

Using effective homological algebra for factoring and decomposing of linear functional systems

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Introduction: factorization and decomposition

- Let $L(\partial)$ be a scalar **ordinary** or **partial differential operator**.
- When is it possible to find $L_1(\partial)$ and $L_2(\partial)$ such that:

$$L(\partial) = L_2(\partial) L_1(\partial)?$$

- We note that $L_1(\partial) y = 0 \Rightarrow L(\partial) y = 0$.
- $L(\partial) y = 0$ is equivalent to the **cascade integration**:

$$L_1(\partial) y = z \quad \& \quad L_2(\partial) z = 0.$$

- When is the integration of $L(\partial) y = 0$ **equivalent** to:

$$L_2(\partial) z = 0 \quad \& \quad L_1(\partial) u = 0?$$

$$(L_1 X + Y L_2 = 1 \Rightarrow L_1(X z) = z \Rightarrow y = u + X z)$$

Introduction: factorization and decomposition

- Let us consider the **first order** ordinary differential system:

$$\partial y = E(t) y, \quad E(t) \in k(t)^{p \times p}. \quad (\star)$$

- When does it exist an **invertible change of variables**

$$y = P(t) z,$$

such that

$$(\star) \Leftrightarrow \partial z = F(t) z,$$

where $F = -P^{-1}(\partial P - E P)$ is either of the **form**:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \quad \text{or} \quad F = \begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}?$$

Factorization: known cases

Square differential systems:

- Beke's algorithm (Beke1894, Schwarz89, Bronstein94, Tsarëv94...)
- Eigenring (Singer96, Giesbrecht98, Barkatou-Pflügel98, Barkatou01 - ideas in Jacobson37...)

Square (q -)difference systems (generalizations):

- Barkatou01, Bomboy01...

Square D -finite partial differential systems (connections):

- Li-Schwarz-Tsarëv03, Wu05...

Same cases in positive characteristic and modular approaches:

- van der Put95, C.03, Giesbrecht-Zhang03, C.-van Hoeij04,06, Barkatou-C.-Weil05...

What about general linear functional systems?

- **Example** (Saint Venant equations): linearized model around the Riemann invariants (Dubois-Petit-Rouchon, ECC99):

$$\begin{cases} y_1(t - 2h) + y_2(t) - 2\dot{y}_3(t - h) = 0, \\ y_1(t) + y_2(t - 2h) - 2\dot{y}_3(t - h) = 0. \end{cases}$$

- Let $D = \mathbb{R} \left[\frac{d}{dt}, \delta \right]$ and consider the system matrix:

$$R = \begin{pmatrix} \delta^2 & 1 & -2\delta \frac{d}{dt} \\ 1 & \delta^2 & -2\delta \frac{d}{dt} \end{pmatrix} \in D^{2 \times 3}.$$

- Question:** $\exists U \in GL_3(D), V \in GL_2(D)$ such that:

$$V R U = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 \end{pmatrix}, \alpha_1, \alpha_2, \alpha_3 \in D?$$

Outline

- **Type of systems:** Partial differential/discrete/differential time-delay... linear systems (LFSs).
- **General topic:** Algebraic study of linear functional systems (LFSs) coming from mathematical physics, engineering sciences...
- **Techniques:** Module theory and homological algebra.
- **Applications:** Equivalences of systems, Galois symmetries, quadratic first integrals/conservation laws, decoupling problem...
- **Implementation:** package MORPHISMS based on OREMODULES:

<http://wwwb.math.rwth-aachen.de/OreModules>.

- 1 A **linear system** is defined by means of a **matrix R with entries in a ring D of functional operators**:

$$Ry = 0. \quad (\star)$$

- 2 We associate a **finitely presented left D -module M** with (\star) .
- 3 A **dictionary** exists between the **properties of (\star) and M** .
- 4 **Homological algebra** allows us to check properties of M .
- 5 **Effective algebra** (non-commutative Gröbner/Janet bases) leads to constructive algorithms.
- 6 **Implementation** (Maple, Singular/Plural, Cocoa...).

I. Ore Module associated with a linear functional system

Ore algebras

Consider a ring A , an automorphism σ of A and a σ -derivation δ :

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

Definition: A non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in

∂ is called **skew** if $\forall a \in A, \quad \partial a = \sigma(a)\partial + \delta(a)$.

Definition: Let us consider $A = k, k[x_1, \dots, x_n]$ or $k(x_1, \dots, x_n)$. The skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \dots [\partial_m; \sigma_m, \delta_m]$ is called an **Ore algebra** if we have:

$$\begin{cases} \sigma_i \delta_j = \delta_j \sigma_i, & 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, & \delta_i(\partial_j) = 0, \quad j < i. \end{cases}$$

$\Rightarrow D$ is generally a **non-commutative polynomial ring**.

Examples of Ore algebras

- **Partial differential operators:** $A = k, k[x_1, \dots, x_n], k(x_1, \dots, x_n),$

$$D = A \left[\partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[\partial_n; \text{id}, \frac{\partial}{\partial x_n} \right],$$

$$P = \sum_{0 \leq |\mu| \leq m} a_\mu(x) \partial^\mu \in D, \quad \partial^\mu = \partial_1^{\mu_1} \dots \partial_n^{\mu_n}.$$

- **Shift operators:**

$$D = A[\partial; \sigma, 0], \quad A = k, k[n], k(n),$$

$$P = \sum_{i=0}^m a_i(n) \partial^i \in D, \quad \sigma(a)(n) = a(n+1).$$

- **Differential time-delay operators:**

$$D = A \left[\partial_1; \text{id}, \frac{d}{dt} \right] \left[\partial_2; \sigma, 0 \right], \quad A = k, k[t], k(t),$$

$$P = \sum_{0 \leq i+j \leq m} a_{ij}(t) \partial_1^i \partial_2^j \in D.$$

Exact sequences

- **Definition:** A sequence of D -morphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be **exact** at M if we have:

$$\ker g = \operatorname{im} f.$$

- **Example:** If $f : M \rightarrow M'$ is a D -morphism, we then have the following **exact sequences**:

$$\textcircled{1} \quad 0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{\rho} \operatorname{coim} f \triangleq M / \ker f \longrightarrow 0.$$

$$\textcircled{2} \quad 0 \longrightarrow \operatorname{im} f \xrightarrow{j} M' \xrightarrow{\kappa} \operatorname{coker} f \triangleq M' / \operatorname{im} f \longrightarrow 0.$$

$$\textcircled{3} \quad 0 \longrightarrow \ker f \xrightarrow{i} M \xrightarrow{f} M' \xrightarrow{\kappa} \operatorname{coker} f \longrightarrow 0.$$

