# Fast algorithms for computing isogenies between elliptic curves

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# Motivations

- original one: SEA;
- later: Kohel, Galbraith, Fouquet/FM (volcanoes);
- more recently: Galbraith/Hess/Smart; Smart; Jao/Miller/Venkatesan; Teske; Rostovtsev/Stolbunov.

**Bibliography:**

- green book (Blake Seroussi Smart). Don't forget to read the original papers, when available...
- Gathen & Gerhard, etc.



# Plan

I. Elliptic curves and isogenies.

II. Fast series computations.

III. Computing the Weierstrass $\wp$-function.

IV. Previous algorithms for computing isogenies.

V. Our fast variant.

VI. Benchmarks.



# I. Elliptic curves and isogenies

$$E : y^2 = x^3 + Ax + B \text{ over } \mathbf{K}, \mathrm{char}(\mathbf{K}) \notin \{2, 3\}.$$

**Def.** (torsion points) For $n \in \mathbb{N}$, $E[n] = \{P \in E(\overline{\mathbf{K}}), [n]P = O_E\}$.

**Division polynomials:**

$$[n](x, y) = \left( \frac{\phi_n(x,y)}{\psi_n(x,y)^2}, \frac{\omega_n(x,y)}{\psi_n(x,y)^3} \right)$$

$$\phi_n = x\psi_n^2 - \psi_{n+1}\psi_{n-1}$$

$$4y\omega_n = \psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2$$

In $\mathbf{K}[x, y]/(y^2 - (x^3 + Ax + B))$, one has:

$$\psi_{2m+1}(x, y) = f_{2m+1}(x), \quad \psi_{2m} = 2y f_{2m}(x)$$

for some $f_m(x) \in \mathbf{K}[A, B, x]$.

$$f_n(x) = \begin{cases} \psi_n(x,y) & \text{for } n \text{ odd} \\ \psi_n(x,y)/(2y) & \text{for } n \text{ even} \end{cases}$$

$$f_{-1} = -1, \quad f_0 = 0, \quad f_1 = 1, \quad f_2 = 1$$

$$f_3(x,y) = 3x^4 + 6Ax^2 + 12Bx - A^2$$

$$f_4(x,y) = x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3$$

$$f_{2n} = f_n(f_{n+2}f_{n-1}^2 - f_{n-2}f_{n+1}^2)$$

$$f_{2n+1} = \begin{cases} f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1}(16y^4) & \text{if } n \text{ is odd} \\ (16y^4)f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1} & \text{otherwise.} \end{cases}$$

$$\deg(f_n(x)) = \begin{cases} (n^2-1)/2 & \text{if } n \text{ is odd} \\ (n^2-4)/2 & \text{otherwise.} \end{cases}$$

**Thm.** $P = (x,y) \in E[\ell] \iff [2]P = O_E$ or $f_\ell(x) = 0$.



# Isogenies

**Def.** $\phi : E \to \tilde{E}$, $\phi(O_E) = O_{\tilde{E}}$; induces a morphism of groups.

**First examples**
1. Separable:
$$[k](x,y) = \left(\frac{\phi_k}{\psi_k^2}, \frac{\omega_k}{\psi_k^3}\right)$$
2. Complex multiplication: $[i](x,y) = (-x, iy)$ on $E : y^2 = x^3 - x$.
3. Inseparable: $\varphi(x,y) = (x^p, y^p)$, $\mathbf{K} = \mathbb{F}_p$.

**In the sequel:** only separable isogenies.

**Thm.** If $F$ is a finite subgroup of $E(\overline{\mathbf{K}})$, then there exists $\phi$ and $\tilde{E}$ s.t.

$$\phi : E \to \tilde{E} = E/F, \quad \ker(\phi) = F.$$



# How does an isogeny look like?

Extending Vélu, Dewaghe. Let

$$D(x) = \prod_{Q \in F^*} (x - x_Q) = x^{\ell-1} - \sigma x^{\ell-2} + \cdots .$$

where $\sigma = \sum_{Q \in F^*} x_Q$.
**Rem.** When $\ell$ is odd, $D(x) = g(x)^2$.

