Stationary Tries and Renewal Theorem

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Contents

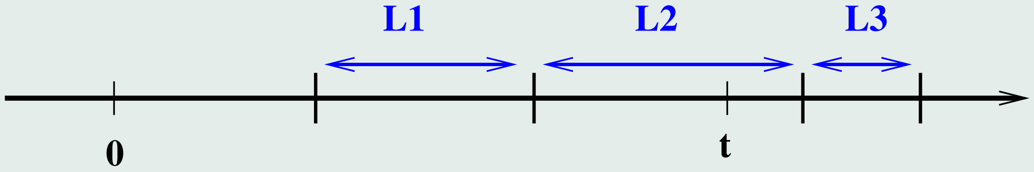
1. Renewal Theorems

Some History

Blackwell (1948):

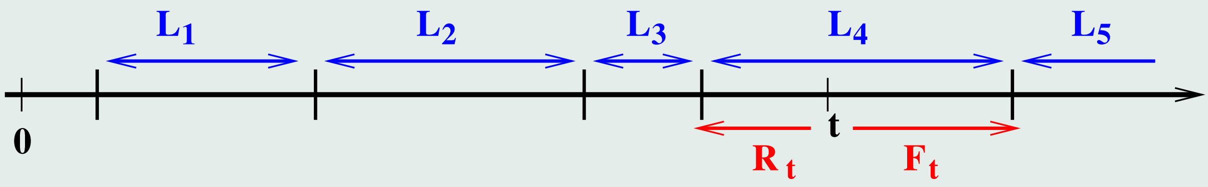
— A light bulb last two years in average.

— How many are necessary for ten years ?



Breiman, Feller, Lindvall, . . .
Babillot (Habilitation).

General Framework



(L_i) i.i.d. non-negative random variables.

— $U(a,b)$: average number of points in $[a,b]$,
U: **Renewal measure**,

$$\text{for } h > 0 \qquad \lim_{t\to+\infty} U(t,t+h)?$$

— Behavior of (R_t, F_t) as $t \to +\infty$?

Renewal Theorem

Non-Lattice Case: $\forall \delta > 0,\ \mathbb{P}(L \in \delta\mathbb{N}) < 1$

$$\lim_{t \to +\infty} U(t, t+h) = \frac{h}{\mathbb{E}(L_1)};$$

$$(R_t, F_t) \overset{\text{dist.}}{\longrightarrow} (R_\infty, F_\infty) :$$

$$\mathbb{E}(f(R_\infty, F_\infty)) = \frac{1}{\mathbb{E}(L_1)} \mathbb{E}\left(\int_0^{L_1} f(u, L_1 - u)\, du \right)$$

F_t has density $\sim x \to \mathbb{P}(L_1 \geq x)/\mathbb{E}(L_1)$

Proofs

— **Renewal Equation:**

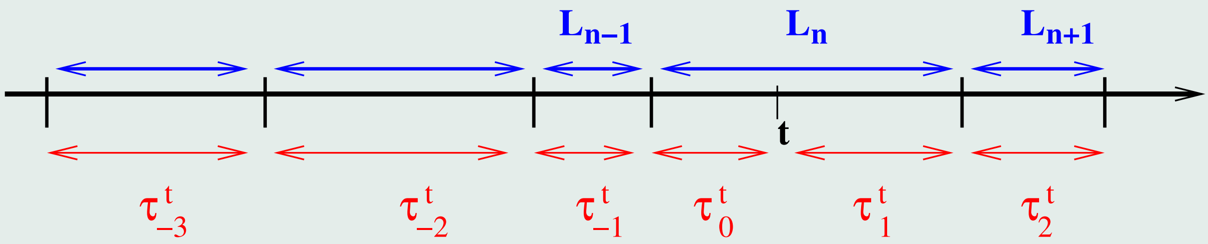
$$U(0,t) = 1 + \int_0^t U(0, t-u) L(du)$$

$\Rightarrow$ Fourier Analysis (Feller).

— **Coupling:**

Lindvall, Athreya and Ney, . . .

A Point Process Point of View



Renewal Theorem:

$$(\tau_i^t, i \in \mathbb{Z}) \xrightarrow{\text{dist.}} (L_i^*, i \in \mathbb{Z})$$

Stationary renewal process.