A left D -module M associated with $R\eta = 0$

- Let D be an Ore algebra, $R \in D^{q \times p}$ and a left D -module \mathcal{F} .
- Let us consider $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$.
- As in number theory or algebraic geometry, we associate with the system $\ker_{\mathcal{F}}(R.)$ the finitely presented left D -module:

$$M = D^{1 \times p} / (D^{1 \times q} R).$$

- **Malgrange's remark:** applying the functor $\text{hom}_D(., \mathcal{F})$ to the finite free resolution (**exact sequence**)

$$\begin{array}{ccccccc} D^{1 \times q} & & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow 0, \\ \lambda = (\lambda_1, \dots, \lambda_q) & \longmapsto & & \lambda R & & & \end{array}$$

we then obtain the **exact sequence**:

$$\begin{array}{ccccccc} \mathcal{F}^q & \xleftarrow{R.} & & \mathcal{F}^p & \xleftarrow{\pi^*} & \text{hom}_D(M, \mathcal{F}) & \longleftarrow 0. \\ R\eta & \longleftarrow & \eta = (\eta_1, \dots, \eta_p)^T & & & & \end{array}$$

Example: Linearized Euler equations

- The **linearized Euler equations** for an incompressible fluid can be defined by the system matrix

$$R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} \in D^{4 \times 4},$$

where $D = \mathbb{R} \left[\partial_1, \text{id}, \frac{\partial}{\partial x_1} \right] \left[\partial_2, \text{id}, \frac{\partial}{\partial x_2} \right] \left[\partial_3, \text{id}, \frac{\partial}{\partial x_3} \right] \left[\partial_t, \text{id}, \frac{\partial}{\partial t} \right]$.

- Let us consider the left D -module $\mathcal{F} = \mathcal{C}^\infty(\Omega)$ (Ω open convex subset of \mathbb{R}^4) and the D -module:

$$M = D^{1 \times 4} / (D^{1 \times 4} R).$$

The solutions of $Ry = 0$ in \mathcal{F} are in 1 – 1 correspondence with the morphisms from M to \mathcal{F} , i.e., with the elements of:

$$\text{hom}_D(M, \mathcal{F}).$$

II. Morphisms between Ore modules finitely presented by two matrices R and R' of functional operators

Morphisms of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- Let us consider the **finitely presented left D -modules**:

$$M = D^{1 \times p} / (D^{1 \times q} R), \quad M' = D^{1 \times p'} / (D^{1 \times q'} R').$$

- We are interested in the **abelian group $\text{hom}_D(M, M')$** of **D -morphisms** from M to M' :

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & & & \downarrow f & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

Morphisms of finitely presented modules

- Let D be an Ore algebra of functional operators.
- Let $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ be two matrices.
- We have the following **commutative exact diagram**:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0. \end{array}$$

$\exists f : M \rightarrow M' \iff \exists P \in D^{p \times p'}, Q \in D^{q \times q'}$ such that:

$$R P = Q R'.$$

Moreover, we have $f(\pi(\lambda)) = \pi'(\lambda P)$, for all $\lambda \in D^{1 \times p}$.

Eigenring: $\partial y = E y$ & $\partial z = F z$

- $D = A[\partial; \sigma, \delta]$, $E, F \in A^{p \times p}$, $R = \partial I_p - E$, $R' = \partial I_p - F$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial_p I - E)} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial I_p - F)} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array}$$

$$(\partial I_p - E) P = Q (\partial I_p - F) \iff \begin{cases} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E P - \sigma(P) F. \end{cases}$$

If $P \in A^{p \times p}$ is **invertible**, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E P).$$

Eigenring: $\partial y = E y$ & $\partial z = F z$

- $D = A[\partial; \sigma, \delta]$, $E, F \in A^{p \times p}$, $R = \partial I_p - E$, $R' = \partial I_p - F$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{\cdot(\partial I - E)} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{\cdot(\partial I_p - F)} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array}$$

$$(\partial I_p - E) P = Q (\partial I_p - F) \iff \begin{cases} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E P - \sigma(P) F. \end{cases}$$

If $P \in A^{p \times p}$ is **invertible**, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E P).$$

- Differential case:** $\delta = \frac{d}{dt}$, $\sigma = \text{id}$:

$$\begin{cases} \dot{P} = E P - P F, \\ F = -P^{-1}(\dot{P} - E P). \end{cases}$$

Eigenring: $\partial y = E y$ & $\partial z = F z$

- $D = A[\partial; \sigma, \delta]$, $E, F \in A^{p \times p}$, $R = \partial I_p - E$, $R' = \partial I_p - F$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{\cdot(\partial I - E)} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{\cdot(\partial I_p - F)} & D^{1 \times p} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array}$$

$$(\partial I_p - E) P = Q (\partial I_p - F) \iff \begin{cases} \sigma(P) = Q \in A^{p \times p}, \\ \delta(P) = E P - \sigma(P) F. \end{cases}$$

If $P \in A^{p \times p}$ is **invertible**, we then have:

$$F = -\sigma(P)^{-1}(\delta(P) - E P).$$

- Discrete case:** $\delta = 0$, $\sigma(k) = k - 1$:

$$\begin{cases} E(k) P(k) - P(k-1) F(k) = 0, \\ B = \sigma(P)^{-1} E P. \end{cases}$$

Computation of $\text{hom}_D(M, M')$

- **Problem:** Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the commutation relation $R P = Q R'$.
- If D is a **commutative ring**, then $\text{hom}_D(M, M')$ is a **D -module**.
- The **Kronecker product** of $E \in D^{q \times p}$ and $F \in D^{r \times s}$ is:

$$E \otimes F = \begin{pmatrix} E_{11} F & \dots & E_{1p} F \\ \vdots & \vdots & \vdots \\ E_{q1} F & \dots & E_{qp} F \end{pmatrix} \in D^{(qr) \times (ps)}.$$

Lemma: If $U \in D^{a \times b}$, $V \in D^{b \times c}$ and $W \in D^{c \times d}$, then we have:

$$U V W = (V_1 \dots V_b) (U^T \otimes W).$$

$$R P I_{p'} = (P_1 \dots P_p) (R^T \otimes I_{p'}), \quad I_q Q R' = (Q_1 \dots Q_q) (I_q \otimes R').$$

We are reduced to compute $\ker_D \left(\begin{pmatrix} R^T \otimes I_{p'} \\ -I_q \otimes R' \end{pmatrix} \right)$.

Computation of $\text{hom}_D(M, M')$

• **Problem:** Given $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, find $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the commutation relation $R P = Q R'$.

• If D is a **non-commutative ring**, then $\text{hom}_D(M, M')$ is an **abelian group** and generally an **infinite-dimensional k -vector space**.