**Fundamental proposition.** The isogeny $I$ can be written as

$$I(x,y) = \left(\frac{N(x)}{D(x)}, y\left(\frac{N(x)}{D(x)}\right)'\right),$$

$$\frac{N(x)}{D(x)} = \ell x - \sigma - (3x^2 + A)\frac{D'(x)}{D(x)} - 2(x^3 + Ax + B)\left(\frac{D'(x)}{D(x)}\right)'$$

$$= \ell x - \sigma - 2\sqrt{x^3 + Ax + B}\left(\sqrt{x^3 + Ax + B}\,\frac{D'(x)}{D(x)}\right)'.$$



# Proof of the fundamental proposition

Vélu gives:

$$x_{I(P)} = x_P + \sum_{Q \in F^*} (x_{P+Q} - x_Q).$$

Injecting the group law:

$$I_x(x) = x + \sum_{Q \in F^*} \left(\frac{3x_Q^2 + A}{x - x_Q} + 2\frac{x_Q^3 + Ax_Q + B}{(x - x_Q)^2}\right),$$

which can be rewritten as

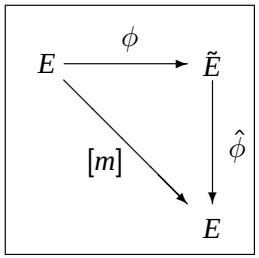
$$I_x(x) \quad = \quad x + \sum_{Q \in F^*} \left(x - x_Q - \frac{3x^2 + A}{x - x_Q} + 2\frac{x^3 + Ax + B}{(x - x_Q)^2}\right).$$

Finish using

$$\sum_{Q \in F^*} \frac{1}{x - x_Q} = \frac{D'(x)}{D(x)}, \quad \sum_{Q \in F^*} \frac{1}{(x - x_Q)^2} = -\left(\frac{D'(x)}{D(x)}\right)'.\square$$

# Dual isogeny

**Thm.** (dual isogeny) There is a unique $\hat{\phi} : \tilde{E} \to E$, $\hat{\phi} \circ \phi = [m]$, $m = \deg\phi$.




**Application to SEA:** $D$ will be a divisor of $\psi_m^2$ (resp. $g \mid f_\ell$).



# Last but not least: Elliptic curves over $\mathbb{C}$

When $\mathbf{K} = \mathbb{C}$, $E = \mathbb{C}/L = \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$, $E : y^2 = x^3 + Ax + B$ can be parametrized using the Weierstrass function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) = \frac{1}{z^2} + \sum_{k=1}^{\infty} c_k(L) z^{2k},$$

where $c_1 = -A/5$, $c_2 = -B/7$.

**Useful remark:** if $E$ and $\tilde{E}$ are $\ell$-isogenous, then

$$\tilde{\wp} = \frac{N}{D} \circ \wp.$$

**Fact.** The curve $\tilde{E} = \mathbb{C}/(\omega_1/\ell, \omega_2)$ is $\ell$-isogenous to $E$. The associated kernel is:

$$F = \{O_E\} \cup \left\{ (\wp(r\omega_1/\ell), \frac{1}{2}\wp'(r\omega_1/\ell)), 1 \leq r \leq \ell - 1 \right\}.$$



# Facts: the Galois point of view

We can put it this way: rewrite

$$f_\ell(X) = \prod_{1 \leq r,s \leq \ell-1} (X - \wp((r\omega_1 + s\omega_2)/\ell)).$$

The factor we build is:

$$D(x) = \prod_{1 \leq r \leq \ell-1} (X - \wp(r\omega_1/\ell))$$

and all its coefficients are in $\mathbf{K}[\sigma]$ where $\sigma = \sum_r \wp(r\omega_1/\ell)$.

$$\begin{array}{c} \mathbf{K}[x]/(f_\ell(x)) \\ | \qquad \ell - 1 \\ \mathbf{K}[x]/(M_\sigma(x)) \\ | \qquad \ell + 1 \\ \mathbf{K}[x] \end{array}$$

If $\sigma$ is rational over $\mathbf{K}$, then $D(x)$ will have rational coefficients.
**Modular equation:** $M_\sigma(x) = \Phi_\ell(x, j(E))$.



