Toward Formal Simplification of Parametric Algebraic Equations via their Lie Point Symmetries

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Outline

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Verhulst's model

Population growth with linear predation

$$\begin{cases} \frac{dx}{dt} = x(a - bx) - cx, \\ \frac{da}{dt} = \frac{db}{dt} = \frac{dc}{dt} = 0. \end{cases} \tag{1}$$

model's description

- $x(t)$ represents a species population at time t;
- dx/dt is its change rate;
- $(a - bx)$ is a per capita birth rate where
  - a denotes the fertility rate;
  - b denotes environment carrying capacity;
- c is a predation rate.

Verhulst's model

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model's characteristics (modeling standpoint)

- only the difference $(a - c)$ between fertility and predation rate is significant. These parameters should be lumped together;
- models should be expressed in dimensionless form
  - used units in the analysis are then unimportant;
  - adjectives small and large have a definite relative meaning;
  - the number of relevant parameters is reduces to *dimensionless groupings* that determine the dynamics;
- using dimensional analysis

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model's characteristics (Lie point symmetries standpoint)

- one parameter group of translations

$$T_\lambda : \left\{\begin{array}{lllllll} t & \rightarrow & t & a & \rightarrow & a+\lambda & \\ x & \rightarrow & x & b & \rightarrow & b & \end{array}\right. \quad c \rightarrow c+\lambda$$

- 2-parameters group of scalings

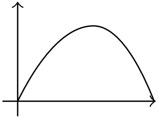
$$S_{\lambda_1,\lambda_2} : \left\{\begin{array}{llllll} t & \rightarrow & t/\lambda_2 & a & \rightarrow & \lambda_2 a \\ x & \rightarrow & \lambda_1 x & b & \rightarrow & \lambda_2 b/\lambda_1 \end{array}\right. \quad c \rightarrow \lambda_2 c$$

Verhulst's model

Population growth with linear predation (canonical form)

$$\frac{d\mathbf{x}}{d\mathbf{t}} = \mathbf{x}(1 - \mathbf{x}).$$

$x = 1$ is a stable fixed point ($1 - 2x < 0$),
$x = 0$ is unstable ($1 - 2x > 0$).



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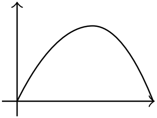

Simplification: from original model to canonical one

Canonical model is obtained after the change of variables:

$$\mathbf{t} = (a - c)t, \quad \mathbf{x} = \frac{b}{a - c}x.$$

Thus $x = (a - c)/b$ is a stable fixed point of the original model if the equalities $0 < b < 2(a - c)$ hold.

These computations can be done in polynomial time w.r.t. inputs size

Theorem (Hubert, Sedoglavic 2006)

Let Σ be a differential system bearing on n state variables and depending on ℓ parameters that is coded by a straight-line program of size L.

There exists a probabilistic algorithm that determines if a Lie point symmetries group of Σ composed of dilatation and translation exists; in that case, a rational set of invariant coordinates is computed and Σ is rewrite in this set with a reduced number of parameters.

The arithmetic complexity of this algorithm is bounded by

$$\mathcal{O}\Big((n+\ell+1)\big(L+(n+\ell+1)(2n+\ell+1)\big)\Big).$$

Simple application of Lie's theory

If one

- considers parameters θ as constant variables $\mathrm{d}\theta/\mathrm{d}t = 0$ i.e. considers extended Lie symmetries;
- uses classical Lie's theory i.e. Lie symmetries and their invariants;
- knows what type of symmetries could be used,

one can

- unify (and extend?) available simplification methods that seem (to me) based on rules of thumb;
- obtain simple effective and efficient algorithmic tools.

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Outline

Representation of point transformation

Vector field representation

$$\left\{ \begin{array}{rcl} dz_1/d\varepsilon & = & g_1(z_1, \ldots, z_n), \\ & \vdots & \\ dz_n/d\varepsilon & = & g_n(z_1, \ldots, z_n). \end{array} \right.$$

Power series representation

$$\left\{ \begin{array}{rcl} z_1(\varepsilon) & = & z_1(0) + g_1(z_1, \ldots, z_n)\varepsilon + \mathcal{O}(\varepsilon^2), \\ & \vdots & \\ z_n(\varepsilon) & = & z_n(0) + g_n(z_1, \ldots, z_n)\varepsilon + \mathcal{O}(\varepsilon^2). \end{array} \right.$$

Representation of point transformation

Infinitesimal representation i.e. derivations acting on the field $\mathbb{K}(z_1, \ldots, z_n)$

$$\delta = \sum_{i=1}^{n} g_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}.$$

Closed form representation (if any) $\sigma(z) = \sum_{i \in \mathbb{N}} \delta^i(z)/i!$

$$\sigma \left\{ \begin{array}{ccl} z_1 & \rightarrow & \zeta_1(z_1, \ldots, z_n), \\ & \vdots & \\ z_n & \rightarrow & \zeta_n(z_1, \ldots, z_n). \end{array} \right.$$

Examples of point transformation

Vector field representation (translation)

$$\begin{cases} dz_1/d\varepsilon &= \alpha_{z_1}, \\ &\vdots \\ dz_n/d\varepsilon &= \alpha_{z_n}, \end{cases}$$

where the α_{z_i}s are numerical constant.