\Rightarrow find a k -basis of morphisms with **given degrees in x_i and in ∂_j** :

- 1 Take an ansatz for P with chosen degrees.
- 2 Compute $R P$ and a Gröbner basis G of the rows of R' .
- 3 Reduce the rows of $R P$ w.r.t. G .
- 4 Solve the system on the coefficients of the ansatz so that all the normal forms vanish.
- 5 Substitute the solutions in P and compute Q by means of a factorization.

Example: Bipendulum

- We consider the Ore algebra $D = \mathbb{R}(g, l) \left[\frac{d}{dt} \right]$.
- We consider the matrix of the bipendulum with $l = l_1 = l_2$:

$$R = \begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{l} & 0 & -\frac{g}{l} \\ 0 & \frac{d^2}{dt^2} + \frac{g}{l} & -\frac{g}{l} \end{pmatrix} \in D^{2 \times 3}.$$

- Let us consider the D -module $M = D^{1 \times 3} / (D^{1 \times 2} R)$.
- We obtain that $\text{end}_D(M)$ is defined by the matrices:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 g \\ \alpha_4 & \alpha_1 + \alpha_2 - \alpha_4 & \alpha_3 g \\ 0 & 0 & \alpha_3 D^2 l + \alpha_1 + \alpha_2 + \alpha_3 g \end{pmatrix},$$

$$Q = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_1 + \alpha_2 - \alpha_4 \end{pmatrix}, \quad \forall \alpha_1, \dots, \alpha_4 \in D.$$

q -dilatation case: $D = \mathbb{R}(q)(x)[H]$ where $H(f(x)) = f(qx)$ and:

$$R = \begin{pmatrix} H & -1 \\ -\frac{1 - q^3 x^2}{1 - qx^2} & \frac{x(1 - q^2)}{1 - qx^2} + H \end{pmatrix} \in D^{2 \times 2}.$$

- Searching for endomorphisms with **degree 0 in H and 2 in x** (both in numerator and denominator), we obtain

$$P = \begin{pmatrix} \frac{-a + bxq - bx + aqx^2}{c(-1 + qx^2)} & \frac{b(-1 + x^2)}{c(-1 + qx^2)} \\ \frac{b(-1 + q^2 x^2)}{c(-1 + qx^2)} & -\frac{a + bxq - bx - aqx^2}{c(-1 + qx^2)} \end{pmatrix},$$

where, a, b, c are **constants** or $P = I_2$ (and corresponding Q).

Saint-Venant equations

- Let $D = \mathbb{Q} \left[\partial_1; \text{id}, \frac{d}{dt} \right] [\partial_2; \sigma, 0]$ be the ring of **differential time-delay operators** and consider the matrix of the **tank model**:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2 \partial_1 \partial_2 \\ 1 & \partial_2^2 & -2 \partial_1 \partial_2 \end{pmatrix} \in D^{2 \times 3}.$$

- The **endomorphisms** of $M = D^{1 \times 3} / (D^{1 \times 2} R)$ are defined by:

$$P_\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 + 2 \alpha_4 \partial_1 + 2 \alpha_5 \partial_1 \partial_2 \\ \alpha_4 \partial_2 + \alpha_5 \end{pmatrix},$$

$$Q_\alpha = \begin{pmatrix} \alpha_2 & 2 \alpha_3 \partial_1 \partial_2 \\ \alpha_1 - 2 \alpha_4 \partial_1 - 2 \alpha_5 \partial_1 \partial_2 & 2 \alpha_3 \partial_1 \partial_2 \\ -\alpha_4 \partial_2 - \alpha_5 & \alpha_1 + \alpha_2 + \alpha_3 (\partial_2^2 + 1) \end{pmatrix},$$

$$Q_\alpha = \begin{pmatrix} \alpha_1 - 2 \alpha_4 \partial_1 & \alpha_2 + 2 \alpha_4 \partial_1 \\ \alpha_2 + 2 \alpha_5 \partial_1 \partial_2 & \alpha_1 - 2 \alpha_5 \partial_1 \partial_2 \end{pmatrix}, \quad \forall \alpha_1, \dots, \alpha_5 \in D.$$

Euler-Tricomi equation

- Let us consider the **Euler-Tricomi equation** (transonic flow):

$$\partial_1^2 u(x_1, x_2) - x_1 \partial_2^2 u(x_1, x_2) = 0.$$

- Let $D = A_2(\mathbb{Q})$, $R = (\partial_1^2 - x_1 \partial_2^2) \in D$ and $M = D/(D R)$.

- $\text{end}_D(M)_{1,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1, \\ Q = (a_1 + 2 a_3) + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1, \end{cases}$$

- $\text{end}_D(M)_{2,0}$ is defined by $P = Q = a_1 + a_2 \partial_2 + a_3 \partial_2^2$.
- $\text{end}_D(M)_{2,1}$ is defined by:

$$\begin{cases} P = a_1 + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1 \\ \quad + a_4 \partial_2^2 + \frac{3}{2} a_5 x_2 \partial_2^2 + a_5 x_1 \partial_1 \partial_2, \\ Q = (a_1 + 2 a_3) + a_2 \partial_2 + \frac{3}{2} a_3 x_2 \partial_2 + a_3 x_1 \partial_1 \\ \quad + a_4 \partial_2^2 + a_5 x_1 \partial_1 \partial_2 + 2 a_5 \partial_2 + \frac{3}{2} a_5 x_2 \partial_2^2. \end{cases}$$

III. A few applications: Galois symmetries, quadratic first integrals of motion and conservation laws

Galois Symmetries

We have the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
 \end{array} \quad (*)$$

If \mathcal{F} is a left D -module, by applying the functor $\text{hom}_D(\cdot, \mathcal{F})$ to $(*)$, we then obtain the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 0 = Q(R'y) = R(Py) & \longleftarrow & Py & & & & \\
 \mathcal{F}^q & & \xleftarrow{\cdot R} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow 0 \\
 \uparrow Q. & & & \uparrow P. & & \uparrow f^* & \\
 \mathcal{F}^{q'} & & \xleftarrow{\cdot R'} & \mathcal{F}^{p'} & \longleftarrow & \ker_{\mathcal{F}}(R'.) & \longleftarrow 0. \\
 0 = R'y & & \longleftarrow & y & & &
 \end{array}$$

$\Rightarrow f^*$ sends $\ker_{\mathcal{F}}(R'.)$ to $\ker_{\mathcal{F}}(R.)$ ($R' = R$: Galois symmetries).