# Application to SEA

**Key point of Elkies:** find prime $\ell$ for which there exists a rational $\ell$-isogeny from $E$ (detected via modular polynomials $\Phi_\ell(X, Y)$; happens with proba $1/2$). Then $g(x) \mid f_\ell(x)$ with $\deg(g) = (\ell - 1)/2$.

**Black box:** imagine we have a way to compute $(\tilde{E}, \sigma)$ given $\mathbf{K}$, $E$, $\ell$, (some) $\Phi_\ell$.

In brief, over $\mathbb{F}_p$ ($p \gg \ell$), do as if we were over $\mathbb{C}$, use modular equations to build $\tilde{E}$ and $\sigma$.

(see FM's 1994/03/07 talk...)

In the remaining part of the talk: compute $D(x)$ as fast as possible.

# Examples

**Ex 1.** $E : y^2 = x^3 + bx$, $F = \langle (0,0) \rangle$;

$$\tilde{E} : y^2 = x^3 - 4bx,$$

$$\phi : (x,y) \mapsto \left( \frac{x^3 + bx}{x^2}, y\frac{x^2 - b}{x^2} \right).$$

**A curiosity:** $E : y^2 = x^3 + x + 3$ defined over $\mathbb{F}_{1009}$; $E$ is 6-isogenous to

$$\tilde{E} : y^2 = x^3 + 830x + 82$$

and $\sigma = 739$ (formulas for prime $\ell$ valid here too!!!) for which

$$\frac{N(x)}{D(x)} = \frac{x^6 + 270\,x^5 + 325\,x^4 + 566\,x^3 + 382\,x^2 + 555\,x + 203}{x^5 + 270\,x^4 + 289\,x^3 + 659\,x^2 + 533\,x + 399}.$$

The denominator factors as

$$(x - 66)\,(x - 23)^2\,(x - 818)^2\,.$$

$x = 66$ is the abscissa of a point of 2-torsion;
23 is the abscissa of a point of 3-torsion;
818 is the abscissa of a primitive point of 6-torsion.



# II. Fast series computations

**Fundamental references:** Brent and Kung; Gathen & Gerhard.
Classical stuff: the multiplication time function M satisfies the following classical super-linearity inequalities for all $n$ and $n'$:

$$\frac{\mathsf{M}(n)}{n} \leq \frac{\mathsf{M}(n')}{n'} \quad \text{if } n \leq n' \qquad \text{and} \qquad \mathsf{M}(nn') \leq n^2\mathsf{M}(n'),$$

which implies:

$$\mathsf{M}(1) + \mathsf{M}(2) + \mathsf{M}(4) + \cdots + \mathsf{M}(2^i) \leq 2\mathsf{M}(2^i).$$

$\Rightarrow$ all algorithms based on Newton's iteration below have complexity in $O(\mathsf{M}(n))$.



**Easy facts:** given a series $A(z) = a_0 + a_1 z + \cdots + O(z^n)$, computing $\int A(z)\,\mathrm{d}z$ or $A'(z)$ is possible in $O(n)$.

**Less easy facts:** computing $A(z)B(z)$, $1/A(z)$, $\log A(z)$, $\exp(A(z))$ may be computed in $O(\mathsf{M}(n))$; use Newton when needed. Idem for recovering a polynomial $f$ of degree $n$ from its first $n$ power sums $p_1, \ldots, p_n$ (Schönhage), since

$$z^n f\left(\frac{1}{z}\right) = \exp_{n+1}\left( -\sum_{i=1}^{n} \frac{p_i}{i} z^i \right).$$

**First order linear differential equation:** with $R(0) = r_0$, one solves $R'(z) + F(z)R(z) \equiv G(z) \bmod z^n$ as:

$$H(z) = \exp\left( \int_0^z F(u)\,\mathrm{d}u \right), R(z) = \left( \frac{1}{H(z)} \right) \left( r_0 + \int_0^z G(u)H(u)\,\mathrm{d}u \right)$$

in time $O(\mathsf{M}(n))$.