1. $(L_i^*, i \leq -1)$, $(L_i^*, i > 1)$, i.i.d. dist. as L_1;
2. $(L_0^*, L_1^*) \stackrel{\text{dist.}}{=} (R_\infty, F_\infty)$.

Lattice Case

If $\mathbb{P}(L_1 \in \delta\mathbb{N}) = 1$ and $\mathbb{P}(L_1 = \delta) > 0$:

— $(\tau_i^t, i \in \mathbb{Z}))$ does not converge as $t \to +\infty$

— for $h > 0$,

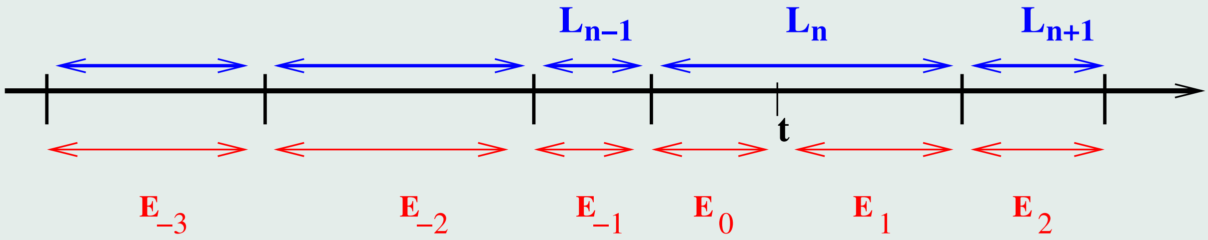
$$(\tau_i^{h+n\delta}, i \in \mathbb{Z})) \overset{n \to +\infty}{\longrightarrow} (L_i^{*h}, i \in \mathbb{Z})$$

— Periodic Behavior

$$(L_i^{*h}, i \in \mathbb{Z}) \overset{\text{dist.}}{=} (L_i^{*(h+\delta)}, i \in \mathbb{Z})$$

The Poisson Process

(L_n) exponential r.v. with parameter 1.
$\mathbb{P}(L_1 \geq x) = \exp(-x)$.



(E_n) also exp. with parameter 1.

Invariant by translation.

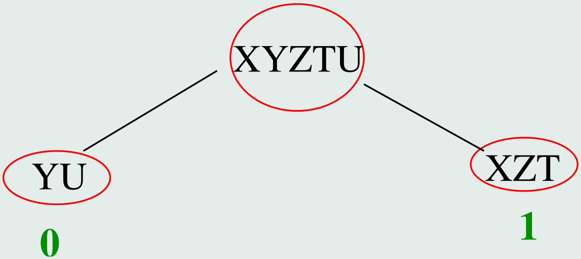
Renewal Theorem holds at time $t = 0$.

2. Tree Algorithms

— **Each item of root node flips a coin X**

$X = 0 \Rightarrow$ **left node**

$X = 1 \Rightarrow$ **right node**



Each item of root node flips a coin

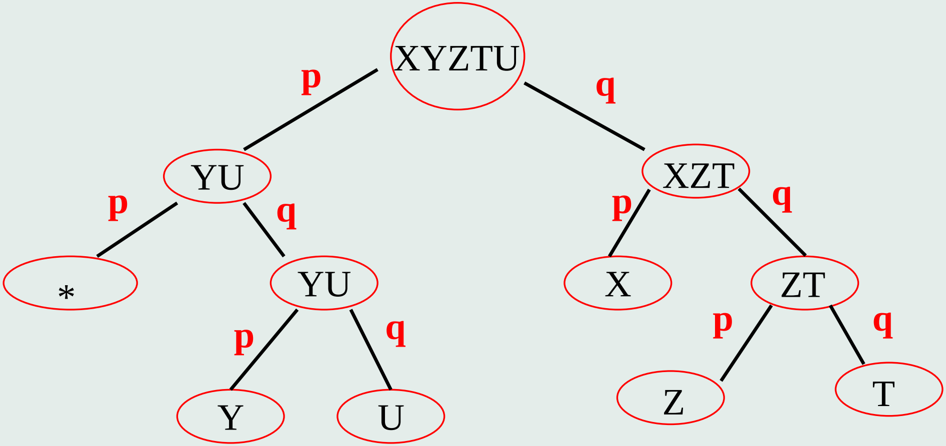
$0 \Rightarrow$ left node

$1 \Rightarrow$ right node

Continue until at most one item per node.