Power series representation

$$\begin{cases} z_1(\varepsilon) &= z_1(0) + \alpha_{z_1}\varepsilon, \\ &\vdots \\ z_n(\varepsilon) &= z_n(0) + \alpha_{z_n}\varepsilon. \end{cases}$$

Examples of point transformation

Infinitesimal representation i.e. derivations acting on the field $\mathbb{K}(z_1, \ldots, z_n)$

$$\varepsilon \sum_{i=1}^{n} \alpha_{z_i} \frac{\partial}{\partial z_i},$$

where the α_{z_i}s are numerical constant.

One-parameter group of $\mathbb{K}(z_1, \ldots, z_n)$ automorphisms

$$\left\{ \begin{array}{ccc} z_1 & \rightarrow & z_1 + \alpha_{z_1}\varepsilon, \\ & \vdots & \\ z_n & \rightarrow & z_n + \alpha_{z_n}\varepsilon. \end{array} \right.$$

Examples of point transformation

Vector field representation (scaling)

$$\begin{cases} dz_1/d\varepsilon &=& \alpha_{z_1} z_1, \\ & \vdots & \\ dz_n/d\varepsilon &=& \alpha_{z_n} z_n, \end{cases}$$

where the α_{z_i}s are numerical constant.

Power series representation

$$\begin{cases} z_1(\varepsilon) &=& z_1(0) + \alpha_{z_1} z_1 \varepsilon + \mathcal{O}(\varepsilon^2), \\ & \vdots & \\ z_n(\varepsilon) &=& z_n(0) + \alpha_{z_n} z_n \varepsilon + \mathcal{O}(\varepsilon^2). \end{cases}$$

Examples of point transformation

Infinitesimal representation i.e. derivations acting on the field $\mathbb{K}(z_1, \ldots, z_n)$

$$\varepsilon \sum_{i=1}^{n} \alpha_{z_i} z_i \frac{\partial}{\partial z_i},$$

where the α_{z_i}s are numerical constant.

One-parameter group of $\mathbb{K}(z_1, \ldots, z_n)$ automorphisms

$$\left\{ \begin{array}{ccc} z_1 & \rightarrow & z_1 \lambda^{\alpha_{z_1}}, \\ & \vdots & \\ z_n & \rightarrow & z_n \lambda^{\alpha_{z_n}}, \end{array} \right. \quad \lambda = \exp(\varepsilon).$$

Determining system

$$\mathfrak{D} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad \delta = \sum_{\rho \in (t,X,P)} \phi_\rho \frac{\partial}{\partial \rho}, \ \phi_\rho \in \mathbb{K}(t, X, P).$$

δ is a symmetry of $\mathfrak{D} \Leftrightarrow [\mathfrak{D}, \delta] = \delta \circ \mathfrak{D} - \mathfrak{D} \circ \delta = \lambda \mathfrak{D}, \lambda \in \mathbb{K}$ i.e.

$$\begin{aligned}
-\frac{\partial \phi_t}{\partial t} - \sum_{i=1}^{n} f_i \frac{\partial \phi_t}{\partial x_i} &= \lambda, \\
\sum_{\rho \in (t,X,P)} \phi_\rho \frac{\partial f_i}{\partial \rho} - \frac{\partial \phi_{x_i}}{\partial t} - \sum_{j=1}^{n} f_j \frac{\partial \phi_{x_i}}{\partial x_j} &= \lambda f_i, \quad \forall i \in \{1, \ldots, n\}, \\
-\frac{\partial \phi_{p_i}}{\partial t} - \sum_{j=1}^{n} f_j \frac{\partial \phi_{p_i}}{\partial x_j} &= 0, \quad \forall i \in \{1, \ldots, m\}.
\end{aligned}$$

There is little hope to solve such a general PDE system.

Determining system

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$$\begin{pmatrix} \frac{\partial f_1}{\partial t} & \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial p_1} & \cdots & \frac{\partial f_1}{\partial p_m} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial t} & \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial p_1} & \cdots & \frac{\partial f_n}{\partial p_m} \end{pmatrix} \begin{pmatrix} \alpha_t \\ \alpha_{x_1} \\ \vdots \\ \alpha_{x_n} \\ \alpha_{p_1} \\ \vdots \\ \alpha_{p_m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

After specialisation of X and P in $\mathbb{K}$, this system is solve by numerical Gaussian elimination in $\mathbb{K}$.

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After specialisation of X in $\mathbb{K}$, this system is solve by polynomial Gaussian elimination in $\mathbb{K}(P)$.

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$$\begin{pmatrix} (t+f_1)\frac{\partial f_1}{\partial t} & (x_1 - f_1)\frac{\partial f_1}{\partial x_1} & \dots & x_n \frac{\partial f_1}{\partial x_n} & p_1 \frac{\partial f_1}{\partial p_1} & \dots & p_m \frac{\partial f_1}{\partial p_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ (t+f_n)\frac{\partial f_n}{\partial t} & x_1 \frac{\partial f_n}{\partial x_1} & \dots & (x_n - f_n)\frac{\partial f_n}{\partial x_n} & p_1 \frac{\partial f_n}{\partial p_1} & \dots & p_m \frac{\partial f_n}{\partial p_m} \end{pmatrix}$$

After specialisation of X and P in $\mathbb{K}$, this system is solve by numerical Gaussian elimination in $\mathbb{K}$.