# Application: expanding $\wp$

$$E = \mathbb{C}/L(\omega_1, \omega_2)$$

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) = \frac{1}{z^2} + \sum_{k=1}^{\infty} c_k(L) z^{2k}.$$

With

$$A = -5c_1, B = -7c_2$$

then $(\wp(z), \wp'(z)/2)$ is a point on $E : y^2 = x^3 + Ax + B$.
By exploiting $\wp'' = 6\wp^2 + 2A$, one gets for $k \geq 3$:

$$c_k = \frac{3}{(k-2)(2k+3)} \sum_{h=1}^{k-2} c_h c_{k-1-h}$$

**Cost:** computing $\wp$ to order $n$ costs $O(n^2)$.

# An $O(\mathrm{M}(d))$ method for $\wp$

**First idea:** (sketch) put $\wp(z) = Q(z)/z^2$, which gives

$$z^2Q'^2 - 4zQQ' - 4Q^3 + 4Q^2 - 4Az^4Q - 4Bz^6 = 0. \qquad (1)$$

$$Q_0(z) = 1 - \frac{A}{5}z^4 - \frac{B}{7}z^6$$

is a solution of (1) at order $s = 8$.With $Q(z) = Q_0 + z^8R(z)$:

$$R'(z) + F(z)R(z) \equiv G(z) \bmod z^8$$

where:

$$F(z) = 5\,\frac{6\,Bz^6 + 7\,Az^4 + 63}{(10\,Bz^6 + 7\,Az^4 + 35)\,z},$$

$$G(z) = \frac{1}{175}\,\frac{105\,A^2Bz^6 + 49\,A^3z^4 + 125\,B^2z^4 + 525\,ABz^2 + 735\,A^2}{(10\,Bz^6 + 7\,Az^4 + 35)\,z}.$$

$$R(z) = \frac{1}{75}\,A^2 + \frac{3}{385}\,ABz^2 + \left(\frac{1}{637}\,B^2 - \frac{2}{4875}\,A^3\right)z^4 - \frac{2}{5775}A^2Bz^6.$$



**Second (better) approach:** Define

$$Q(z) = \frac{1}{\wp(z)} \in z^2 + z^6\mathbf{K}[[z^2]] \quad \text{and} \quad R(z) = \sqrt{Q(z)} \in z + z^5\mathbf{K}[[z^2]].$$

The differential equation satisfied by $R$ is

$$R'(z)^2 = B\,R(z)^6 + A\,R(z)^4 + 1.$$

First terms:

$$R(z) = z + \frac{A}{10}z^5 + \frac{B}{14}z^7 + O(z^8) = z\left(1 + \frac{A}{10}z^4 + \frac{B}{14}z^6 + O(z^7)\right).$$

**Algorithm:**

1. Compute $R(z) \bmod z^{2n+4}$ using linearization;
2. Compute $Q(z) = R(z)^2 \bmod z^{2n+5}$;
3. Compute $\wp(z) = 1/Q(z) \bmod z^{2n+1}$.

**Thm.** The first $n$ coefficients of the expansion of $\wp$ can be computed in $O(\mathrm{M}(n))$ field operations.



# Benchmarks

NTL C++ library on an AMD 64 Processor 3400+ (2.4GHz).

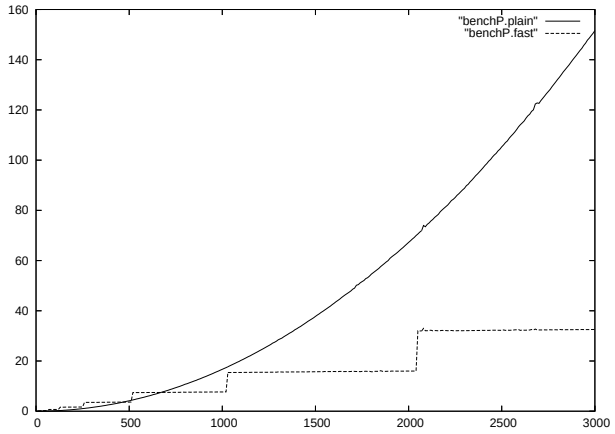



Figure: Timings for computing $\wp$ on $E : y^2 = x^3 + 4589x + 91128$ over $\mathbb{F}_{10^{2004}+4683}$



# IV. Previous algorithms for computing isogenies

**Elementary remark:**

$$\frac{N}{D} \circ \frac{\hat{N}}{\hat{D}} = \frac{\phi_\ell}{\psi_\ell^2}$$

but all methods to solve this require factoring $\psi_\ell$, which is just too costly. . .