Independent coin tossings.

A Non-Symmetrical Tree Algorithm



$$\mathbb{P}(X = 0) = p,\ \mathbb{P}(X = 1) = q$$

Applications

— Communication protocols;

— Search and Sort Algorithms;

— Leader election problems;

— Statistical tests,

— . . .

Asymptotic behavior

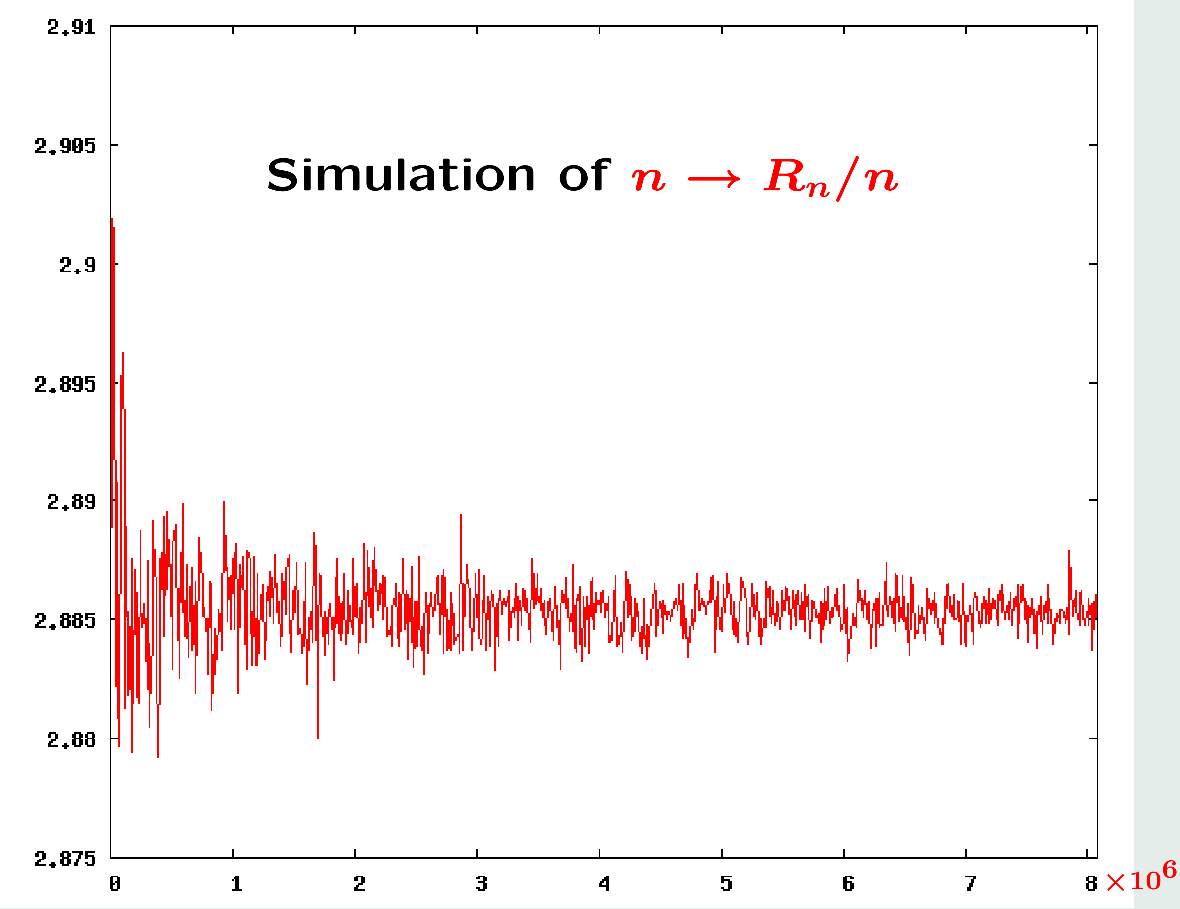
Cost Function. Starting with n items at root