Example: Linear elasticity

- Consider the **Killing operator for the euclidian metric** defined by:

$$R = \begin{pmatrix} \partial_1 & 0 \\ \partial_2/2 & \partial_1/2 \\ 0 & \partial_2 \end{pmatrix}.$$

- The system $Ry = 0$ admits the following **general solution**:

$$y = \begin{pmatrix} c_1 x_2 + c_2 \\ -c_1 x_1 + c_3 \end{pmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}. \quad (\star)$$

- We find that $\text{end}_D(D^{1 \times 2} / (D^{1 \times 3} R))$ is defined by:

$$P = \begin{pmatrix} \alpha_1 & \alpha_2 \partial_1 \\ 0 & 2\alpha_3 \partial_1 + \alpha_1 \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \in D.$$

- Applying P to (\star) , we then get the **new solution**:

$$\bar{y} = Py = \begin{pmatrix} \alpha_1 c_1 x_2 + \alpha_1 c_2 - \alpha_2 c_1 \\ -\alpha_1 c_1 x_1 + \alpha_1 c_3 - 2\alpha_3 c_1 \end{pmatrix}, \quad \text{i.e., } R\bar{y} = 0.$$

Quadratic first integrals of motion

Let us consider a **morphism f** from \tilde{N} to M defined by:

$$\begin{array}{ccccccc} 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial I_p + E^T)} & D^{1 \times p} & \xrightarrow{\pi} & \tilde{N} & \longrightarrow & 0 \\ & & \downarrow .P & & \downarrow .P & & \downarrow f & & \\ 0 & \longrightarrow & D^{1 \times p} & \xrightarrow{.(\partial I_p - E)} & D^{1 \times p} & \xrightarrow{\pi'} & M & \longrightarrow & 0. \end{array}$$

We then have:

$$\dot{P} + E^T P + P E = 0.$$

If $V(x) = x^T P x$, then $\dot{V}(x) = x^T (\dot{P} + E^T P + P E) x$ so that:

$$\dot{P} + E^T P + P E = 0 \iff V(x) = x^T P x \text{ first integral.}$$

\Rightarrow **Morphisms from \tilde{N} to M give quadratic first integrals.**

If E is a **skew-symmetric matrix**, i.e., $E = -E^T$, then we have:

$$(\partial I_p + E^T) = (\partial I_p - E), \quad \tilde{N} = M, \quad \text{hom}_D(\tilde{N}, M) = \text{end}_D(M).$$

Example: Landau & Lifchitz (p. 117)

- Consider $R = \partial I_4 - E$, where $E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & \alpha \end{pmatrix}$.

- We find that the morphisms from \tilde{N} to M are defined by

$$P = \begin{pmatrix} c_1 \omega^4 & c_2 \omega^2 & -\omega^2 (c_1 \alpha + c_2) & c_1 \omega^2 \\ -c_2 \omega^2 & c_1 \omega^2 & -c_1 \omega^2 + c_2 \alpha & -c_2 \\ -\omega^2 (c_1 \alpha - c_2) & -c_1 \omega^2 - c_2 \alpha & c_1 (\alpha^2 + \omega^2) & -c_1 \alpha + c_2 \\ c_1 \omega^2 & c_2 & -c_1 \alpha - c_2 & c_1 \end{pmatrix},$$

which leads to the **quadratic first integral** $V(x) = x^T P x$:

$$\begin{aligned} V(x) = & c_1 \omega^4 x_1(t)^2 - 2 x_1(t) \omega^2 x_3(t) c_1 \alpha + 2 x_1(t) c_1 \omega^2 x_4(t) \\ & + x_2(t)^2 c_1 \omega^2 - 2 x_2(t) c_1 x_3(t) \omega^2 + c_1 x_3(t)^2 \alpha^2 \\ & + c_1 x_3(t)^2 \omega^2 - 2 x_3(t) x_4(t) c_1 \alpha + c_1 x_4(t)^2. \end{aligned}$$

Formal adjoint

- Let $D = A \left[\partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \dots \left[\partial_n; \text{id}, \frac{\partial}{\partial x_n} \right]$ be the **ring of differential operators** with coefficients in A (e.g., $k[x_1, \dots, x_n]$, $k(x_1, \dots, x_n)$).
- The **formal adjoint** $\tilde{R} \in D^{p \times q}$ of $R \in D^{q \times p}$ is defined by:

$$\langle \lambda, R \eta \rangle = \langle \tilde{R} \lambda, \eta \rangle + \sum_{i=1}^n \partial_i \Phi_i(\lambda, \eta).$$

- The **formal adjoint** \tilde{R} can be defined by $\tilde{R} = (\theta(R_{ij}))^T \in D^{p \times q}$, where $\theta : D \rightarrow D$ is the **involution** defined by:

- 1 $\forall a \in A, \quad \theta(a) = a.$
- 2 $\theta(\partial_i) = -\partial_i, \quad i = 1, \dots, n.$

Involution: $\theta^2 = \text{id}_D, \quad \forall P_1, P_2 \in D: \quad \theta(P_1 P_2) = \theta(P_2) \theta(P_1).$

Conservation laws

- Let us consider the left D -modules:

$$M = D^{1 \times p} / (D^{1 \times q} R), \quad \tilde{N} = D^{1 \times q} / (D^{1 \times p} \tilde{R}).$$

- Let $f : \tilde{N} \rightarrow M$ be a **morphism** defined by the matrices P and Q .
- Let \mathcal{F} be a left D -module and the **commutative exact diagram**:

$$\begin{array}{ccccccc}
 \mathcal{F}^p & \xleftarrow{\tilde{R}.} & \mathcal{F}^q & \longleftarrow & \ker_{\mathcal{F}}(\tilde{R}.) & \longleftarrow & 0 \\
 \uparrow Q. & & \uparrow P. & & \uparrow f^* & & \\
 \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) & \longleftarrow & 0.
 \end{array}$$

- $\eta \in \mathcal{F}^p$ **solution** of $R\eta = 0 \Rightarrow \lambda = P\eta$ is a **solution** of $\tilde{R}\lambda = 0$.

$$\Rightarrow \langle P\eta, R\eta \rangle - \langle \tilde{R}(P\eta), \eta \rangle = \sum_{i=1}^n \partial_i \Phi_i(P\eta, \eta) = 0,$$

i.e., $\Phi = (\Phi_1(P\eta, \eta), \dots, \Phi_n(P\eta, \eta))^T$ satisfies **div $\Phi = 0$** .

Example: Laplacian operator

- Let us consider the **Laplacian operator** $\Delta y(x_1, x_2) = 0$, where:

$$\Delta = \partial_1^2 + \partial_2^2 \in D = \mathbb{Q} \left[\partial_1; \text{id}, \frac{\partial}{\partial x_1} \right] \left[\partial_2; \text{id}, \frac{\partial}{\partial x_2} \right].$$

- The **formal adjoint** \tilde{R} of R is then defined by:

$$\lambda(\Delta \eta) - (\Delta \lambda) \eta = \partial_1 (\lambda(\partial_1 \eta) - (\partial_1 \lambda) \eta) + \partial_2 (\lambda(\partial_2 \eta) - (\partial_2 \lambda) \eta).$$

- $R = \Delta = \tilde{R} \in D \Rightarrow \text{hom}_D(\tilde{N}, M) = \text{end}_D(M) = D$.