**Remember:** if $E$ and $\tilde{E}$ are $\ell$-isogenous, then

$$\tilde{\wp} = \frac{N}{D} \circ \wp.$$

Indeterminate coefficients, linear system $\Rightarrow O(\ell^\omega)$, $2 \le \omega \le 3$.
Use

$$\frac{N}{D} = \tilde{\wp} \circ \wp^{-1},$$

but modular composition is too costly ($O(\mathrm{M}(\ell)\sqrt{\ell} + \ell^{\frac{\omega+1}{2}})$ or $O(\mathrm{M}(\ell)\sqrt{\ell \log \ell})$).

# A) Stark's method (1972)

INPUT: $E$ and $\tilde{E}$ related via an $\ell$-isogeny [$\sigma$ not required]
OUTPUT: $I(x) = N(x)/D(x)$.
ALGORITHM: develop $\tilde{\wp}(z)$ as a continued fraction in $\wp(z)$.

1. First write $\wp(z) = \wp(Z)$ with $Z = z^2$; $T := \tilde{\wp}(Z) + O(Z^\ell)$;
2. $n := 1$; $q_0 := 1$; $q_1 := 0$;
3. **while** $\deg(q_n) < \ell - 1$ **do**
   {inv: $T(Z) = t_{-r}Z^{-r} + \cdots + t_0 + t_1 Z + \cdots + O(Z^{(\ell - \deg q_n - r) - 1})$}
   3.1 $n := n + 1$; $a_n := 0$;
   3.2 **while** $r \geq 1$ **do**
   $a_n := a_n + t_{-r} z^r$;
   $T := T - t_{-r}\wp^r = t_{-s}Z^{-s} + \cdots$;
   $r := s$
   3.3 $q_n := a_n q_{n-1} + q_{n-2}$;
   3.4 $T := 1/T$;
4. Return $D := q_n$.

**Cost:** $O(\ell \mathrm{M}(\ell)) + O(\mathrm{M}(\ell))$.



$E : y^2 = x^3 + x + 1 \bmod 503$, $\tilde{E} : \tilde{y}^2 = \tilde{x}^3 + 463\tilde{x} + 478 \bmod 503$, $\ell = 11$.

$$\wp(z) = z^{-2} + 201\,z^2 + 431\,z^4 + 389\,z^6 + 260\,z^8 + O(z^{10}),$$

$$\tilde{\wp} = z^{-2} + 8\,z^2 + 291\,z^4 + 189\,z^6 + 452\,z^8 + O(z^{10}).$$

$$n = 0 : \tilde{\wp} = \wp + 310\,z^2 + 363\,z^4 + 303\,z^6 + 192\,z^8 + O(z^{10}),$$

$$\begin{aligned} & 1/(310\,z^2 + 363\,z^4 + 303\,z^6 + 192\,z^8 + O(z^{10})) \\ = \;& 86\,z^{-2} + 266 + 419\,z^2 + 22\,z^4 + 427\,z^6 + O(z^8) \\ = \;& 86\wp + 266 + \cdots \end{aligned}$$

etc. Finally:

$$I(x) = \frac{x^{11} + 335\,x^{10} + 288\,x^9 + 161\,x^8 + 221\,x^7 + 288\,x^6 + 110\,x^5 + 112\,x^4 + 434\,x^3 + 125\,x^2 + 291\,x + 260}{x^{10} + 335\,x^9 + 481\,x^8 + 69\,x^7 + 382\,x^6 + 150\,x^5 + 221\,x^4 + 92\,x^3 + 324\,x^2 + 366\,x + 4}$$

and denominators factors as:

$$(x + 488)^2\,(x + 128)^2\,(x + 103)^2\,(x + 99)^2\,(x + 104)^2\,.$$



# B) Atkin

By integrating some formula involving $\zeta(z) = -\int \wp$, one gets:

$$D(\wp(z)) = z^{2-2\ell} \exp(F(z))$$

where

$$F(z) = -\sigma z^2 + 2\left(\sum_{k=1}^{\infty} (\ell c_k - \tilde{c}_k) \frac{z^{2k+2}}{(2k+1)(2k+2)}\right).$$

1. Compute the series $P_i(Z) = \wp(Z)^i$ at order $\ell - 1$, for $1 \leq i \leq \ell - 1$;
2. Compute $G(Z) = \exp_\ell(F(Z)) = \exp(F(Z)) \bmod Z^\ell$;
3. $T := G$; $D := 0$;
4. **for** $i := \ell - 1$ **downto** $0$ **do**
   {at this point, $T = tZ^{-i} + \cdots$}
   4.1 $D := D + tz^i$;
   4.2 $T := T - tP_i$.

