L. Barabási  
Hierarchical Organization of Modularity in Metabolic Networks  
Science **297**, 1551 (2002)

## A similar model - Apollonian networks



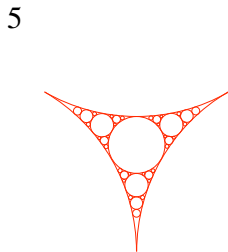
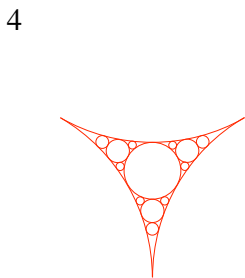
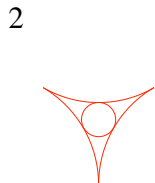
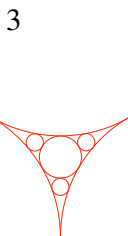
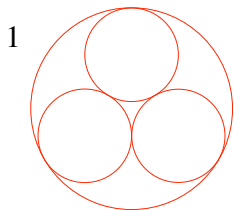
- Same properties as the hierarchical networks
- Inspired from the apollonian packings



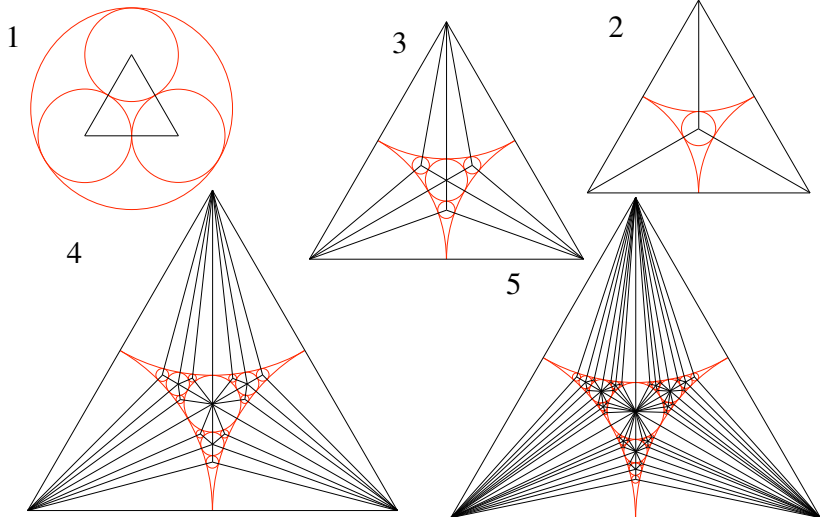
J. S. Andrade, Jr., H. J. Herrmann,  
R. F. S. Andrade & L. R. da Silva

Apollonian Networks : Simultaneously Scale-Free, Small  
World, Euclidean, Space Filling, and with Matching Graphs  
*Phys. Rev. Lett.* **94**, 018702 (2005)

# Apollonian packings, Apollonian networks



# Apollonian packings, Apollonian networks



# random Apollonian networks



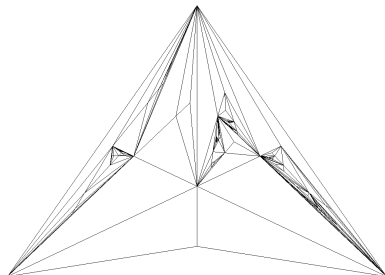
Tao Zhou, Gang Yan, & Bing-Hong Wang

Maximal planar networks with large clustering coefficient  
and power-law degree distribution

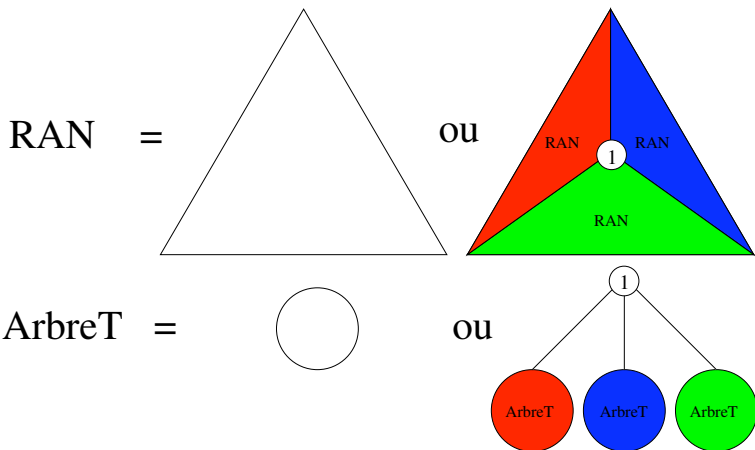
Physical Review E **71**, 046141 (2005)