- if \mathcal{F} is a **D -module** (e.g., $C^\infty(\Omega)$), then we have:

$$\forall \alpha \in D, \forall \eta \in \ker_{\mathcal{F}}(\Delta.), \quad \lambda = \alpha y \in \ker_{\mathcal{F}}(\Delta.).$$

$$\Rightarrow \text{div } \Phi = \partial_1 \Phi_1 + \partial_2 \Phi_2 = 0, \quad \Phi = \begin{pmatrix} (\alpha y)(\partial_1 y) - y(\partial_1 \alpha y) \\ (\alpha y)(\partial_2 y) - y(\partial_2 \alpha y) \end{pmatrix}.$$

IV. Factorization of linear functional systems

Kernel and factorization

$$\begin{array}{ccccccc}
 & & \lambda & \longmapsto & y & & \\
 D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 \downarrow \cdot Q & & \downarrow \cdot P & & \downarrow f & & \\
 D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0 \\
 \exists \mu & \longmapsto & \mu R = \lambda P & \longmapsto & 0 & &
 \end{array}$$

- $\ker_D \left(\cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1 \times r} (S \quad - T)$

$$\Rightarrow \{ \lambda \in D^{1 \times p} \mid \lambda P \in D^{1 \times q} R \} = D^{1 \times r} S$$

$$\Rightarrow \ker f = (D^{1 \times r} S) / (D^{1 \times q} R).$$

- $(D^{1 \times q} (R \quad - Q)) \in \ker_D \left(\cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) \Rightarrow (D^{1 \times q} R) \subseteq (D^{1 \times r} S).$

$$\exists V \in D^{q \times r} : R = V S.$$

Kernel and factorization

We have the following **commutative exact diagram**:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \ker f & & \\ & & & & \downarrow i & & \\ D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow \cdot V & & \parallel & & \downarrow \kappa & & \\ D^{1 \times r} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\pi'} & M / \ker f & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Example: Linearized Euler equations

- Let $R = \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix}$ over $D = \mathbb{R}[\partial_1, \partial_2, \partial_3, \partial_t]$.
- Let us consider $f \in \text{end}_D(M)$ defined by:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \partial_3^2 & -\partial_2 \partial_3 & 0 \\ 0 & -\partial_2 \partial_3 & \partial_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Computing $\ker_D \left(\begin{pmatrix} P \\ R \end{pmatrix} \right)$ and factorizing R by S , we obtain:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix}.$$

Example: Linearized Euler equations

- We have $R = VS$ where:

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 & 0 \\ \partial_t & 0 & 0 & \partial_1 \\ 0 & \partial_t & 0 & \partial_2 \\ 0 & 0 & \partial_t & \partial_3 \end{pmatrix} = \begin{pmatrix} \partial_1 & 1 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 & \partial_1 \\ 0 & 0 & -1 & 0 & \partial_2 \\ 0 & 0 & 0 & 1 & \partial_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_3 & 0 \\ 0 & -\partial_t & 0 & 0 \\ 0 & 0 & \partial_t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The solutions of $Sy = 0$ are particular solutions of $Ry = 0$.
- \Rightarrow Integrating S , we obtain the following solutions of $Ry = 0$:

$$y(x_1, x_2, x_3, t) = \begin{pmatrix} 0 \\ -\frac{\partial}{\partial x_3} \xi(x_1, x_2, x_3) \\ \frac{\partial}{\partial x_2} \xi(x_1, x_2, x_3) \\ 0 \end{pmatrix}, \quad \forall \xi \in C^\infty(\Omega).$$

Free modules & similarity transformations

- **Definition:** A left D -module M is **free** if there exists $l \in \mathbb{Z}_+$ s.t.:

$$M \cong D^{1 \times l}.$$

- **Proposition:** Let $P \in D^{p \times p}$. We have the equivalences:
 - 1 $\ker_D(.P)$ and $\text{coim}_D(.P)$ are **free** left D -modules of rank p and $p - m$.
 - 2 There exists a **unimodular matrix** $U \in D^{p \times p}$, i.e., $U \in \text{GL}_p(D)$, such that:

$$J \triangleq U P U^{-1} = \begin{pmatrix} 0 \\ J_2 \end{pmatrix}, \quad J_2 \in D^{(p-m) \times p}.$$

$$\Rightarrow U = (U_1^T \quad U_2^T)^T, \quad \begin{cases} \ker_D(.P) = D^{1 \times m} U_1 \\ \text{coim}_D(.P) = \pi'(D^{1 \times (p-m)} U_2). \end{cases}$$

A useful proposition

- **Proposition:** Let $R \in D^{q \times p}$ and $P \in D^{p \times p}$, $Q \in D^{q \times q}$ be two matrices satisfying:

$$R P = Q R.$$

Let $U \in \text{GL}_p(D)$ and $V \in \text{GL}_q(D)$ such that

$$\begin{cases} P = U^{-1} J_P U, \\ Q = V^{-1} J_Q V, \end{cases}$$

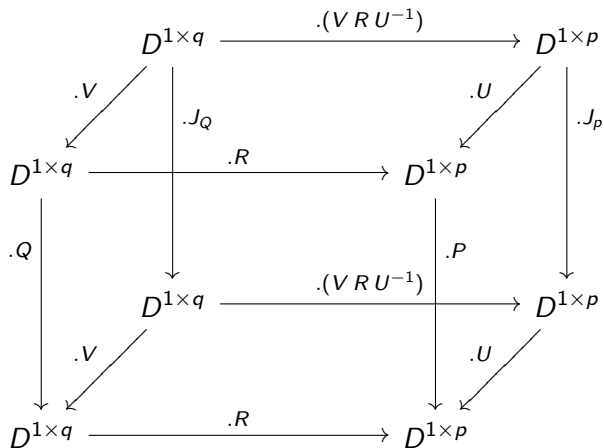
for certain $J_P \in D^{p \times p}$ and $J_Q \in D^{q \times q}$.

Then, the matrix $\bar{R} = V R U^{-1}$ satisfies:

$$\bar{R} J_P = J_Q \bar{R}.$$

A commutative diagram

The following commutative diagram



implies $(VRU^{-1})J_P = J_Q(VRU^{-1})$.

Block triangular decomposition

- **Theorem:** Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $f \in \text{end}_D(M)$ defined by P and Q satisfyin $RP = QR$.