R_n : nb. of nodes of the tree

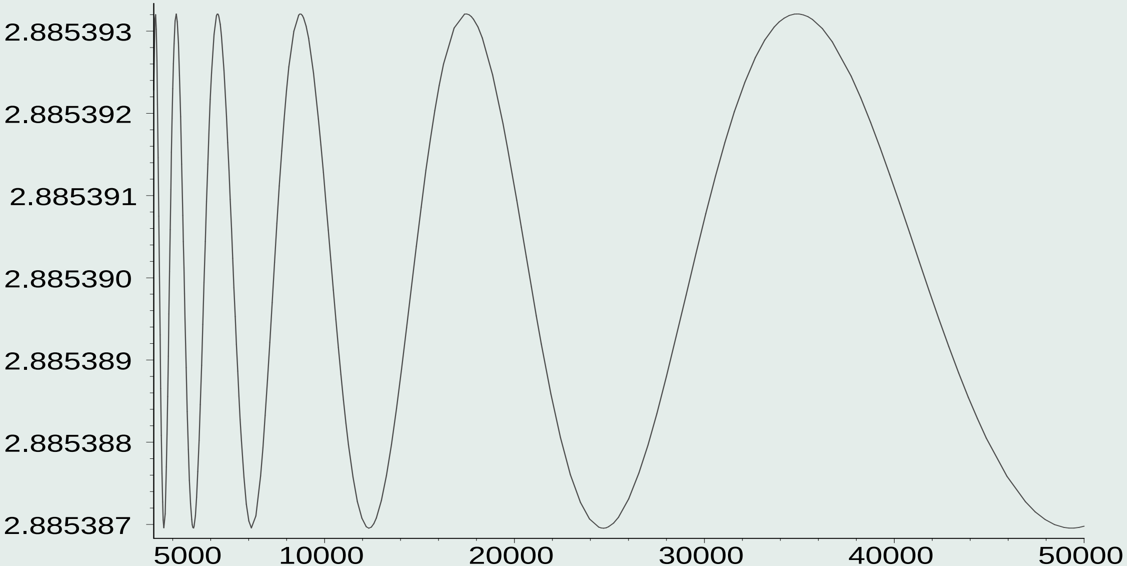
$\frac{\mathbb{E}(R_n)}{n}$: Average cost to process 1 item.

Law of Large Numbers: $\lim_{n \to +\infty} \frac{\mathbb{E}(R_n)}{n}$?

the limit does not always exist !


Simulation of $n \to R_n/n$
2.91
2.905
2.9
2.895
2.89
2.885
2.88
2.875
0
1
2
3
4
5
6
7
8
$\times 10^6$

The sequence $n \to \mathbb{E}(R_n)/n$



Asymptotic behavior

$$\left(\frac{\mathbb{E}(R_n)}{n}\right) : \begin{cases} \text{converges} & \text{if } \dfrac{\log p}{\log q} \not\in \mathbb{Q} \\ \text{oscillates} & \text{otherwise} \end{cases}$$

Literature: Complex analysis methods

FLajolet and co-authors.

3. A Probabilistic Point of View

Recurrence Relation

$R_0 = R_1 = 1$. For $n \geq 2$,

$$R_n \overset{\text{dist.}}{=} 1 + R_{X_n} + \overline{R}_{n-X_n}$$

with

$$X_n = B_1 + B_2 + \cdots + B_n.$$

$(B_i) \in \{0, 1\}$ i.i.d. Bernoulli parameter p.

$(\overline{R}_n)$ same dist. as (R_n) independent of (R_n)

Equation for the Poisson Transform

$$R_n \overset{\text{dist.}}{=} 1 + R_{X_n} + \overline{R}_{n-X_n} - 2 \times 1_{\{n \leq 1\}}$$

If $(N_x, x \geq 0)$ Poisson process with rate 1

$$\mathbb{E}(R_{N_x}) = \mathbb{E}(R_{N_{px}}) + \mathbb{E}(R_{N_{qx}}) + 1 - 2\mathbb{P}(t_2 \geq x)$$

t_2 second point of Poisson process (N_x)

If $r(x) = \mathbb{E}(R_{N_x}) - 1$

$$r(x) = r(px) + r(qx) + 2\mathbb{P}(t_2 < x)$$

Poisson Transform (II)

$$r(x) = r(px) + r(qx) + 2\mathbb{P}(t_2 < x)$$

If $A_1 \in \{p, q\}$ **r.v. such that** $\mathbb{P}(A_1 = p) = p$

$$r(x) = \mathbb{E}\left(\frac{r(A_1 x)}{A_1}\right) + 2\mathbb{E}\left(1_{\{t_2 < x\}}\right)$$