### Algorithm

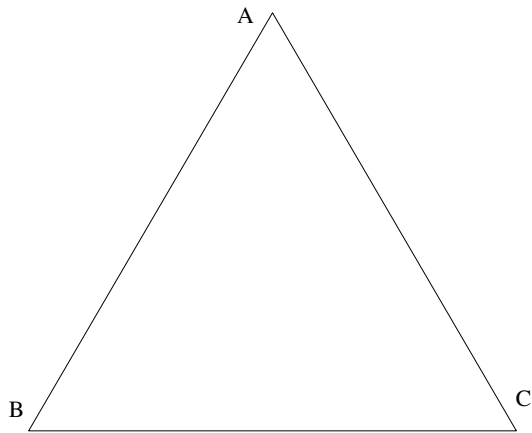
- Initial state : a triangle
- Iterative state : Choose a triangle and add to it a point and link it to the three vertices of the triangle



# A bijection with ternary increasing trees

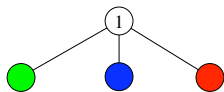
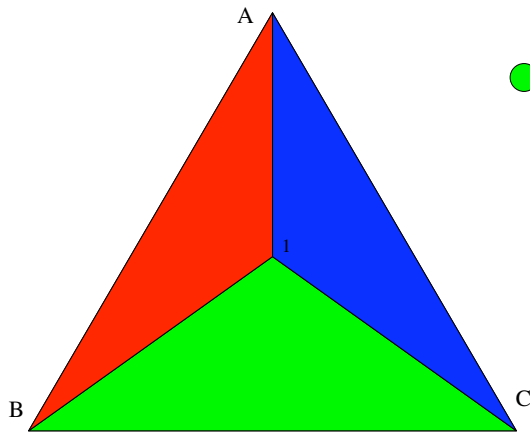


# The bijection Random Apollonian Networks $\leftrightarrow$ Ternary Increasing Trees

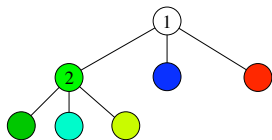
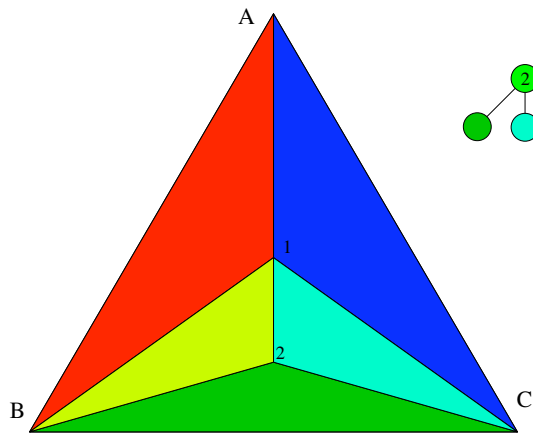


The bijection

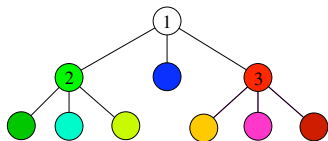
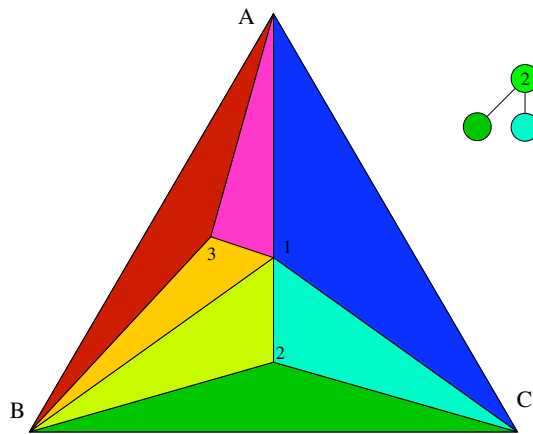
Random Apollonian Networks  $\leftrightarrow$  Ternary Increasing Trees