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After specialisation of X in $\mathbb{K}$, this system is solve by polynomial Gaussian elimination in $\mathbb{K}(P)$.

Higher orders constraints

Generically, there is no symmetries; but previously we consider systems composed of $(n + \ell + 1)$ unknowns and n relations!

One can consider a prolongated field $\mathbb{K}\langle t, X, \Theta\rangle$ and induced derivations ($\mathfrak{S}$ is supposed to be a scaling):

$$\mathfrak{D}_\infty = \mathfrak{D} + \sum_{j \in \mathbb{N}^\star \setminus \{1\}} \sum_{i=1}^{n} \mathfrak{D}^j f_i \frac{\partial}{\partial x_i^{(j)}}, \quad \mathfrak{S}_\infty = \mathfrak{S} + \sum_{j \in \mathbb{N}^\star} \sum_{\rho \in (X, \Theta)} (\alpha_\rho - j\alpha_t)\rho^{(j)} \frac{\partial}{\partial \rho^{(j)}}$$

to obtain an infinite determining system $[\mathfrak{S}_\infty, \mathfrak{D}_\infty] = \lambda \mathfrak{D}_\infty$.

Nevertheless, computations does not rely on power series expansion but only on multiple specialisation.

Some examples

Consider the system

$$\left\{ \begin{array}{rcl} & \dot{a} = \dot{b} = \dot{c} = \dot{d} = 0, & \\ \dot{x} & = & c(x - x^3/3 - y + d), \\ \dot{y} & = & (x + a - by)/c, \end{array} \right.$$

it's infinitesimal symmetries form a vector field spanned by:

$$\frac{\partial}{\partial y} + b\frac{\partial}{\partial a} + \frac{\partial}{\partial d},$$

and the associated one-parameters group of automorphisms:

$$\begin{array}{rcl} y & \rightarrow & y + \lambda, \\ a & \rightarrow & a + b\lambda, \\ d & \rightarrow & d + \lambda. \end{array}$$

Some examples

Consider the system

$$\left\{ \begin{array}{l} \dot{a} = \dot{b} = \dot{c} = \dot{d} = 0, \ \dot{u} \neq 0, \\ \dot{x} \ = \ u - (a+c)x + by, \\ \dot{y} \ = \ ax - (b+d)y, \end{array} \right.$$

it's infinitesimal symmetries form a vector field spanned by:

$$y\frac{\partial}{\partial y} + a\frac{\partial}{\partial a} - a\frac{\partial}{\partial c} - b\frac{\partial}{\partial b} + b\frac{\partial}{\partial d},$$

and the associated one-parameters group of automorphisms:

$$\begin{array}{rcl} a & \rightarrow & \lambda a, \\ y & \rightarrow & \lambda y, \\ b & \rightarrow & b/\lambda, \end{array} \qquad \begin{array}{rcl} c & \rightarrow & c + (1 - 1/\lambda)a, \\ d & \rightarrow & d + (1 - 1/\lambda)b. \end{array}$$

Some examples

Consider the system

$$\left\{\begin{array}{rcl} & \dot{a} = \dot{b} = \dot{d} = \dot{c} = \dot{e} = 0, & \\ \dot{x} & = & (c - a - dx)x + by, \\ \dot{y} & = & ax - (e + b)y, \end{array}\right.$$

it's infinitesimal symmetries form a vector field spanned by:

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - d\frac{\partial}{\partial d},$$
$$y\frac{\partial}{\partial y} + a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} + a\frac{\partial}{\partial c} + b\frac{\partial}{\partial e},$$
$$a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c} + d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} - t\frac{\partial}{\partial t},$$

and the associated 3-parameters group of automorphisms:

$$\begin{array}{rclrcl} t & \rightarrow & t/\lambda_3, & b & \rightarrow & \lambda_3 b/\lambda_2, \\ x & \rightarrow & \lambda_1 x, & c & \rightarrow & \lambda_3 c + \lambda_3 a(\lambda_2 - 1), \\ y & \rightarrow & \lambda_1\lambda_2 y, & d & \rightarrow & \lambda_3 d/\lambda_1, \\ a & \rightarrow & \lambda_3 a, & e & \rightarrow & \lambda_3 e + \lambda_3 b(1 - 1/\lambda_2). \end{array}$$

Outline

Symmetries of Verhulst's model

$$\begin{aligned}
\mathcal{T} &= \frac{\partial}{\partial a} + \frac{\partial}{\partial c}, \\
\mathcal{S}_1 &= t\frac{\partial}{\partial t} - a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c}, \\
\mathcal{S}_2 &= x\frac{\partial}{\partial x} - b\frac{\partial}{\partial b}.
\end{aligned}$$

We consider the Lie algebra spanned by these generators. Its commutation table is:

	$\frac{\partial}{\partial t}$	$\mathcal{D}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_2$
$\frac{\partial}{\partial t}$	0	0	0	$\frac{\partial}{\partial t}$	0
$\mathcal{D}$	0	0	0	$-\mathcal{D}$	0
$\mathcal{T}$	0	0	0	$-\mathcal{T}$	0
$\mathcal{S}_1$	$-\frac{\partial}{\partial t}$	$\mathcal{D}$	$\mathcal{T}$	0	0
$\mathcal{S}_2$	0	0	0	0	0

