Analysis of Approximate Median Selection

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Finding the median efficiently — a difficult problem.

A deterministic algorithm for the exact median was improved in 5/99 by Dor & Zwick, requiring (in the worst case) $\approx 2.942n$.

Extremely involved . . .

For expected number of comparisons: Floyd & Rivest showed (1975) it can be done in $(1.5+o(1))n$. Cunto & Munro (1989): this bound is tight.

Our algorithm was developed in 1998 by Cantone — and only much later we discovered that several formulated various analogues earlier — as early as 1978!

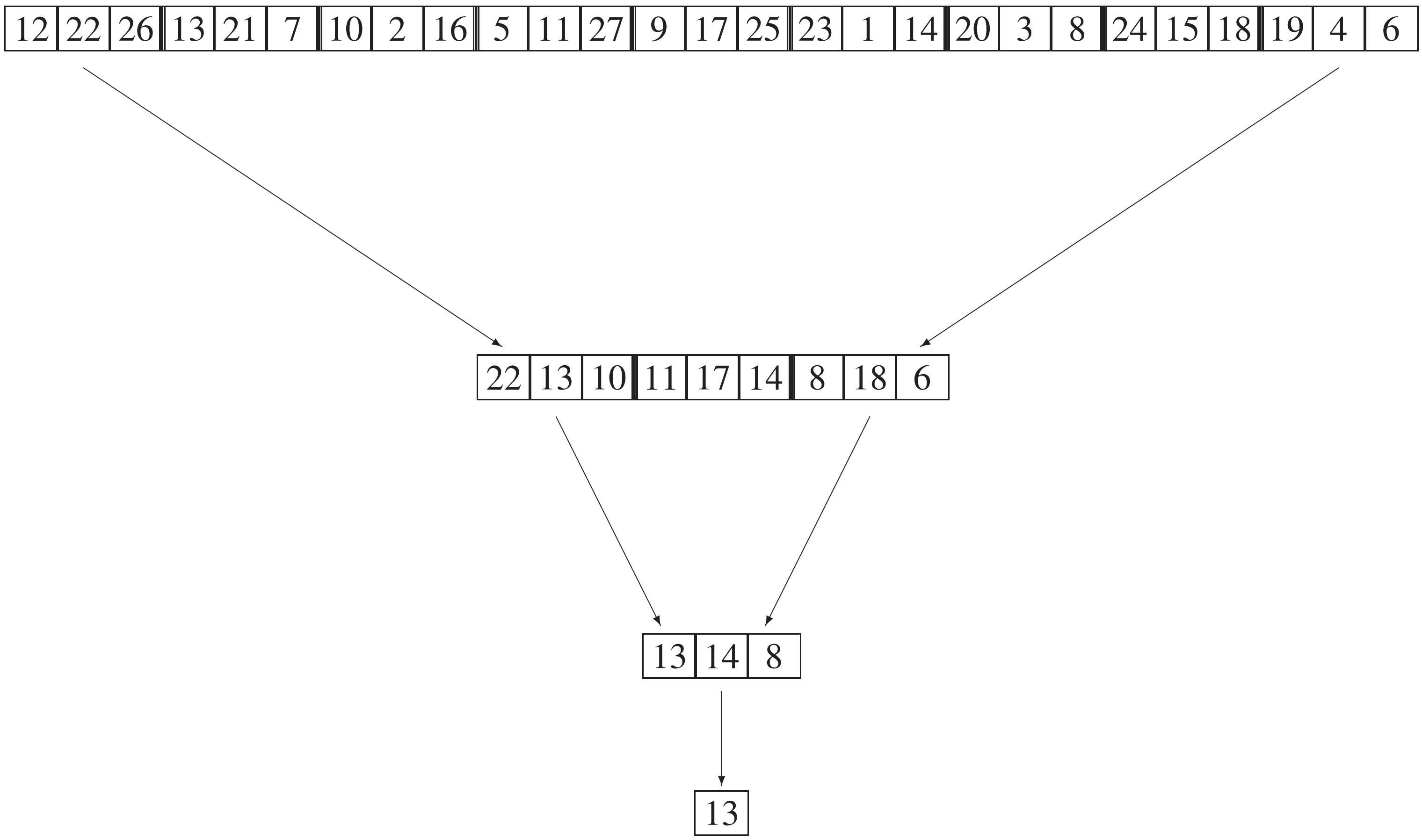
Deterministic, uses at most $1.5n$ comparisons,

and the expected number is $4/3n$.