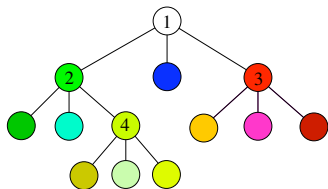
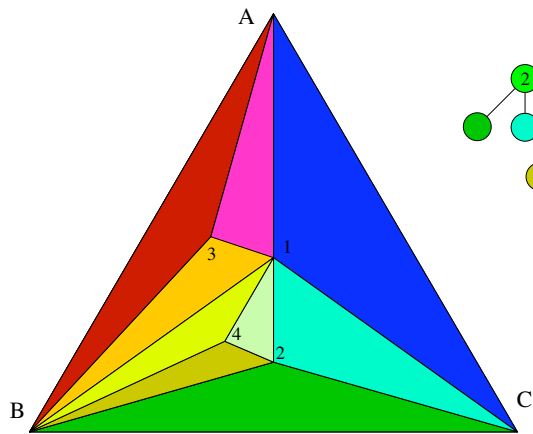
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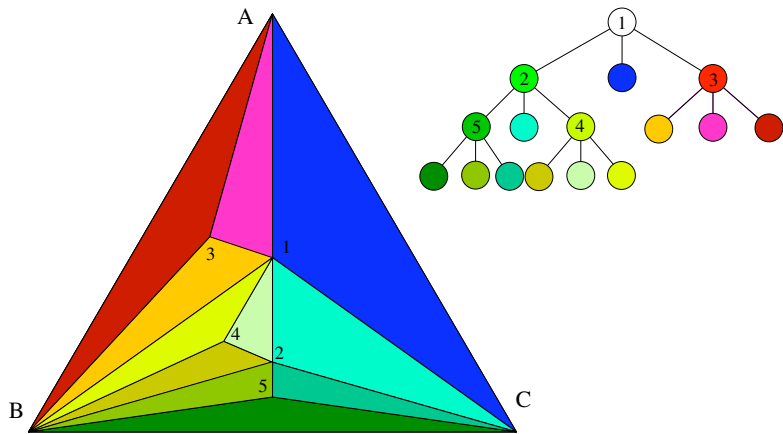
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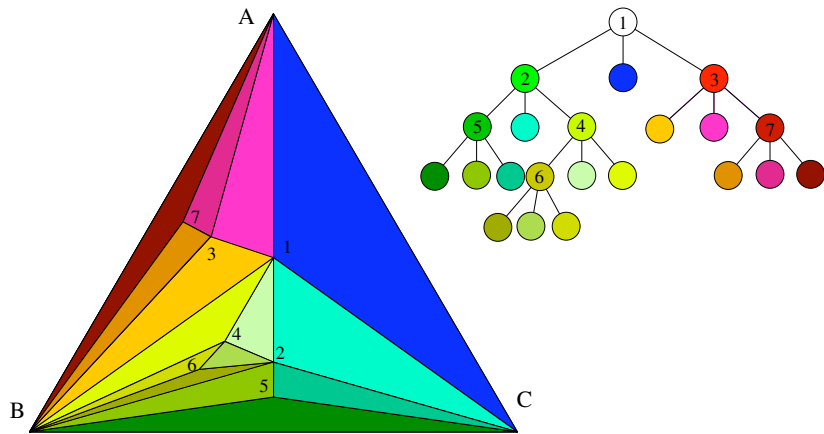