If the **left D -modules**

$$\ker_D(.P), \quad \text{coim}_D(.P), \quad \ker_D(.Q), \quad \text{coim}_D(.Q)$$

are **free** of rank m , $p - m$, l , $q - l$, then there exist two matrices $U = (U_1^T \quad U_2^T)^T \in GL_p(D)$ and $V = (V_1^T \quad V_2^T)^T \in GL_q(D)$ such that

$$\bar{R} = VRU^{-1} = \begin{pmatrix} V_1 R W_1 & 0 \\ V_2 R W_1 & V_2 R W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: OD system

- Let $D = k[t] [\partial; \text{id}, \frac{d}{dt}]$ and $M = D^{1 \times 4} / (D^{1 \times 4} R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4 \times 4}.$$

- An endomorphism f of M is defined by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

- We can prove that the left D -modules $\ker_D(.P)$, $\text{coim}_D(.P)$, $\ker_D(.Q)$ and $\text{coim}_D(.Q)$ are free of rank 2.

Example: OD system

- We obtain:

$$\begin{cases} U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ V_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \end{pmatrix}, & V_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \end{cases}$$
$$\Rightarrow U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & t-1 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

we then obtain that \bar{R} is equivalent to:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} -\partial & 1 & 0 & 0 \\ t\partial - t & -\partial - t & 0 & 0 \\ \partial + t & \partial - 1 & \partial & -t \\ -\partial & 1 & 0 & \partial \end{pmatrix}.$$

Example: Saint-Venant equations

- We consider $D = \mathbb{Q} [\partial_1; \text{id}, \frac{d}{dt}] [\partial_2; \sigma, 0]$ and:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2 \partial_1 \partial_2 \\ 1 & \partial_2^2 & -2 \partial_1 \partial_2 \end{pmatrix}.$$

- A **endomorphism f of M** is defined by:

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 2 \partial_1 \partial_2 & -2 \partial_1 \partial_2 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 2 \partial_1 \partial_2 & -2 \partial_1 \partial_2 \end{pmatrix}.$$

- We can check that $\ker_D(.P)$, $\text{coim}_D(.P)$, $\ker_D(.Q)$ and $\text{coim}_D(.Q)$ are **free D -modules** of rank respectively 2, 1, 1, 1.

$$\Rightarrow \begin{cases} U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \partial_1 \partial_2 \end{pmatrix}, & U_2 = (0 \ 0 \ 1), \\ V_1 = (1 \ 0), & V_2 = (0 \ 1). \end{cases}$$

Example: Saint-Venant equations

- If we denote by

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\partial_1\partial_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we then obtain

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 2\partial_1\partial_2 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(D),$$

and the matrix R is equivalent to:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} \partial_1^2 & -1 & 0 \\ 1 & -\partial_1^2 & 2\partial_1\partial_2(\partial_1^2 - 1) \end{pmatrix}.$$

Ker f , im f , coim f and coker f

- **Proposition:** Let $M = D^{1 \times p} / (D^{1 \times q} R)$, $M' = D^{1 \times p'} / (D^{1 \times q'} R')$ and $f : M \rightarrow M'$ be a **morphism** defined by $R P = Q R'$.

Let us consider the matrices $S \in D^{r \times p}$, $T \in D^{r \times q'}$, $U \in D^{s \times r}$ and $V \in D^{q \times r}$ satisfying $R = V S$, $\ker_D(\cdot S) = D^{1 \times s} U$ and:

$$\ker_D \left(\cdot \begin{pmatrix} P \\ R' \end{pmatrix} \right) = D^{1 \times r} (S \quad -T).$$

Then, we have:

- $\ker f = (D^{1 \times r} S) / (D^{1 \times q} R) \cong D^{1 \times r} / \left(D^{1 \times (q+s)} \begin{pmatrix} U \\ V \end{pmatrix} \right)$,
- $\text{coim } f \triangleq M / \ker f = D^{1 \times p} / (D^{1 \times r} S)$,
- $\text{im } f = D^{1 \times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} / (D^{1 \times q} R) \cong D^{1 \times p} / (D^{1 \times r} S)$,
- $\text{coker } f \triangleq M' / \text{im } f = D^{1 \times p'} / \left(D^{1 \times (p+q')} \begin{pmatrix} P \\ R' \end{pmatrix} \right)$.

Equivalence of systems

- **Corollary:** Let us consider $f \in \text{hom}_D(M, M')$. Then, we have:
 - ① f is **injective** iff one of the assertions holds:
 - There exists $L \in D^{r \times q}$ such that $S = L R$.
 - $\begin{pmatrix} U \\ V \end{pmatrix}$ admits a **left-inverse**.
 - ② f is **surjective** iff $\begin{pmatrix} P \\ R' \end{pmatrix}$ admits a **left-inverse**.
 - ③ f is an **isomorphism**, i.e., $M \cong M'$, iff 1 and 2 are satisfied.

Pommaret's example

- Equivalence of the systems defined by the following R and R' ?

$$R = \begin{pmatrix} \partial_1^2 \partial_2^2 - 1 & -\partial_1 \partial_2^3 - \partial_2^2 \\ \partial_1^3 \partial_2 - \partial_1^2 & -\partial_1^2 \partial_2^2 \end{pmatrix}, \quad R' = (\partial_1 \partial_2 - 1 \quad -\partial_2^2).$$

- We find a **morphism** given by $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 + \partial_1 \partial_2 \\ \partial_1^2 \end{pmatrix}$.
- $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 + \partial_1 \partial_2 \\ \partial_1^2 \end{pmatrix}$ admits the **left-inverse** $(1 - \partial_1 \partial_2 \quad \partial_2^2)$.
- $\begin{pmatrix} P \\ R' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 \partial_2 - 1 & -\partial_2^2 \end{pmatrix}$ admits the **left-inverse** $(I_2 \quad 0)$.

$$\Rightarrow M = D^{1 \times 2} / (D^{1 \times 2} R) \cong M' = D^{1 \times 2} / (D R').$$

V. Decomposition of linear functional systems

Projectors of $\text{end}_D(M)$

- **Lemma:** An endomorphism f of $M = D^{1 \times p} / (D^{1 \times q} R)$, defined by the matrices P and Q , is a **projector**, i.e., $f^2 = f$, iff there exist $Z \in D^{p \times q}$ and $Z' \in D^{q \times t}$ such that

$$\begin{cases} P^2 = P + Z R, \\ Q^2 = Q + R Z + Z' R_2, \end{cases}$$

where $R_2 \in D^{t \times q}$ satisfies $\ker_D(\cdot R) = D^{1 \times t} R_2$.