Major virtue: extremely easy to implement

(and understand) — but it only approximates

the median.



This is performed in situ.

Essentially the same algorithm can be done "on-line:" processing a stream of values and using work-area of $4\log_3 n$ positions.

Analysis — Cost of search

Finding median of three requires

2 comparisons in 2 permutations,

3 comparisons in 4 permutations,

— out of the 6 possible permutations.

Hence $E[C_3] = 8/3$.
The expected total number of comparisons when looking in a list of size n:

$$C_3(n) = \frac{n}{3} \cdot \frac{8}{3} + C_3(\frac{n}{3}), \qquad C_3(1) = 0$$

Result: $C_3(n) = \frac{4}{3}(n-1)$.

The number of elements that are moved is similarly $E_3(n) = \frac{1}{3}(n-1)$.

The number of three-medians computed: $\frac{1}{2}(n-1)$.

Analysis — Probabilities of selection

To show that the selected median – X_n – is likely to be close to the true median we need to compute the distribution of the rank of the selected entry, X_n.

Let $n = 3^r$.

The key quantity is $q_{a,b}^{(r)} \stackrel{\text{def}}{=}$ the number of permutations, out of the $n!$ possible ones, in which the entry which is the a^{th} smallest in the array is:
(i) selected, and
(ii) has rank b (= is the b^{th} smallest) in the next set, that has $\frac{n}{3} = 3^{r-1}$ entries.

The counting is performed in two steps:
1. Count permutations in which a is chosen in the bth triplet, and all the entries chosen in the first $b-1$ triplets are smaller than a, and all the items chosen in the rightmost $n/3-b$ triplets are larger that a.

2. Compensate for this restriction: multiply the result of step one by the number of rearrangements of

such permutations: $\frac{(n/3)!}{(b-1)!(\frac{n}{3}-b)!} = \frac{n}{3}\binom{\frac{n}{3}-1}{b-1}$.

The first step is not that simple, and it produces the following expression,

$$2n(a-1)!(n-a)!3^{a-b}\sum_i \binom{b-1}{i}\binom{\frac{n}{3}-b}{a-2b-i}\frac{1}{9^i}.$$

We find:

$$q^{(r)}_{a,b} = 2n(a-1)!(n-a)!\binom{\frac{n}{3}-1}{b-1}3^{a-b-1}$$

$$\times \sum_i \binom{b-1}{i}\binom{\frac{n}{3}-br}{a-2b-i}\frac{1}{9^i}.$$

The related probability: $p^{(r)}_{a,b} = q^{(r)}_{a,b}/n!$:

$$p^{(r)}_{a,b} = \frac{2}{3}\cdot\frac{3^{-b}\binom{\frac{n}{3}-1}{b-1}}{3^{-a}\binom{n-1}{a-1}} \times \sum_i \binom{b-1}{i}\binom{\frac{n}{3}-b}{a-2b-i}\frac{1}{9^i}$$

$$= \frac{2}{3}\cdot\frac{3^{-b}\binom{\frac{n}{3}-1}{b-1}}{3^{-a}\binom{n-1}{a-1}} \times [z^{a-2b}](1+\frac{z}{9})^{b-1}(1+z)^{\frac{n}{3}-b}.$$

Finally, $P_a^{(r)}$: the probability that the algorithm chooses a from an array holding $1, \ldots, n = 3^r$.