Symmetries of Verhulst's model

This commutation table show that $\mathrm{Der}(\mathbb{K}(t, X, \Theta)/\mathbb{K})$ is solvable:

$$\{0\} \subset \mathcal{L}\left(\frac{\partial}{\partial t}, \mathcal{D}\right) \subset \mathcal{L}\left(\frac{\partial}{\partial t}, \mathcal{D}, \mathcal{T}\right) \subset \underset{\displaystyle \mathrm{Der}(\mathbb{K}(t, X, \Theta)/\mathbb{K}).}{\mathcal{L}\left(\tfrac{\partial}{\partial t}, \mathcal{D}, \mathcal{T}, \mathcal{S}_2, \mathcal{S}_1\right) =}$$

Groups of automorphisms are associated to these algebras; each group have an invariant field:

$$\mathbb{K}(t, x, a, b, c) \hookleftarrow \mathbb{K}(t, x, a, b, c)^{T_\lambda} \hookleftarrow \left(\mathbb{K}(t, x, a, b, c)^{T_\lambda}\right)^{S_{\lambda_1, \lambda_2}} \hookleftarrow \{0\}.$$

Using notations:

$$\mathfrak{a} = a - c, \quad \mathfrak{x} = bx/\mathfrak{a}, \quad \mathfrak{t} = (a - c)\, t,$$

we have

$$\mathbb{K}(t, x, a, b, c) \hookleftarrow \mathbb{K}(t, x, \mathfrak{a}, b) \hookleftarrow \mathbb{K}(\mathfrak{t}, \mathfrak{x}) \hookleftarrow \{0\}.$$

Geometric point of view

Theorem

Let M be a smooth n-dimensional manifold. Suppose G is a local transformation group that acts regularly on M with s-dimensional orbits. There exist a smooth $(m-s)$-dimensional manifold M/G (the quotient of M by G's orbits) together with a projection $\pi : M \to M/G$ such that:

- *π is a smooth map between manifolds;*
- *points x and y lie in the same orbit of G in M if, and only if, the relation $\pi(x) = \pi(y)$ holds;*
- *if $\mathfrak{g}$ denotes the Lie algebra of infinitesimal generators of G's action then the linear map $\mathrm{d}\pi : TM|_x \to T(M/G)|_{\pi(x)}$ is onto, with kernel $\mathfrak{g}|_x = \{s|_x | s \in \mathfrak{g}\}$.*

Furthermore, local coordinates on the quotient manifold M/G are provided by a complete set of functionally independant invariants for the group action.

Mainly, there is just 2 main geometric remarks:

- by rewriting original system in an invariants coordinates set, we reduce the number of parameters.
- these computations—invariants' computation and system rewriting—could be done in a single step.

All forthcoming is classical application of invariant theory.

We are going to recall some fact from invariant theory and illustrate these assertions by an example (for general case—i.e. use of Gröbner bases—see Hubert Kogan 2005).

Example: scaling treatment

We focus our attention on parameters and on the graph of automorphisms' action:

$$\forall y \in (\Theta), \quad \sigma_{(\lambda_1,\ldots,\lambda_m)}(y) = y \prod_{i=1}^{m} \lambda_i^{a_{y,i}}$$

By classical canonical homomorphism, these multiplicative relations could be considered as a module represented by the following matrix (the determinant of the submatrix $(a_{\theta_i,j})_{i=1,\ldots m}^{j=1,\ldots,m}$ is supposed different from 0):

$$\begin{array}{ccccccccc} \lambda_1 & \ldots & \lambda_m & \theta_1 & \ldots & \theta_\ell & \sigma(\theta_1) & \ldots & \sigma(\theta_\ell) \end{array}$$

$$\left(\begin{array}{ccccccc|ccc}
a_{\theta_1,1} & \ldots & a_{\theta_1,m} & 1 & 0 & \ldots & 0 & 1 & \ldots & 0 \\
a_{\theta_2,1} & \ldots & a_{\theta_2,m} & 0 & 1 & & & & \vdots & \\
\vdots & & \vdots & \vdots & & \ddots & & & & \\
a_{\theta_{\ell-1},1} & \ldots & a_{\theta_{\ell-1},m} & & & 1 & 0 & & \ddots & \\
a_{\theta_\ell,1} & \ldots & a_{\theta_\ell,m} & 0 & \ldots & 0 & 1 & & & 1
\end{array}\right).$$

A Gaussian elimination performed on this matrix and terminated at $m+1$ column position leads to the matrix :

$$\left(\begin{array}{ccccccccc}
1 & 0 & \dots & 0 & \gamma_{1,\theta_1} & \dots & \gamma_{1,\theta_\ell} & \\
0 & \ddots & \ddots & \vdots & \vdots & & \vdots & \\
\vdots & \ddots & \ddots & 0 & \vdots & & \vdots & \\
0 & \dots & 0 & 1 & \gamma_{m,\theta_1} & \dots & \gamma_{m,\theta_\ell} & \\
\hline
0 & & \dots & 0 & \beta_{m+1,\theta_1} & \dots & \beta_{m+1,\theta_\ell} & \text{representation} \\
\vdots & & & \vdots & \vdots & & \vdots & \text{of} \\
0 & & \dots & 0 & \beta_{\ell,\theta_1} & \dots & \beta_{\ell,\theta_\ell} & \text{orbits}
\end{array}\right).$$