- Some **projectors** of $\text{end}_D(M)$ can be computed when a **family of endomorphisms of M** is known.
- **Example:** $D = A_1(\mathbb{Q})$, $R = (\partial^2 - t\partial - 1)$, $M = D^{1 \times 2} / (D R)$.

$$P = \begin{pmatrix} -(t+a)\partial + 1 & t^2 + at \\ 0 & 1 \end{pmatrix}, \quad P^2 = P + \begin{pmatrix} (t+a)^2 \\ 0 \end{pmatrix} R.$$

Projectors of $\text{end}_D(M)$ & Idempotents

- **Particular case:** $(R_2 = 0 \text{ and } P^2 = P) \implies Q^2 = Q$.
- **Lemma:** Let us suppose that $R_2 = 0$ and $P^2 = P + ZR$. If there exists a solution $\Lambda \in D^{p \times q}$ of the **Riccatti equation**

$$\Lambda R \Lambda + (P - I_p) \Lambda + \Lambda Q + Z = 0, \quad (\star)$$

then the matrices $\bar{P} = P + \Lambda R$ and $\bar{Q} = Q + R \Lambda$ satisfy:

$$R \bar{P} = \bar{Q} R, \quad \bar{P}^2 = \bar{P}, \quad \bar{Q}^2 = \bar{Q}.$$

- **Example:** $\Lambda = (a t \quad a \partial - 1)^T$ is a solution of (\star)

$$\Rightarrow \bar{P} = \begin{pmatrix} a t \partial^2 - (t + a) \partial + 1 & t^2 (1 - a \partial) \\ (a \partial - 1) \partial^2 & -a t \partial^2 + (t - 2a) \partial + 2 \end{pmatrix}, \bar{Q} = 0,$$

then satisfy $\bar{P}^2 = \bar{P}$ and $\bar{Q}^2 = \bar{Q}$.

Projectors of $\text{end}_D(M)$

- **Proposition:** f is a **projector of $\text{end}_D(M)$** , i.e., $f^2 = f$, iff there exists a matrix $X \in D^{p \times s}$ such that $P = I_p - X S$ and we have the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \ker f & & \\
 & & & & \downarrow i & & \\
 & & D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M \longrightarrow 0 \\
 & & \cdot T \uparrow \downarrow \cdot V & & \cdot P \uparrow \downarrow \cdot I_p & & f \uparrow \downarrow \kappa \\
 D^{1 \times s} & \xrightarrow{\cdot U} & D^{1 \times r} & \xrightarrow{\cdot S} & D^{1 \times p} & \xrightarrow{\pi'} & M / \ker f \longrightarrow 0 \\
 & & & \xleftarrow{\cdot X} & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

$$\Rightarrow M \cong \ker f \oplus \text{im } f \quad \& \quad S - S X S = T R. \quad (*)$$

- **Corollary:** If $\ker_D(\cdot S) = 0$, then $R = V S$ satisfies:

$$S X - T V = I_r.$$

Decomposition of solutions

- **Corollary:** Let us suppose that \mathcal{F} is an **injective left D -module**. Then, we have the following **commutative exact diagram**:

$$\begin{array}{ccccccc}
 & & Vz = 0 = Ry & \longleftarrow & y & & \\
 & & \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(R.) \longleftarrow 0 \\
 & & \uparrow v. & & \parallel & & \uparrow f^* \\
 \mathcal{F}^s & \xleftarrow{U.} & \mathcal{F}^r & \xleftarrow{S.} & \mathcal{F}^p & \longleftarrow & \ker_{\mathcal{F}}(S.) \longleftarrow 0. \\
 & & & \xrightarrow{X.} & & & \\
 0 = Uz & \longleftarrow & z = Sy & \longleftarrow & y & &
 \end{array}$$

Moreover, we have: $Ry = 0 \Leftrightarrow \begin{pmatrix} U \\ V \end{pmatrix} z = 0, \quad Sy = z.$

General solution: $y = u + Xz$ where $Su = 0$ and $\begin{pmatrix} U \\ V \end{pmatrix} z = 0.$

Example: OD system

- Let $D = k[t] \left[\partial; \text{id}, \frac{d}{dt} \right]$ and $M = D^{1 \times 4} / (D^{1 \times 4} R)$, where:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix} \in D^{4 \times 4}.$$

- We obtain the following **idempotent**:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in k^{4 \times 4} : P^2 = P.$$

- We obtain the **factorization** $R = VS$, where:

$$S = \begin{pmatrix} \partial & -t & 0 & 0 \\ 0 & \partial & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & t & \partial \\ 1 & t & \partial & -1 \\ 1 & 0 & \partial + t & \partial - 1 \\ 1 & 1 & t & \partial \end{pmatrix}.$$

Example

- Using the fact that we must have $I_p - P = X S$, we then obtain:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R y = 0 \Leftrightarrow y = u + X z : \quad V z = 0, \quad S u = 0.$$

- The **solution of $S u = 0$** is defined by:

$$u_1 = \frac{1}{2} C_1 t^2 + C_2, \quad u_2 = C_1, \quad u_3 = 0, \quad u_4 = 0.$$

- The **solution of $V z = 0$** is defined by: $z_1 = 0, z_2 = 0$ and
 $z_3(t) = C_3 \text{Ai}(t) + C_4 \text{Bi}(t), \quad z_4(t) = C_3 \partial \text{Ai}(t) + C_4 \partial \text{Bi}(t).$
- The **general solution** of $R y = 0$ is then given by:

$$y = u + X z = \left(\frac{1}{2} C_1 t^2 + C_2 \quad C_1 \quad z_3(t) \quad z_4(t) \right)^T.$$

Idempotents & Projective modules

- **Definition:** A left D -module M is **projective** if there exists a left D -module N and $l \in \mathbb{Z}_+$ such that $M \oplus N \cong D^{1 \times l}$.
- **Lemma:** If $P \in D^{p \times p}$ is an **idempotent**, then:
 - $\ker_D(.P)$ and $\operatorname{im}_D(.P)$ are **projective left D -modules** of rank m and $p - m$.
 - $\operatorname{im}_D(.P) = \ker_D(. (I_p - P))$.
- **Proposition:** Let $P \in D^{p \times p}$ be an **idempotent**. $1 \Leftrightarrow 2$:
 - 1 $\ker_D(.P)$ and $\operatorname{im}_D(.P)$ are **free modules** of rank m and $p - m$.
 - 2 $\exists U \in \operatorname{GL}_p(D)$ satisfying $U P U^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} \end{pmatrix}$

$$\Rightarrow U = (U_1^T \quad U_2^T)^T, \quad \begin{cases} \ker_D(.P) = D^{1 \times m} U_1, \\ \operatorname{im}_D(.P) = D^{1 \times (p-m)} U_2. \end{cases}$$