$$\boxed{P_a^{(r)} = \sum_{b_r} p_{a,b_r}^{(r)} P_{b_r}^{(r-1)}} = \sum_{b_r, b_{r-1}, \cdots, b_3} p_{a,b_r}^{(r)} p_{b_r,b_{r-1}}^{(r-1)} \cdots p_{b_3,2}^{(2)}$$

For $2^{j-1} \leq b_j \leq 3^{j-1} - 2^{j-1} + 1$.

$$P_a^{(r)} = \left(\frac{2}{3}\right)^r \frac{3^{a-1}}{\binom{n-1}{a-1}}$$

$$\times \sum_{b_r, b_{r-1}, \cdots, b_3} \prod_{j=2}^{r} \sum_{i_j \geq 0} \binom{b_j - 1}{i_j} \binom{3^{j-1} - b_j}{b_{j+1} - 2b_j - i_j} \frac{1}{9^{i_j}}$$

$b_j \in [2^{j-1} \,..\, 3^{j-1} - 2^{j-1} + 1]$, $b_2 = 2$ and $b_{r+1} \equiv a$.

No known reduction ...

Numerical calculations produced:

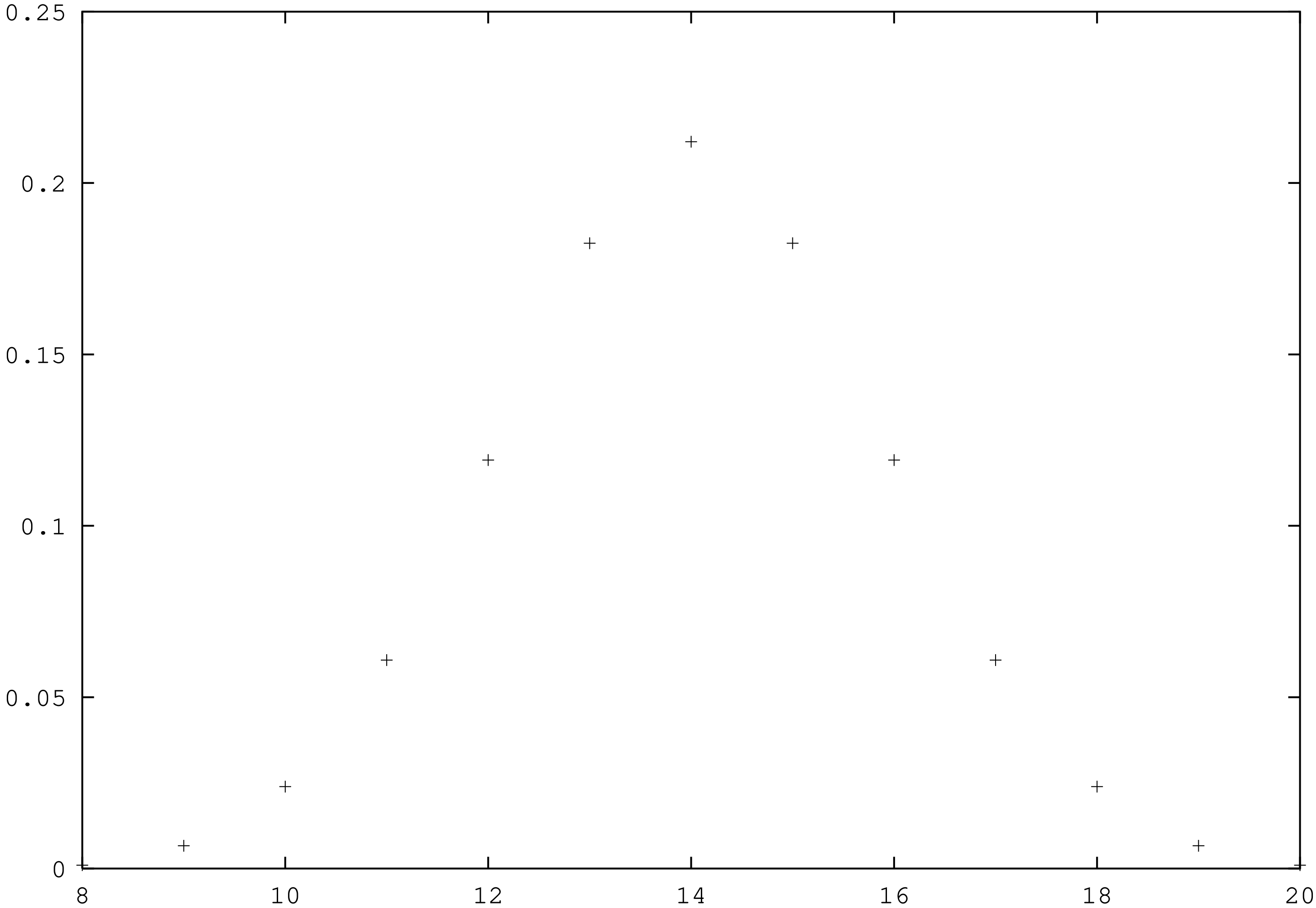
n	$r = \log_3 n$	Avg.	σ_d	$\sigma_d / n^{2/3}$
9	2	0.428571	0.494872	0.114375
27	3	1.475971	1.184262	0.131585
81	4	3.617240	2.782263	0.148619
243	5	8.096189	6.194667	0.159079
729	6	17.377167	13.282273	0.163979
2187	7	36.427027	27.826992	0.165158