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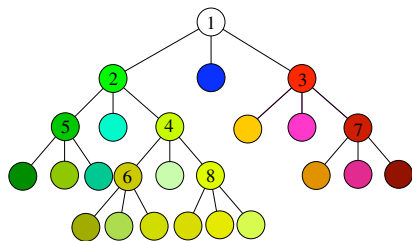
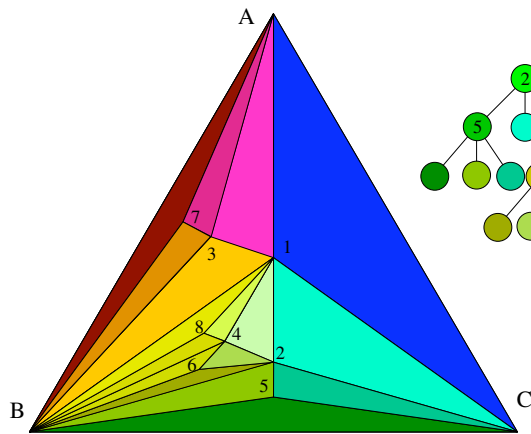




# The bijection Random Apollonian Networks $\leftrightarrow$ Ternary Increasing Trees



# The bijection Random Apollonian Networks $\leftrightarrow$ Ternary Increasing Trees



# Random Apollonian network structures

## Replace Ternary Increasing Trees with Ternary Trees

### Properties

- Same bijection
- Same class of graphs
- Different probability distribution
- Properties preserved
- Simple combinatorial description of the model

### What for ?

- General methods for sampling
- Efficient generation (Boltzmann)
- Greater flexibility

# Ternary tree generation using the Boltzmann model



# The Boltzmann model

## Specifiable combinatorial classes

- Basic operations :  
**Union, Product**, Sequence, Cycle, Set
- **Recursive definitions**

## Properties

- Uniform generation
- Approximate size
- Efficiency



P. Duchon, P. Flajolet, G. Louchard, G. Schaeffer  
Boltzmann samplers for the random generation of  
combinatorial structures

## Algorithm for the generation of a ternary tree

$$T(z) = z + zT(z)^3$$

Algorithm : TernaryTree( $\rho$ )

**if** rand(0..1) <  $\rho$  **then**

  Leaf

**else**

  Node(TernaryTree( $\rho$ ), TernaryTree( $\rho$ ), TernaryTree( $\rho$ ))

**end if**

$$\rho = x/T(x), x \leq \rho$$

- Aim at mean value :  $x < \rho$
- Singular sampling :  $x = \rho$
- Pointing

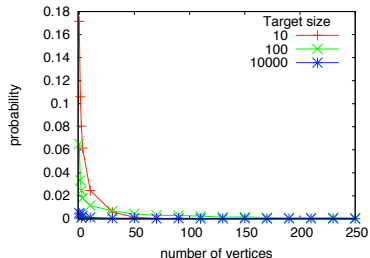
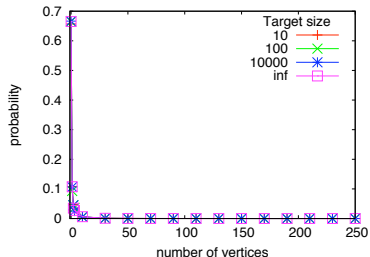
# Distribution of the sizes of the generated trees

## Using the straightforward algorithm

- Most generated trees are leaves
- A few very big trees
- Power law distribution

## With pointing

- Many small trees still present, but
- Less disparity



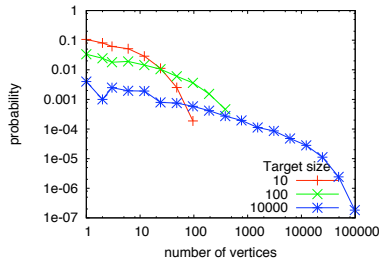
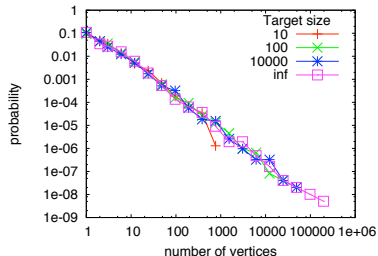
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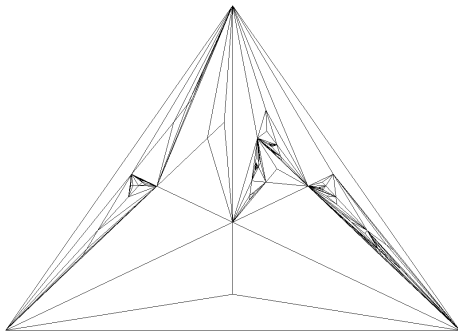
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# Back to the network properties