This computation is sufficient to determine the following generators of the multiplicative set of rational invariants:

$$\sigma_{(\lambda_1,\dots,\lambda_m)}\left(\prod_{j=1}^{\ell} {\theta_j}^{\beta_{h,\theta_j}}\right) = \prod_{j=1}^{\ell} {\theta_j}^{\beta_{h,\theta_j}}, \quad h = m+1, \dots, \ell.$$

A Gaussian elimination performed on this matrix and terminated at $m+1$ column position leads to the matrix :

$$\left(\begin{array}{ccccccc|c} 1 & 0 & \dots & 0 & \gamma_{1,\theta_1} & \dots & \gamma_{1,\theta_\ell} & \text{cross section} \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots & \text{of orbits are} \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots & \text{chosen there} \\ 0 & \dots & 0 & 1 & \gamma_{m,\theta_1} & \dots & \gamma_{m,\theta_\ell} & \\ \hline 0 & & \dots & 0 & \beta_{m+1,\theta_1} & \dots & \beta_{m+1,\theta_\ell} & \\ \vdots & & & \vdots & \vdots & & \vdots & \\ 0 & & \dots & 0 & \beta_{\ell,\theta_1} & \dots & \beta_{\ell,\theta_\ell} & \end{array}\right).$$

$$\sigma_{(\lambda_1,\dots,\lambda_m)}(\theta_i) = \theta_i \prod_{h=1}^{m} \left(\prod_{j=1}^{\ell} \theta_j^{-\gamma_{h,\theta_j}} \right)^{a_{\theta_i,h}} = \theta_i \prod_{j=1}^{\ell} \theta_j^{-\sum_{h=1}^{m} a_{\theta_i,h}\gamma_{h,\theta_j}} = 1.$$

Example of application

Let us consider the following two-species oscillator:

$$\left\{\begin{array}{rcl} \mathrm{d}x/\mathrm{d}t & = & a - k_1 x + k_2 x^2 y, \\ \mathrm{d}y/\mathrm{d}t & = & b - k_2 x^2 y, \\ \mathrm{d}a/\mathrm{d}t = \dot{b} = \dot{k}_1 = \dot{k}_2 = 0. & & \end{array}\right.$$

One can remark that the following two parameters group of scale symmetries:

$$\begin{array}{rcl} t & \rightarrow & \lambda\, t, \\ x & \rightarrow & \lambda\mu\, x, \\ y & \rightarrow & \lambda\mu\, y, \end{array} \qquad \begin{array}{rcl} a & \rightarrow & \mu a, \\ b & \rightarrow & \mu b, \\ k_1 & \rightarrow & k_1/\lambda, \\ k_2 & \rightarrow & k_2/\lambda^3\mu^2 \end{array}$$

leaves invariant solutions of this system.

Using previous computational strategy, one can deduce that the specialization:

$$
\begin{array}{rcl}
t & \rightarrow & \mathfrak{t} = k_1\, t, \\
x & \rightarrow & \mathfrak{x} = k_1 \left(\sqrt{{k_1}^3/k_2}\right) x, \\
y & \rightarrow & \mathfrak{y} = k_1 \left(\sqrt{{k_1}^3/k_2}\right) y,
\end{array}
\quad
\begin{array}{rcl}
a & \rightarrow & \mathfrak{a} = \left(\sqrt{{k_1}^3/k_2}\right) a, \\
b & \rightarrow & \mathfrak{b} = \left(\sqrt{{k_1}^3/k_2}\right) b, \\
k_1 & \rightarrow & 1 = k_1/k_1, \\
k_2 & \rightarrow & 1 = k_2/k_1^3 \left(\sqrt{{k_1}^3/k_2}\right)^2
\end{array}
$$

leads to the system:

$$
\left\{
\begin{array}{rcl}
\mathrm{d}\mathfrak{x}/\mathrm{d}\mathfrak{t} & = & \mathfrak{a} - \mathfrak{x} + \mathfrak{x}^2\mathfrak{y}, \\
\mathrm{d}\mathfrak{y}/\mathrm{d}\mathfrak{t} & = & \mathfrak{b} - \mathfrak{x}^2\mathfrak{y}.
\end{array}
\right.
$$

The choice of an invariant coordinates set is arbitrary.
Assuming that $\beta := b/a > 0, \kappa := k_2/(a^2 {k_1}^3) > 0$, we have

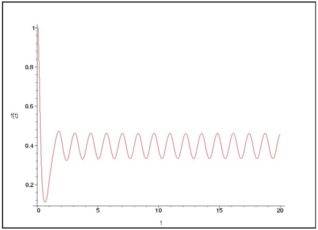
$$\begin{aligned} \xi_0 &= \beta + 1, & \det &= \kappa(\beta+1)^2, \\ \chi_0 &= \frac{\beta}{(\beta+1)^2\kappa}, & \text{trace} &= \frac{\beta - \kappa(\beta+1)^3 - 1}{\beta+1}. \end{aligned}$$