Variance ratios for the median selection as function of array size

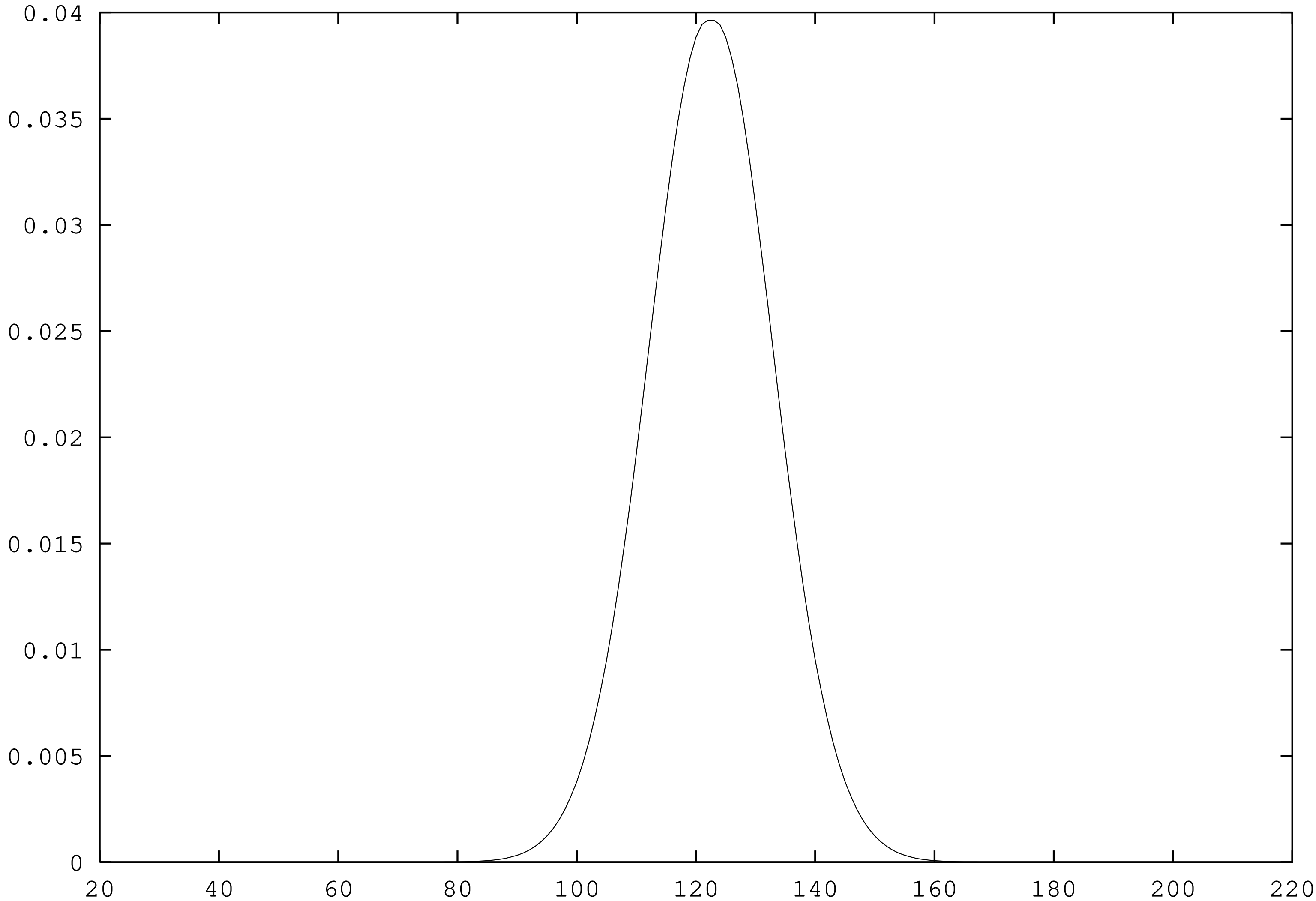
d is the error of the approximation:

$$d \equiv \left| X_n - \frac{n+1}{2} \right|$$

What can we expect when n grows?



Plot of the median probability distribution for n=27



Plot of the median probability distribution for n=243

To answer the last question we look at a "similar" situation, where we look at n independent random variables:

$$\Xi = (\xi_1, \xi_2, \ldots, \xi_n), \quad \xi_j \sim U(0,1).$$

Ξ is a permutation of their sorted order, $S(\Xi)$:

$$S(\Xi) = (\xi_{(1)} \le \xi_{(2)} \le \cdots \le \xi_{(n)}).$$

Observation:
If the Sicilian algorithm operates on this permutation of $\mathbb{N}_n$, and returns $X_n = k$,
then sicking it on Ξ would return $Y_n = \xi_{(k)}$.

The idea: Y_n tracks $\frac{X_n}{n}$, but—due to the indpendence of the n variables ξ_i—it has a simpler distribution.

How good is the tracking? Condition on the sampled value:

$$E_S\left[\left(Y_n-\frac{1}{2}\right)-\left(\frac{X_n-\frac{n+1}{2}}{n}\right)\right]^2 = E_S\left[Y_n-\frac{X_n-1/2}{n}\right]^2$$

$$= E_k\left[\xi_{(k)}-\frac{k-1/2}{n}\right]^2 \lesssim \frac{1}{4n}.$$

And the variance of $|D_n|/n$ is larger, and decreases more slowly!

We said Y_n is simpler... How simple is it?

$F_r(x) \equiv \Pr(Y_n - 1/2 \le x), \qquad -1/2 \le x \le 1/2, \qquad n = 3^r.$

$F_0(x) = x + 1/2.$

Now we need a recurrence:

Y_{3n} is the median of 3 independent values $\sim Y_n$, hence

$$\begin{aligned} F_{r+1}(x) &= \Pr(Y_{3n} \le x+1/2) = 3F_r^2(x)(1-F_r(x)) + F_r^3(x) \\ &= 3F_r^2(x) - 2F_r^3(x). \end{aligned}$$

A simpler form is obtained by shifting $F_r(\cdot)$ by 1/2;

$$G_r(x) \equiv F_r(x) - 1/2 \quad \Longrightarrow \quad G_0(x) = x,$$

We get our first key equation:

$$\boxed{G_{r+1}(x) = \tfrac{3}{2}G_r(x) - 2G_r^3(x).}$$

But it is not interesting! it is satisfied by

$$G_r(x) = \begin{cases} -\frac{1}{2} & x < a \\ 0 & x = a \\ \frac{1}{2} & x > 0 \end{cases}$$

This says: $\frac{D_n}{n} \longrightarrow 0, \qquad D_n \stackrel{\text{def}}{=} X_n - \frac{n+1}{2}.$

Need change of scale.