# Properties of the generated networks

## By construction :

- Number of edges  
Equal to  $3v - 6$ , where  $v$  the number of vertices
- Connectivity  
A single component
- Mean degree  
= 6

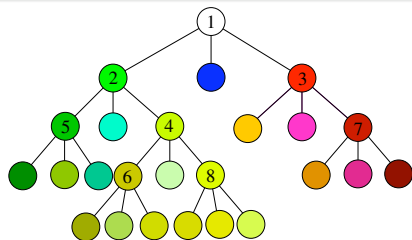
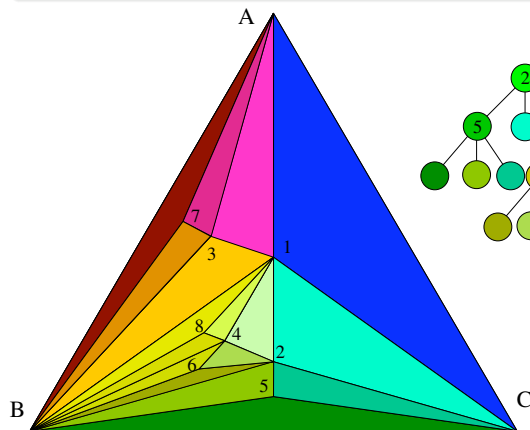
## Needing further investigation :

- Degree distribution
- Clustering
- Mean distance

# Degree

## Neighborhood of a vertex

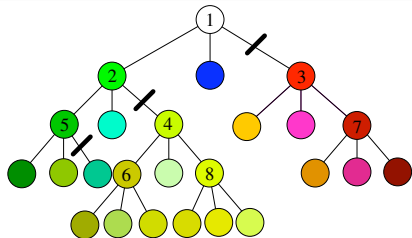
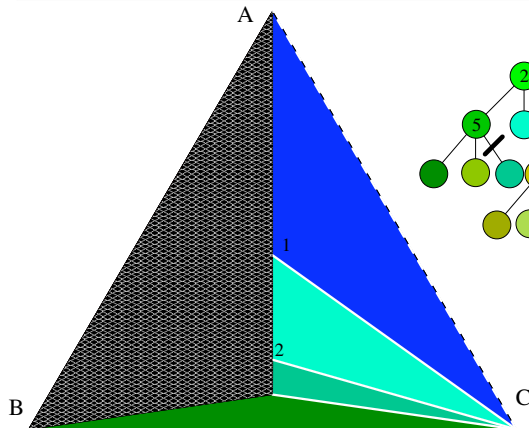
- 3 ancestors + root neighborhood



# Degree

## Neighborhood of a vertex

- 3 ancestors + root neighborhood







## Bivariate generating functions

$u$  marks the neighbors

- of the center (root) :  $RD(z, u) = zu^3 T^3(z, u)$

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# Bivariate generating functions

$u$  marks the neighbors

- of the center (root) :  $RD(z, u) = zu^3 T^3(z, u)$
- of an external node :  $T(z, u) = 1 + zuT^2(z, u)T(z)$

The distribution of the value of a parameter on Boltzmann generated objects

$$\begin{aligned}\Pr(\Omega = k) &= \sum_n \Pr(\Omega = k/N = n) \times \Pr(N = n) \\ &= \sum_n \frac{C_{n,k}}{C_n} \times \frac{C_n x^n}{C(x)} = \frac{\sum_n C_{n,k} x^n}{C(x)} = \frac{[u^k]C(x, u)}{C(x, 1)}\end{aligned}$$

# Degree distribution

## Proposition : Statistical properties

Same for :