Poisson Transform (II)

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If $A_1 \in \{p, q\}$ r.v. such that $\mathbb{P}(A_1 = p) = p$

$$r(x) = \mathbb{E}\left(\frac{r(A_1 x)}{A_1}\right) + 2\mathbb{E}\left(\mathbf{1}_{\{t_2 < x\}}\right)$$

$$= \mathbb{E}\left(\frac{r(A_2 A_1 x)}{A_2 A_1}\right) + 2\mathbb{E}\left(\frac{1}{A_1}\mathbf{1}_{\{t_2 < x A_1\}}\right) + 2\mathbb{E}\left(\mathbf{1}_{\{t_2 < x\}}\right)$$

$$= 2\sum_{k=0}^{+\infty} \mathbb{E}\left(\frac{1}{\prod_1^k A_i}\mathbf{1}_{\{t_2 < x \prod_1^k A_i\}}\right)$$

Poisson Transform (III)

$$\mathbb{E}\left(R_{N_x}\right) = 1 + 2\sum_{k=0}^{+\infty} \mathbb{E}\left(\frac{1}{\prod_1^k A_i} \mathbf{1}_{\{t_2 < x\prod_1^k A_i\}}\right)$$

$$= 1 + 2\sum_{k=0}^{+\infty} \mathbb{E}\left(\sum_{n=2}^{+\infty} \frac{1}{\prod_1^k A_i} \mathbf{1}_{\{t_2 < x\prod_1^k A_i, N_x = n\}}\right)$$

Given $\{N_x = n\}$ the n points of Poisson proc.: n i.i.d. uniformly dist. r.v. on $[0, x]$

$U_{2,n}$: second smallest value of n i.i.d. uniform r.v. on $[0, 1]$

$$\mathbb{P}\left(t_2 \leq \frac{x}{\mathcal{A}}, N_x = n\right)$$

$$= \mathbb{P}\left(t_2 \leq x/\mathcal{A} \mid N_x = n\right) \frac{x^n}{n!} e^{-x}$$

$$\mathbb{P}\left(t_2 \leq x/\mathcal{A} \mid N_x = n\right) = \mathbb{P}\left(U_{2,n} \leq 1/\mathcal{A}\right)$$

Probabilistic de-Poissonization

$$\mathbb{E}\left(R_{N_x}\right) = \sum_{n\geq 0} \mathbb{E}(R_n)\frac{x^n}{n!}\,e^{-x}$$

$$= 1 + 2\sum_{k=0}^{+\infty}\sum_{n=2}^{+\infty} \mathbb{E}\left(\frac{1}{\prod_1^k A_i}\mathbf{1}_{\{t_2<x\prod_1^k A_i, N_x=n\}}\right)$$

$$\mathbb{E}\left(R_{N_x}\right) = \sum_{n\geq 0} \frac{x^n}{n!}\,e^{-x}$$

$$+2\sum_{n=2}^{+\infty}\sum_{k=0}^{+\infty} \mathbb{E}\left(\frac{1}{\prod_1^k A_i}\mathbf{1}_{\{U_{2,n}<\prod_1^k A_i\}}\right)\frac{x^n}{n!}\,e^{-x}$$

An Associated Random Walk

$U_{2,n}$: 2th min. of n ind. uniform r.v. on $[0,1]$

$$\mathbb{E}(R_n) = 1+2\mathbb{E}\left(\sum_{k\geq 0}\frac{1}{\prod_1^k A_i}1_{\{U_{2,n}<\prod_1^k A_i\}}\right), \; n \geq 2.$$

$$\frac{\mathbb{E}(R_n)-1}{2n} = \mathbb{E}\left(\sum_k e^{-\log n+\sum_{i=1}^k -\log(A_i)} \times 1_{\left\{-\sum_1^k \log(A_i)\leq -\log U_{2,n}\right\}}\right)$$

The Use of Renewal Theorem

$$L_i = -\log A_i \qquad U_{2,n} \sim 1/n$$

$$\Delta_n = \mathbb{E}\left(\sum_k e^{-\left(\log n - \sum_{i=1}^k L_i\right)} \mathbf{1}_{\left\{\sum_1^k L_i \le \log n\right\}}\right)$$

$$\Delta_n \sim \mathbb{E}\left(\sum_{k\le 0} e^{-\sum_{i=k}^0 \tau_i^{\log n}}\right) \sim \mathbb{E}\left(\sum_{k\le 0} e^{-\sum_{i=k}^0 L_i^*}\right)$$

if L_0 non-lattice.