We showed,

$$\mu^{2r} E\left[\left(Y_n - \tfrac{1}{2}\right) - D_n/n\right]^2 \longrightarrow 0 \qquad \forall \mu \in [0, \sqrt{3}).$$

Hence we can track $\mu^r(D_n/n)$ with $\mu^r(Y_n - 1/2)$.

We pick a convenient value, $\mu = 3/2$ and show:

Theorem [Svante Janson]

Let $n = 3^r$, $r \in \mathbb{N}$.

X_n — approximate median of random permutation of N_n.

Then a random variable X exists, such that

$$\left(\frac{3}{2}\right)^r \frac{X_n - \frac{n+1}{2}}{n} \quad \longrightarrow \quad X,$$

where X has the distribution $F(\cdot)$;

with the same shift

$$F(x) \equiv G(x) + 1/2,$$

we get the equation

$$G(\frac{3}{2}x) = \frac{3}{2}G(x) - 2G^3(x), \qquad -\infty < x < \infty$$

Moreover:

The distribution function $F(\cdot)$ is strictly increasing throughout.

The value 3/2 is inherent in the problem!

The proof of the Theorem uses the technical lemma

Lemma Let $a \in (0, \infty)$ and ϕ that maps $[0, a]$ into $[0, a]$
For $x > a$ we define $\phi(x) = x$. Assume

(i) $\phi(0) = 0$
(ii) $\phi(a) = a$
(iii) $\phi(x) > x$, for all $x \in (0, a)$.
(iv) $\phi'(0) = \mu > 1$, and continuous there;
$\phi(\cdot)$ is continuous and strictly increasing on $[0, a)$.
(v) $\phi(x) < \mu x, \quad x \in (0, a)$.

Let $\phi_r(t) = \phi(\phi_{r-1}(t))$, the rth iterate of $\phi(\cdot)$.
Then

$$\text{as } r \longrightarrow \infty, \qquad \phi_r(x/\mu^r) \longrightarrow \psi(x), \qquad x \geq 0.$$

$\psi(x)$ is well defined, strictly monotonic increasing for all x, increases from 0 to a, and satisfies the equation $\psi(\mu x) = \phi(\psi(x))$.

Proof:
From Property (v): $\qquad \phi(x/\mu^{r+1}) < x/\mu^r,$
Since iteration preserves monotonicity,

$$\phi_{r+1}(x/\mu^{r+1}) = \phi_r(\phi(x/\mu^{r+1})) < \phi_r(x/\mu^r).$$

Hence a limit $\psi(\cdot)$ exists.

The properties of $\psi(x)$ depend on the behavior of $\phi(\cdot)$ near $x = 0$.
Since $\phi'(x)$ is continuous at $x = 0$,
$\psi(\cdot)$ is continuous throughout. Since it is bounded, the convergence is uniform on $[0, \infty]$. Hence, since $\phi(\cdot)$ and all its iterates are strictly monotonic, so is $\psi(\cdot)$ itself.

We have then the equation

$$G(\frac{3}{2}x) = \frac{3}{2}G(x) - 2G^3(x), \qquad -\infty < x < \infty$$

but we have no explicit solution for it.

What can we do?

Several things.
We can calculate a power expansion for it; From $G_0(\cdot)$ and the iteration, all $G_r(\cdot)$ are odd, hence we can write

$$G(x) = \sum_{k \geq 1} b_k x^{2k-1}.$$

b_1 is avaiable from the iteration: The derivatives of $G_r(x/\mu^r)$ are all 1, hence this is also the derivative there of $G(x)$.