- the set of all subtrees of a random tree
- a set of random trees independently generated with a Boltzmann sampler

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Same for :

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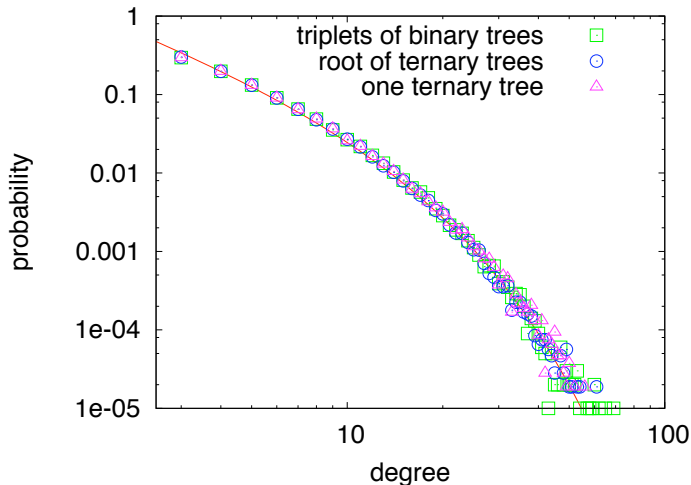
## Theorem : degree distribution in RANS

Mean value 6 and a Catalan form for the pgf :

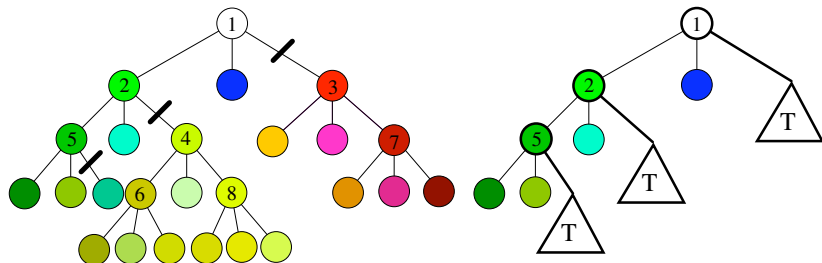
$$\Pr(D = 3 + k) = \frac{8}{9} \frac{1}{k+3} \binom{2k+2}{k} \sim C \left(\frac{8}{9}\right)^k (k+3)^{-3/2}$$

# Degree distribution

$$P(D = 3 + k) \sim C \left(\frac{8}{9}\right)^k (k + 3)^{-3/2}$$



## Sketch of proof



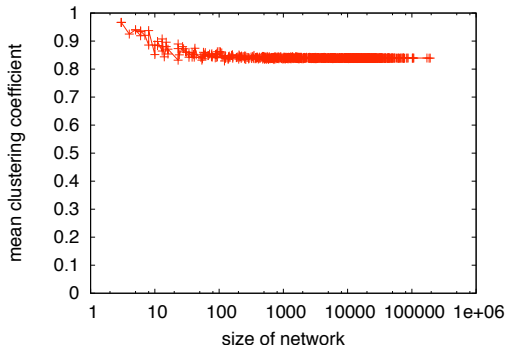
- Ternary trees marked for degree :  
 $T(z, u) = 1 + zuT^2(z, u)T(z)$
- Simulated by binary trees :  
 $T(z, u) = B(zuT(z))$ , where  $B(t) = \sum B_n t^n$
- Schema is subcritical :  $\rho_T < 1/4$
- $[u^k]B(zuT(z)) = \rho^k \tau^k \frac{1}{k+1} \binom{2k}{k}$

# Clustering

Definition : Clustering coefficient of a vertex of degree  $k$

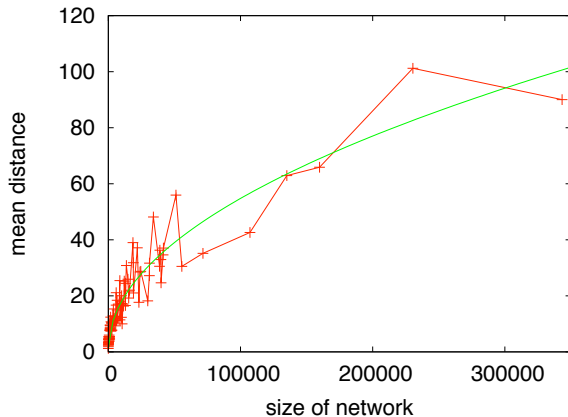
$$C(k) = \frac{\text{number of links between neighbors}}{k(k-1)}$$