Renewal Theorems

— Dist. of $-\log A$ non-lattice: $\log p / \log q \notin \mathbb{Q}$.
Continuous Renewal Theorem.
Convergence of $\mathbb{E}(R_n)/n$.

— Dist. of $-\log A$ lattice: $\log p / \log q \in \mathbb{Q}$.
Discrete Renewal Theorem.
Periodic Fluctuations of $\mathbb{E}(R_n)/n$.

General Splitting Scheme

Algorithm $\mathcal{A}(n)$

— $n < D \Rightarrow$ STOP.

Otherwise:

— Take a r.v. $G \in \mathbb{N}$; Branching variable

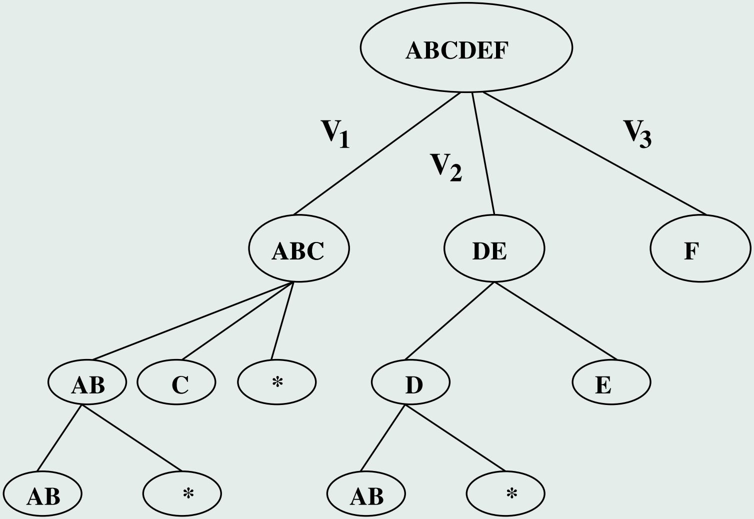
— Take a random probability vector $\mathcal{V} = (V_1, \ldots, V_G)$, $V_1 + \cdots + V_G = 1$;

Weights on arcs

— Split n into G subgroups $(n_1, \ldots n_G)$ randomly, according to vector $\mathcal{V}$.

— Apply $\mathcal{A}(n_1)$, . . . , $\mathcal{A}(n_G)$.

General Splitting Algorithms



Splitting Measure

Probability Measure on $[0, 1]$:

$$\int_0^1 f(x)\,\mathcal{W}(dx) = \mathbb{E}\left(\sum_{i=1}^{G} V_{i,G} f(V_{i,G})\right).$$

Theorem. [Mohamed and R. (2005)] If

$$\int_0^1 \frac{|\log(y)|}{y} \mathcal{W}(dy) < +\infty,$$

— if $\log \mathcal{W}$ non-lattice,

$$\lim_{n \to +\infty} \mathbb{E}(R_n)/n = \frac{\mathbb{E}(G)}{(D-1)\int_0^1 |\log(y)|\, \mathcal{W}(dy)}.$$

— $\log \mathcal{W}$ lattice: $\mathbb{E}(R_n)/n \sim F(\log n/\lambda)$

$$F(x) = C \int_0^{+\infty} \exp\left(-\lambda \left\{ x - \frac{\log y}{\lambda} \right\}\right) \frac{y^{D-2}}{(D-1)!} e^{-y}\, dy$$

Some connections

— Fragmentation processes;

— Random Fractal subsets of $[0, 1]$;

— Branching processes:
Multiplicative martingales;

— Dynamical systems: counting problems.