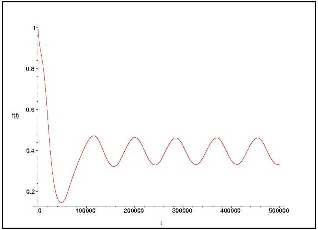
There is a bifurcation for $\kappa = (\beta-1)/(\beta+1)^3$. We perform another change of variable $\kappa = (\beta-1)/(\beta+1)^3 + \epsilon$. If $\epsilon < 0$, the fixed point is attractive; otherwise, according to Poincaré-Bendixon theorem, our system presents a limit cycle.

$$\left\{ \begin{array}{rcl} \mathrm{d}\xi/\mathrm{d}\tau &=& 1 - \xi + \frac{\beta-1}{(\beta+1)^3}\xi^2\chi + \epsilon\xi^2\chi, \\ \mathrm{d}\chi/\mathrm{d}\tau &=& \beta - \frac{\beta-1}{(\beta+1)^3}\xi^2\chi - \epsilon\xi^2\chi, \\ \dot{\epsilon} = \dot{\beta} = 0, && \end{array} \right. \qquad \epsilon = \frac{k_2}{a^2 {k_1}^3} + a^2\frac{a-b}{(a+b)^3}.$$

One can use ϵ as a perturbing parameter for a Poincaré-Lindstedt expansion.

If one can deduce a parameters' set for which the system oscillate:





one can change its oscillation period using time dilatation:

$$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - k_1\frac{\partial}{\partial k_1} - 3k_2\frac{\partial}{\partial k_2}.$$

(work in progress with F. Lemaire and A. Ürgüplü)

Outline

Determining system for Lie symmetries of polynomial parametric systems

One can seek for Lie point symmetries of polynomial parametric system $F(X, \Theta) = 0$ as follow:

$$X(\epsilon) = X + \epsilon \Phi_X(X, \Theta) + \mathcal{O}(\epsilon^2), \quad \Theta(\epsilon) = \Theta + \epsilon \Phi_\Theta(X, \Theta) + \mathcal{O}(\epsilon^2), \qquad F(X(\epsilon), \Theta(\epsilon)) = 0 + \mathcal{O}(\epsilon^2).$$

In that case, infinitesimal generators is $\delta = \sum_{\rho \in (t, X, \Theta)} \phi_\rho \frac{\partial}{\partial \rho}$, and determining equations are:

$$\delta F = \frac{\partial F}{\partial X} \Phi_X(X, \Theta) + \frac{\partial F}{\partial \Theta} \Phi_\Theta(X, \Theta) = 0 \bmod F(X, \Theta),$$

and their computation could be done by polynomial elimination. Rewriting of original system in an invariant coordinates set could also be done by elimination (see Hubert Kogan 2005).

Example

Let us apply the presented method to the following positive dimensional system:

$$\begin{cases} s_i = \sin\theta_i, \\ c_i = \cos\theta_i, \\ \ell_1 c_1 + \ell_2 c_2 = x, \\ \ell_1 s_1 + \ell_2 s_2 = y, \\ {c_1}^2 + {s_1}^2 = 1, \\ {c_2}^2 + {s_2}^2 = 1. \end{cases}$$

$$c_1\frac{\partial}{\partial x} + s_1\frac{\partial}{\partial y} + \frac{\partial}{\partial \ell_1},$$
$$c_2\frac{\partial}{\partial x} + s_2\frac{\partial}{\partial y} + \frac{\partial}{\partial \ell_2},$$
$$s_1\frac{\partial}{\partial c_1} - c_1\frac{\partial}{\partial s_1} + \ell_1 s_1\frac{\partial}{\partial x} - \ell_1 c_1\frac{\partial}{\partial y},$$
$$s_2\frac{\partial}{\partial c_2} - c_2\frac{\partial}{\partial s_2} + \ell_2 s_2\frac{\partial}{\partial x} - \ell_2 c_2\frac{\partial}{\partial y}.$$

$$\begin{aligned} y &\to y + \lambda_1 s_1 + \lambda_2 s_2, \\ x &\to x + \lambda_1 c_1 + \lambda_2 c_2, \\ \ell_1 &\to \ell_1 + \lambda_1, \\ \ell_2 &\to \ell_2 + \lambda_2. \end{aligned}$$

$$r_1 = y - s_1(\ell_1 - 1), \quad r_2 = x - c_1(\ell_1 - 1).$$

$$\begin{cases} c_1 + \ell_2 c_2 = r_2, \\ s_1 + \ell_2 s_2 = r_1, \\ {c_1}^2 + {s_1}^2 = 1, \\ {c_2}^2 + {s_2}^2 = 1. \end{cases}$$

Fixed point of ordinary differential system

Study of qualitative features of system:

$$\left\{ \begin{array}{rcl} d\mathfrak{x}/dt & = & \mathfrak{a} - \mathfrak{x} + \mathfrak{x}^2\mathfrak{y}, \\ d\mathfrak{y}/dt & = & \mathfrak{b} - \mathfrak{x}^2\mathfrak{y}, \end{array} \right.$$

is a purely algebraic problem (elimination and eigenvalues computation).
The one-parameter group associated to

$$\mathfrak{x}\frac{\partial}{\partial\mathfrak{x}} - \mathfrak{y}\frac{\partial}{\partial\mathfrak{y}} + \mathfrak{a}\frac{\partial}{\partial\mathfrak{a}} + \mathfrak{b}\frac{\partial}{\partial\mathfrak{b}}$$

is a symmetries group of the algebraic problem but not of the differential one.
The algebraic problem could be further simplified.