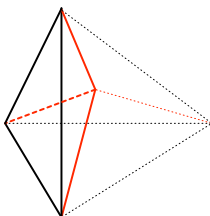
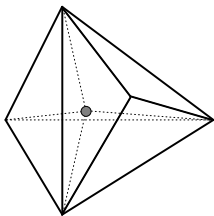
- $C(k) = 3 \frac{2k-d-1}{k(k-1)}$
- Mean value over all vertices independent of size



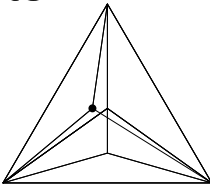
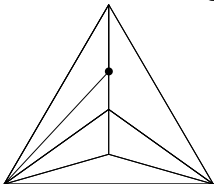
## Mean distance

Simulation confirms a small mean distance (order  $\sqrt{N}$ )

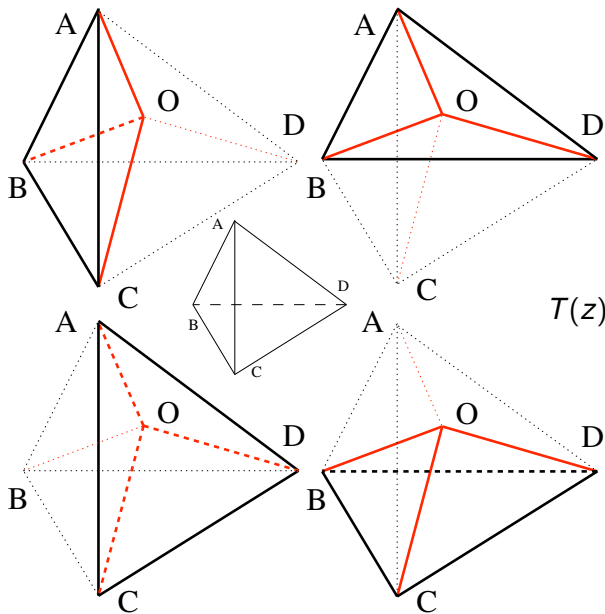




More flexibility :  
**Variants**



## Add a dimension



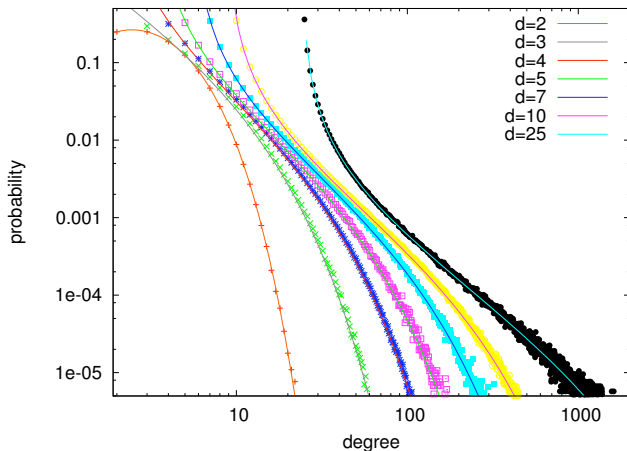
$$T(z) = z + zT(z)^4$$

# Higher dimension RANS

$$\Pr(D_d = d + k) \sim C\alpha^k \left(k + \frac{d}{d-2}\right)^{-\frac{3}{2}}$$