Block diagonal decomposition

- **Theorem:** Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $f \in \text{end}_D(M)$ defined by P and Q satisfying:

$$P^2 = P, \quad Q^2 = Q.$$

If the left D -modules

$$\ker_D(.P), \quad \text{im}_D(.P), \quad \ker_D(.Q), \quad \text{im}_D(.Q)$$

are free of rank m , $p - m$, l , $q - l$, then there exist two matrices $U = (U_1^T \quad U_2^T)^T \in \text{GL}_p(D)$ and $V = (V_1^T \quad V_2^T)^T \in \text{GL}_q(D)$ such that

$$\bar{R} = V R U^{-1} = \begin{pmatrix} V_1 R W_1 & 0 \\ 0 & V_2 R W_2 \end{pmatrix} \in D^{q \times p},$$

where $U^{-1} = (W_1 \quad W_2)$, $W_1 \in D^{p \times m}$, $W_2 \in D^{p \times (p-m)}$ and:

$$U_1 \in D^{m \times p}, \quad U_2 \in D^{(p-m) \times p}, \quad V_1 \in D^{l \times q}, \quad V_2 \in D^{(q-l) \times q}.$$

Example: OD system

- Let us consider the matrix again:

$$R = \begin{pmatrix} \partial & -t & t & \partial \\ \partial & t\partial - t & \partial & -1 \\ \partial & -t & \partial + t & \partial - 1 \\ \partial & \partial - t & t & \partial \end{pmatrix}.$$

- A projector $f \in \text{end}_D(M)$ is defined by the idempotents

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} t+1 & 1 & -1 & -t \\ 1 & 1 & -1 & 0 \\ t+1 & 1 & -1 & -t \\ t & 1 & -1 & -t+1 \end{pmatrix}.$$

i.e., P and Q satisfy:

$$RP = QR, \quad P^2 = P, \quad Q^2 = Q.$$

- **Computing bases** of the left D -modules

$$\ker_D(.P), \quad \text{im}_D(.P), \quad \ker_D(.P), \quad \text{im}_D(.Q),$$

we obtain the **unimodular matrices**:

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -t & -1 & 1 & t \\ t+1 & 1 & -1 & -t \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

- R is then equivalent to the following **block diagonal matrix**:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} \partial & -1 & 0 & 0 \\ t & \partial & 0 & 0 \\ 0 & 0 & \partial & -t \\ 0 & 0 & 0 & \partial \end{pmatrix}.$$

Example: Saint-Venant equations

- We consider $D = \mathbb{Q} [\partial_1; \text{id}, \frac{d}{dt}] [\partial_2; \sigma, 0]$ and:

$$R = \begin{pmatrix} \partial_2^2 & 1 & -2 \partial_1 \partial_2 \\ 1 & \partial_2^2 & -2 \partial_1 \partial_2 \end{pmatrix}.$$

- A **projector** $f \in \text{end}_D(M)$ is defined by the **idempotents**

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

i.e., P and Q satisfy:

$$RP = QR, \quad P^2 = P, \quad Q^2 = Q.$$

Example: Saint-Venant equations

$$\begin{cases} U_1 = \ker_D(.P) = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \\ U_2 = \operatorname{im}_D(.P) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ V_1 = \ker_D(.Q) = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\ V_2 = \operatorname{im}_D(.Q) = \begin{pmatrix} 1 & 1 \end{pmatrix}, \end{cases}$$

and we obtain the following two **unimodular matrices**:

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

- We easily check that we have the following **block diagonal matrix**:

$$\bar{R} = V R U^{-1} = \begin{pmatrix} \partial_2^2 - 1 & 0 & 0 \\ 0 & 1 + \partial_2^2 & -4 \partial_1 \partial_2 \end{pmatrix}.$$

Corollary

• **Corollary:** Let $R \in D^{q \times p}$, $M = D^{1 \times p} / (D^{1 \times q} R)$ and $f \in \text{end}_D(M)$ be defined by P and Q and satisfying $P^2 = P$ and $Q^2 = Q$. Let us suppose that one of the conditions holds:

- 1 $D = A[\partial; \sigma, \delta]$, where A is a field and σ is injective,
- 2 $D = k[\partial_1; \sigma_1, \delta_1] \dots [\partial_n; \sigma_n, \delta_n]$ is a commutative Ore algebra,
- 3 $D = A[\partial_1; \text{id}, \delta_1] \dots [\partial_n; \text{id}, \delta_n]$, where $A = k[x_1, \dots, x_n]$ or $k(x_1, \dots, x_n)$ and k is a field of characteristic 0, and:

$$\begin{aligned} \text{rank}_D(\ker_D(.P)) &\geq 2, & \text{rank}_D(\text{im}_D(.P)) &\geq 2, \\ \text{rank}_D(\ker_D(.Q)) &\geq 2, & \text{rank}_D(\text{im}_D(.Q)) &\geq 2. \end{aligned}$$

Then, there exist $U \in \text{GL}_p(D)$ and $V \in \text{GL}_q(D)$ such that $\bar{R} = V R U^{-1}$ is a block diagonal matrix.

Example: Flexible rod

- Let us consider the **flexible rod** (Mounier 95):

$$R = \begin{pmatrix} \partial_1 & -\partial_1 \partial_2 & -1 \\ 2 \partial_1 \partial_2 & -\partial_1 \partial_2^2 - \partial_1 & 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 + \partial_2^2 & -\frac{1}{2} \partial_2^2 (1 + \partial_2) & 0 \\ 2 \partial_2 & -\partial_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -\frac{1}{2} \partial_2 \\ 0 & 0 \end{pmatrix},$$

$$\Rightarrow U = \begin{pmatrix} -2 \partial_2 & \partial_2^2 + 1 & 0 \\ -2 & \partial_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 2 & -\partial_2 \end{pmatrix},$$

$$\Rightarrow \bar{R} = V R U^{-1} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 (\partial_2^2 - 1) & -2 \end{pmatrix}.$$

V. Implementation: the Maple MORPHISMS package

The MORPHISMS package

- The algorithms have been implemented in a **Maple package** called **MORPHISMS** based on the library OREMODULES developed by Chyzak, Q. and Robertz:

<http://wwwb.math.rwth-aachen.de/OreModules>

- **List of functions:**
 - Morphisms, MorphismsConst, MorphismsRat, MorphismsRat1.
 - Projectors, ProjectorsConst, ProjectorsRat, Idempotents.
 - KerMorphism, ImMorphism, CokerMorphism, CoimMorphism.
 - TestSurj, TestInj, TestBij.
 - QuadraticFirstIntegralConst. . .
- **It will be soon available with a library of examples**

Conclusion

- Contributions:

- We use **constructive homological algebra** to provide algorithms for studying general LFSs (e.g., factoring or decomposing).
- We apply the obtained results in control theory.

- Work in progress:

Using morphism computations for factoring and decomposing general linear functional systems, in the proceedings of the Mathematical Theory of Networks and Systems (MTNS), Kyoto (Japan), 2006, rapport INRIA.

- Open questions:

- Bounds in the general case.
- Criteria for choosing the right P .
- Existence of a solution to the Riccati equation.
- Formulas for connections. . .