Successive calculations are easy:

k	b_k
1	$1.00000000000000 \times 10^{+00}$
2	$-1.06666666666667 \times 10^{+00}$
3	$1.05025641025641 \times 10^{+00}$
4	$-8.42310905468800 \times 10^{-01}$
5	$5.66391554459281 \times 10^{-01}$
6	$-3.29043692201665 \times 10^{-01}$
7	$1.69063219329527 \times 10^{-01}$
8	$-7.82052123482121 \times 10^{-02}$
9	$3.30170547707520 \times 10^{-02}$
10	$-1.28576608229956 \times 10^{-02}$
11	$4.65739657183461 \times 10^{-03}$
12	$-1.57980373987906 \times 10^{-03}$
13	$5.04579631846217 \times 10^{-04}$
14	$-1.52443954167610 \times 10^{-04}$
15	$4.37348017371645 \times 10^{-05}$
20	$-4.33903859413399 \times 10^{-08}$
25	$1.70629958951577 \times 10^{-11}$
30	$-3.20126276232555 \times 10^{-15}$
40	$-1.94773425996709 \times 10^{-23}$
50	$-1.85826863188012 \times 10^{-32}$
60	$-4.03988860877434 \times 10^{-42}$

The fit of $F(\cdot)$—calculated using the first 150 b_k—to the distribution of X_n/n is poor for n in the low hundreds but improves very fast.

We show an example later.

Fact: it is very close to Normal, with mean zero and $\sigma = 1/\sqrt{2\pi}$ – but not quite!

We can investigate how similar it is by looking at the tail of the distribution — the complementary function $g(x) \equiv 1 - F(x)$.
It satisfies

$$g(x\mu) = 3g^2(x) - 2g^3(x) \Longrightarrow 3g(x\mu) = (3g(x))^2 \left(1 - \frac{2}{3}g(x)\right).$$

since $0 < g(x) < 1$:

$$\frac{1}{3}(3g(x))^2 < 3g(x\mu) < (3g(x))^2$$

This is all we need in order to show that the tail of the distribution of X_n/n is

$$e^{-dt^v} < g(t) < \frac{1}{3}e^{-ct^v} \quad c \approx 3.8788, \quad d \approx 4.9774 \quad v \approx 1.70951139.$$

$$c = \ln(1 - F(1)); \quad v = \ln 2/\ln(3/2), \quad d = c + \ln 3,$$

whereas the Normal distribution decays much faster: its tail is

$$1 - \Phi(x) \sim e^{-0.5x^2}/(2\pi x).$$

Example: From simulation, at $n = 1000$, 95% of the values of D_{1000} fell in the interval $[-58, 58]$.
From the "tracking claim" we have $D_n \sim nX/\mu^r$.

Also $n/\mu^r = 3^r/(3/2)^r = 2^r = n^{\ln(2)/\ln(3)} \approx n^{0.63092975}$.

And then
$\Pr[|D_n| \le d] \approx \Pr[|X| \le d\mu^r/n]$
$\Longrightarrow \Pr[|D_{1000}| \le 58] \approx \Pr[|X| \le 0.7424013] \approx 0.934543$.

This was calculated using the power series development.

When using the upper bound on the tail we similarly find
$\Pr[|D_{1000}| \le 58] = 1 - 2g(0.7424013)$

$$\approx 1 - \frac{2}{3}\exp\left(-3.878797 \times 0.7424^{1.7095113}\right) \approx 0.935205.$$

Open problems:

1. A better characterization of the solution for $G(x)$.

2. An explicit value for the variance of the limiting distribution; From the relation $\mu^r(Y_n - 1/2) \xrightarrow{d} X$ we can numerically iterate the transformation and find that it is about 2–3% larger than $1/2\pi$, but an exact value is not easy.

3. A much taller order: compute the quality of a derivative algorithm, that produces an approximate fractile $X_{k/n}$ for any $1 \leq k \leq n$.

This can be done by filtering the initial values: for example, by picking the 23rd from each set of 28 initial values, and then finding their median, we approximates the fractile $X_{0.8n}$ of the original data.