4. Stationary tries

Stationary sequences

— $X = (X_k, k \in \mathbb{Z}) \in \{0,1\}^{\mathbb{Z}}$
a stationary sequence;

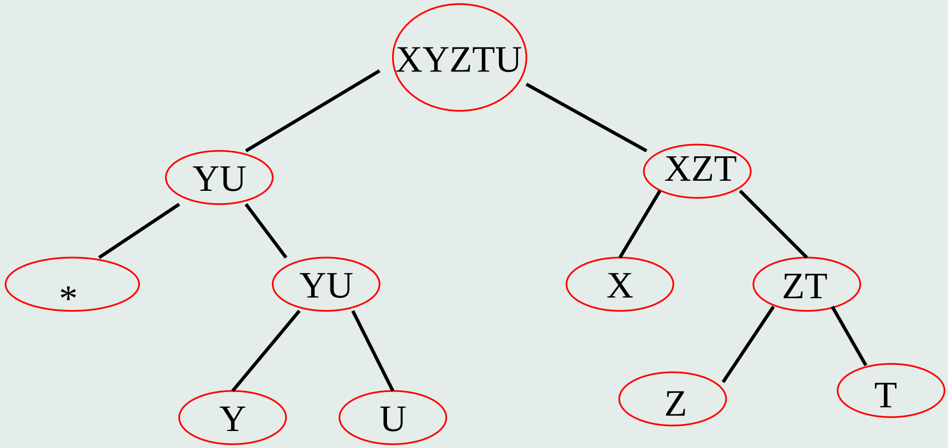
— Each item draws a sequence dist. as X;

X_i: "coin" (possibly) used at level i ;

— R_n size of the tree with n items.

Asymptotic behavior of $\mathbb{E}(R_n)/n$?

$\mathbf{X} = \mathbf{10},\ \mathbf{Y} = \mathbf{010},\ \mathbf{Z} = \mathbf{1110},\ \mathbf{T} = \mathbf{1111},\ \mathbf{U} = \mathbf{011}.$



Functional Analysis Approach

Context

— A function $\phi[0,1] \to [0,1]$;

— A Partition $[0,1] = \cup_i I_i$;

— $X_n = p$ if $\phi^{(n)}(X_0) \in I_p$.

Methods

— Ruelle's Transfer Operator;

— Functional Transforms.

Baladi, Bourdon, Clément, Flajolet, Vallée, . . .

Cost Function

$$C_n(f) = \sum_{\alpha \in \Sigma^*} f(np_\alpha)$$

Σ^* finite vectors in $\{0, 1, \ldots, C\}$
p_α proba. of vector $\alpha \in \Sigma^*$;
$f(0) = 0$.

Exemple: Clément, Flajolet and Vallée (2001)

$$R_n = \sum_{\alpha \in \Sigma^*} \left(1 - (1 + np_\alpha)(1 - p_\alpha)^n\right)$$

$$\sim \sum_{\alpha \in \Sigma^*} \int_0^{np_\alpha} u e^{-u} \, du$$

Probabilistic rewriting of cost function

$$C_n(f) = \sum_{\alpha\in\Sigma^*} f(np_\alpha)$$

$$= \sum_{k\geq 1} \sum_{\substack{\alpha\in\Sigma^* \\ |\alpha|=k}} \frac{f(np_\alpha)}{p_\alpha} p_\alpha$$

$$= \sum_{k\geq 1} \mathbb{E}\left(\frac{f(np_{\kappa_k})}{p_{\kappa_k}}\right)$$

κ_k projection on k first coord.

Rewriting of cost function: Fubini's Theorem

$$C_n(f) = \sum_{k\geq 1} \mathbb{E}\left(\frac{f(np_{\kappa_k})}{p_{\kappa_k}}\right)$$

$$= \mathbb{E}\left(\sum_{k\geq 1} \frac{1}{p_{\kappa_k}} \int_0^{+\infty} 1_{\{u\leq np_{\kappa_k}\}} f'(u)\, du\right)$$

$$= \int_0^{+\infty} \mathbb{E}\left(G_n(u)\right) f'(u)\, du,$$

with

$$G_n(u) = \sum_{k\geq 1} \frac{1}{p_{\kappa_k}} 1_{\{u\leq np_{\kappa_k}\}}.$$

A rough estimation

Shannon's Theorem: $-\log p_{\kappa_k} \sim kH$, a.s.
H entropy.

$$G_n(u) = \sum_{k\geq 1} \frac{1}{p_{\kappa_k}} 1_{\{u\leq n p_{\kappa_k}\}}$$
$$= \sum_{k\geq 1} e^{-\log p_{\kappa_k}} 1_{\{-\log p_{\kappa_k} \leq \log(n/u)\}}$$
$$\text{"}\sim\text{"} \sum_{k\geq 1} e^{kH} 1_{\{k\leq \log(n/u)/H\}}$$
$$= \frac{e^{\lceil \log(n/u)/H \rceil H} - 1}{e^H - 1} \sim \frac{n}{u(e^H - 1)}.$$

G_n of order n.