Fixed point of ordinary differential system

In invariant coordinates, we consider the following system:

$$x = \mathfrak{x}/\mathfrak{a}, \quad y = \mathfrak{a}\mathfrak{y}, \quad b = \mathfrak{b}/\mathfrak{a}, \qquad \begin{cases} 1 - x + x^2 y = 0, \\ b - x^2 y = 0, \end{cases}$$

and compute the following relations:

$$x = b + 1, \qquad y = \frac{b}{(b+1)^2},$$

$$\det = (b+1)^2, \quad \operatorname{trace} = -\frac{2+2b+3b^2+b^3}{b+1}.$$

Fixed point of ordinary differential system

Study of qualitative features of system:

$$\begin{cases} d\mathfrak{x}/d\mathfrak{t} &= \mathfrak{a} - \mathfrak{x} + \mathfrak{x}^2\mathfrak{y}, \\ d\mathfrak{y}/d\mathfrak{t} &= \mathfrak{b} - \mathfrak{x}^2\mathfrak{y}, \end{cases}$$

is a purely algebraic problem (elimination and eigenvalues computation).
From previous specialization, we deduce that:

$$\mathfrak{x} = \mathfrak{b} + \mathfrak{a}, \qquad \mathfrak{y} = \frac{\mathfrak{b}}{(\mathfrak{b}+\mathfrak{a})^2},$$

$$\forall(\mathfrak{a}, \mathfrak{b}), \det > 0, \quad \text{trace} > 0 \Leftrightarrow \mathfrak{b} > r\mathfrak{a} \; (r \approx 2.52).$$

Others specializations should also be used in order to return in the original coordinates space.

Axe effect

The following algebraic system

$$p_2 + p_1 x_1 = 0, \quad x_2(p_2 + p_1 x_1) + p_2 {x_2}^2 + 1 = 0,$$

presents a trivial and a non trivial symmetries:

$$x_1 \frac{\partial}{\partial x_1} - p_1 \frac{\partial}{\partial p_1}, \quad g := p_2 + p_1 x_1 + 2 p_2 x_2, \ \frac{g}{p_1} \frac{\partial}{\partial x_1} + {x_2}^2 \frac{\partial}{\partial x_2} - g \frac{\partial}{\partial p_2}.$$

There is little hope to find an automorphism associated to this last derivation.

Axe effect

But, the more general algebraic system

$$p_2 + p_1 x_1 = 0, \quad x_2(p_2 + p_1 x_1) + p_2 x_2^2 + p_3 = 0,$$

presents only trivial symmetries:

$$x_1 \frac{\partial}{\partial x_1} - p_1 \frac{\partial}{\partial p_1}, \quad \frac{\partial}{\partial p_1} - x_1 \frac{\partial}{\partial p_2} + x_2^2 x_1 \frac{\partial}{\partial p_3}, \quad \frac{\partial}{\partial x_2} - g \frac{\partial}{\partial p_3},$$

that leads to automorphisms.

Action of point transformation on Lie algebra

Following Hydon 1998, suppose that the Lie algebra $\mathcal{L}$ spanned by infinitesimal generators of system's Lie point symmetries is not trivial. Any point transformation σ—discrete or not—induces an automorphism of the Lie algebra $\mathcal{L}$ and there exists a constant $\dim\mathcal{L} \times \dim\mathcal{L}$ matrix (m_i^j) such that $\mathcal{S}_i = m_i^k \mathcal{S}_k$ i.e.

$$\begin{pmatrix} \mathcal{S}_1 \\ \vdots \\ \mathcal{S}_{\dim\mathcal{L}} \end{pmatrix} = \begin{pmatrix} m_1^1 & \cdots & m_1^{\dim\mathcal{L}} \\ \vdots & & \vdots \\ m_{\dim\mathcal{L}}^1 & \cdots & m_{\dim\mathcal{L}}^{\dim\mathcal{L}} \end{pmatrix} \begin{pmatrix} \sigma \circ \mathcal{S}_1 \circ \sigma^{-1} \\ \vdots \\ \sigma \circ \mathcal{S}_{\dim\mathcal{L}} \circ \sigma^{-1} \end{pmatrix}.$$

This automorphism preserves structure constants c_{kl}^n taken from the commutation table and thus, the following relations hold:

$$c_{kl}^n m_i^k m_j^l = c_{ij}^k m_k^n, \quad 1 \leq i < j \leq \dim\mathcal{L}, \quad 1 \leq n \leq \dim\mathcal{L}.$$

Consider the single second order differential equation:

$$a\ddot{x} + bx + c = 0 \Leftrightarrow \left\{ \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & -\frac{bx+c}{a}. \end{array} \right.$$

Its infinitesimal generator of symmetries are $\mathcal{S}_0 = \frac{\partial}{\partial t}$ and:

$$\begin{array}{rclcrcl} \mathcal{S}_1 & = & \frac{\partial}{\partial x} - b\frac{\partial}{\partial c}, & & \mathcal{S}_3 & = & a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}, \\ \mathcal{S}_2 & = & x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + c\frac{\partial}{\partial c}, & & \mathcal{S}_4 & = & t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b}. \end{array}$$