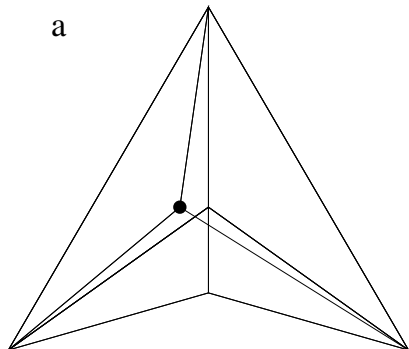
$$RD_d(z, u) = zu^d T_d^d(z, u)$$

$$T_d(z, u) = T_{d-1}(uzT_d(z))$$

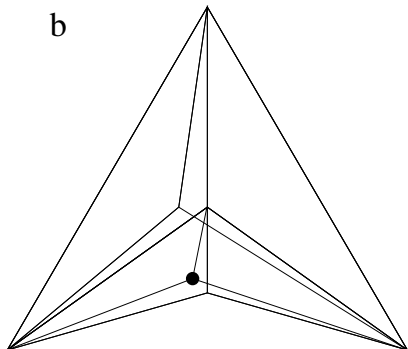


# Reuse triangles

a

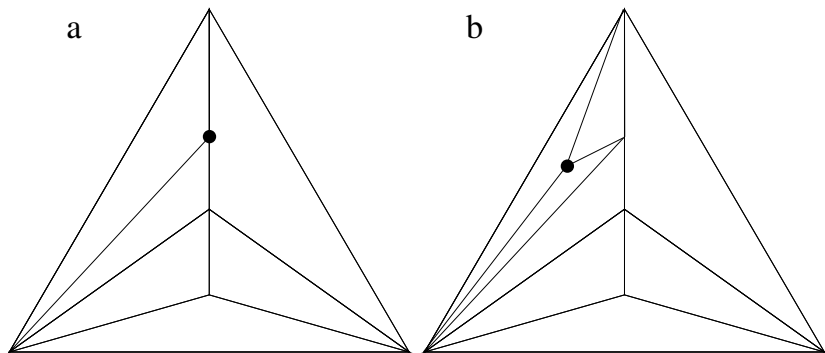


b



$$T(z) = z + zT(z)^4$$

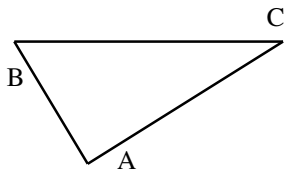
## Remove siblings



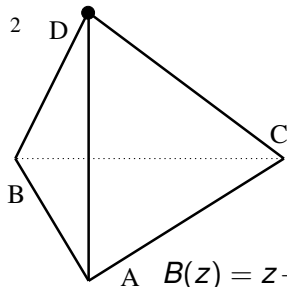
$$T(z) = z + 3zT(z) + 3zT(z)^2 + zT(z)^3$$

# Combine models

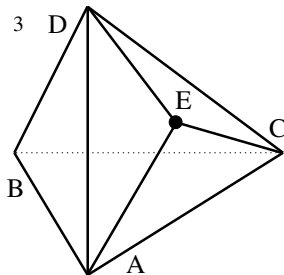
1



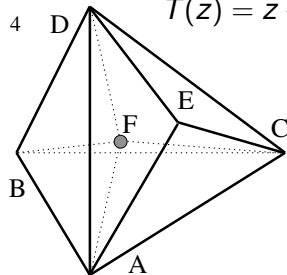
2



3



4



$$B(z) = z + zB(z)^3 + zB(z)^3T(z)$$
$$T(z) = z + zB(z)^4T(z)^4$$

# Implementation

$$T(z) = z + zT(z)^d$$

- 1000 lines of C
- Tree and Network generation
- Parameter computation

## Sampling of

$10^6$  generated trees  
 $10^6$  maximum “usable” size  
 $10^9$  maximum size  
in a few seconds time. . .

## Simple families of trees $T_i(z) = z + \phi(\langle T(z) \rangle)$

- Sampler compiler
- Written in Maple, using Combstruct
- Sampler in C, eventually using Maple as a co-routine

## Conclusion and Perspectives

- A simple and extensible model
- Similar models :
  - $k$ -trees
  - stacked triangulations
- More adequate to model real graphs ?
- Different strategies to generate trees
- Tree sampling has many more applications (eg. XML)

# Image references

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