More Rigor

$$p_{\kappa_n(\omega)} = \mathbb{P}(X_n{=}\omega_n, \ldots, X_0 = \omega_0 \mid X_k{=}w_k, k < 0)$$

$$= \prod_{i=1}^{n} \mathbb{P}(X_i{=}\omega_i \mid X_i{=}\omega_i, \ldots, X_0{=}\omega_0, X_k{=}\omega_k, k < 0)$$

$$-\log p_{\kappa_n} = \sum_{i=1}^{n} h \circ \theta^i(\omega)$$

θ: shift on sequences;

$$h(\omega) = -\log \mathbb{P}(X_0 = \omega_0 \mid X_k = \omega_k, k < 0).$$

h entropy function, $H = \mathbb{E}(h)$

More Rigor (II)

$$-\log p_{\kappa_n} = \sum_{i=1}^{n} h \circ \theta^i(\omega)$$

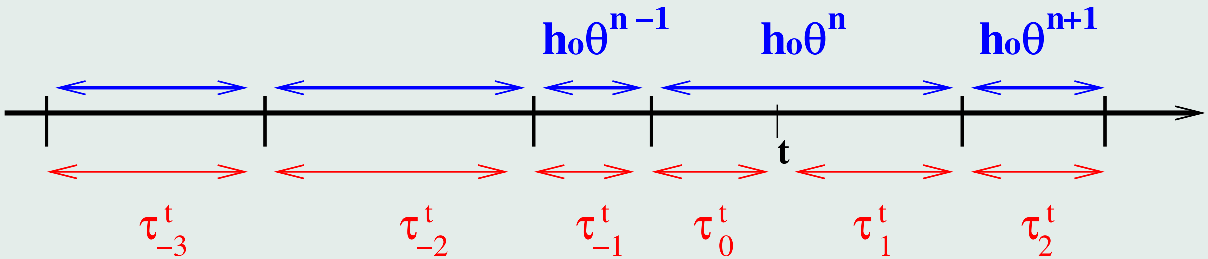
$$G_n(u) = \sum_{k \geq 1} \frac{1}{p_{\kappa_k}} \mathbf{1}_{\{u \leq n p_{\kappa_k}\}}$$

$$\frac{G_n(u)}{n} = \frac{1}{u} \sum_{k \geq 1} \exp\left[-\left(\log(n/u) - \sum_1^k h \circ \theta^i\right)\right]$$

$$\mathbf{1}_{\left\{\sum_1^k h \circ \theta^i \leq \log(n/u)\right\}}$$

$\Rightarrow$ Renewal Theorem Context.

Renewal Theory for Stationary Sequences



Renewal Theorem: $(h \circ \theta^n)$ stationary ergodic,

$$(\tau_i^t, i \in \mathbb{Z}) \xrightarrow{\text{dist.}} ?$$

Renewal Theory for Stationary Sequences (II)

Blanchard (1976) Delasnerie and Neveu (1977)
Equivalence between

— Renewal thm. for sequence $(h \circ \theta^n)$;

— Mixing property of special flow under h.

— The variable h is non lattice.

Lalley (1989) and Guivarc'h

— Limit theorems for counting measures of stationary sequences.

A convergence result

Entropy function

$$h(x) = -\log(P(X_0 = x \mid X_{-1}, X_{-2}, \ldots))$$

Entropy $H = \mathbb{E}(h(X_0))$.

Theorem:
If the distribution of $h(X_0)$ is not lattice,

$$\lim_{n \to +\infty} \frac{\mathbb{E}(C_n(f))}{n} = \frac{1}{H} \int_0^{+\infty} \frac{f'(u)}{u} \, du.$$

The End