	$\mathcal{D}$	$\mathcal{S}_0$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
$\mathcal{D}$	0	0	0	0	0	$\mathcal{D}$
$\mathcal{S}_0$	0	0	0	0	0	$\mathcal{S}_0$
$\mathcal{S}_1$	0	0	0	$\mathcal{S}_1$	0	$\mathcal{S}_1$
$\mathcal{S}_2$	0	0	$-\mathcal{S}_1$	0	0	0
$\mathcal{S}_3$	0	0	0	0	0	0
$\mathcal{S}_4$	$-\mathcal{D}$	$-\mathcal{S}_0$	$-\mathcal{S}_1$	0	0	0

aa

A discrete symmetry $\bar{\rho} = \psi(t, x, y, a, b, c)$ for all ρ in (t, x, y, a, b, c), is such that the following relations holds:

$$\begin{pmatrix} \mathcal{S}_1(\bar{t}) & \mathcal{S}_1(\bar{x}) & \mathcal{S}_1(\bar{y}) & \mathcal{S}_1(\bar{a}) & \mathcal{S}_1(\bar{b}) & \mathcal{S}_1(\bar{c}) \\ \mathcal{S}_2(\bar{t}) & \mathcal{S}_2(\bar{x}) & \mathcal{S}_2(\bar{y}) & \mathcal{S}_2(\bar{a}) & \mathcal{S}_2(\bar{b}) & \mathcal{S}_2(\bar{c}) \\ \mathcal{S}_3(\bar{t}) & \mathcal{S}_3(\bar{x}) & \mathcal{S}_3(\bar{y}) & \mathcal{S}_3(\bar{a}) & \mathcal{S}_3(\bar{b}) & \mathcal{S}_3(\bar{c}) \\ \mathcal{S}_4(\bar{t}) & \mathcal{S}_4(\bar{x}) & \mathcal{S}_4(\bar{y}) & \mathcal{S}_4(\bar{a}) & \mathcal{S}_4(\bar{b}) & \mathcal{S}_4(\bar{c}) \end{pmatrix} = \begin{pmatrix} m_1^1 & m_1^2 & m_1^3 & m_1^4 \\ m_2^1 & m_2^2 & m_2^3 & m_2^4 \\ m_3^1 & m_3^2 & m_3^3 & m_3^4 \\ m_4^1 & m_4^2 & m_4^3 & m_4^4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & -\bar{b} \\ 0 & \bar{x} & \bar{y} & 0 & 0 & \bar{c} \\ 0 & 0 & 0 & \bar{a} & \bar{b} & \bar{c} \\ \bar{t} & \bar{x} & 0 & \bar{a} & -\bar{b} & 0 \end{pmatrix}.$$

We do not try to compute all discrete point symmetries by solving these equations. We are just interested in a discrete point transformation acting only on time. Thus, we suppose that:

$$\bar{t} = t + \psi(a, b, c), \quad \forall \rho \in (X, \Theta),\ \bar{\rho} = \rho.$$

In that case, our system is composed by an purely algebraic system:

$$
\begin{aligned}
&1 - m_1^1 - (m_1^2 - m_1^4)x, m_1^2, m_4^2, m_3^2, m_1^3 + m_1^4, m_1^3 - m_1^4, (m_1^1 - 1)b - (m_1^2 + m_1^3)c, (1 - m_2^2 - m_2^4)x - m_2^1,\\
&m_2^3 - m_2^4, (1 - m_2^2 - m_2^3)c + m_2^1 b, m_3^1 + (m_3^2 + m_3^4)x, 1 - m_3^3 - m_3^4, 1 - m_3^3 + m_3^4, (1 - m_3^2 - m_3^3)c + m_3^1 b,\\
&(1 - m_4^2 - m_4^4)x - m_4^1, 1 - m_4^3 - m_4^4, -1 - m_4^3 + m_4^4, m_4^1 b - (m_4^2 + m_4^3)c, 1 - m_2^2,\\
&m_1^1(1 - m_4^2 - m_4^4) + m_1^4(m_1^2 + m_1^4), m_1^1(1 - m_2^2 - m_2^4) + m_2^1(m_1^2 + m_1^4),
\end{aligned}
$$

—which could be easily solved—and by the partial differential system:

$$
\begin{aligned}
&\frac{\partial}{\partial c}\psi(a, b, c),\\
&a\frac{\partial}{\partial a}(a, b, c) + b\frac{\partial}{\partial b}\psi(a, b, c) + c\frac{\partial}{\partial c}\psi(a, b, c),\\
&a\frac{\partial}{\partial a}\psi(a, b, c) - b\frac{\partial}{\partial b}\psi(a, b, c) - \psi(a, b, c),
\end{aligned}
$$

whose solution is cste$\sqrt{a/b}$.

Open questions

- What kind of symmetries occur in practice?
- Could we find dedicated kernel computations of reasonable complexity?
- Are axe (and other) effects algorithmic?
- Could we extend this symmetry based approach to systems of differential-difference equation?
- Connexion with Galois theory of parameterized differential equation (Cassidy/Singer 2004)?