**Cost:** $O(\ell^2) + O(\ell \mathrm{M}(\ell))$.

**Pb:** rather huge constants. We need $\sigma$.



**A better algorithm:** rewrite

$$D\left(\frac{1}{x}\right) = \mathcal{I}^{2-2\ell}\left((\exp \circ F) \circ \mathcal{I}\right), \qquad (2)$$

with $\mathcal{I}(x) = \wp^{-1}(1/x)$, where $\wp^{-1}$ is the functional inverse of $\wp$, where

$$\mathcal{I}'(x)^2 = \frac{1}{4x(1 + Ax^2 + Bx^3)} \quad \text{or} \quad \mathcal{I}'(x) = \frac{1}{2\sqrt{x}} \frac{1}{\sqrt{1 + Ax^2 + Bx^3}}$$

$\mathcal{I}(x) = x^{\frac{1}{2}} \sum_{i \geq 0} \frac{a_i}{2i+1} x^i$, with

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -\frac{A}{2},$$

$$a_{i+1} = \frac{B a_{i-2} - 2Bi a_{i-2} - 2A i a_{i-1}}{2i + 2} \quad \text{for} \quad i \geq 2.$$

AtkinModComp:

1. Compute $G(Z) = \exp_\ell(F(Z))$;
2. Compute $\mathcal{I}(x)$;
3. Compute $G(\mathcal{I})$ by modular composition (which is possible since $G$ is in $\mathbf{K}[[Z]] = \mathbf{K}[[x^2]]$);
4. Deduce $D$ using Equation (2).

**Cost:** dominated by the composition of the series $G = \exp \circ F$ and $\mathcal{I}$, which can be done in $O(\mathsf{M}(\ell)\sqrt{\ell} + \ell^{\frac{\omega+1}{2}})$ or $O(\mathsf{M}(\ell)\sqrt{\ell \log \ell})$ operations in $\mathbf{K}$.



# C) Elkies92 (case $D = g^2$)

$$\frac{d^{2k}\wp(z)}{dz^{2k}} = \mu_k(k+1)\wp^{k+1} + \cdots + \mu_k(0)$$

**Cost:** computing all coefficients up to $k = d$ costs $O(d^2)$ but with very small constants.
**Idea:** compute $p_k = \sum_r \wp(r\omega_1/\ell)^k$ via:

$$(2k)!(\tilde{c}_k - c_k) = 2(\mu_k(0)p_0 + \cdots + \mu_k(k+1)p_{k+1})$$

for $1 \le k \le d$. In particular:

$$A - \tilde{A} = 5(6p_2 + 2Ap_0), \quad B - \tilde{B} = 7(10p_3 + 6Ap_1 + 4Bp_0).$$

Once all $p_k$ are known, use Newton's formulas; by Schönage, this costs $O(\mathsf{M}(d))$.
**Total cost:** $O(d^2)$ but with very small constants (again!).



# D) Elkies98

Start from

$$\frac{N(x)}{D(x)} = \tilde{\wp} \circ \wp^{-1}(x)$$

and apply the chain rule to get:

$$(x^3 + Ax + B)\left(\frac{N(x)}{D(x)}\right)'^2 = \left(\frac{N(x)}{D(x)}\right)^3 + \tilde{A}\left(\frac{N(x)}{D(x)}\right) + \tilde{B}.$$

A second differentiation leads to

$$(3x^2 + A)\left(\frac{N(x)}{D(x)}\right)' + 2(x^3 + Ax + B)\left(\frac{N(x)}{D(x)}\right)'' = 3\left(\frac{N(x)}{D(x)}\right)^2 + \tilde{A}.$$



Writing

$$\frac{N(x)}{D(x)} = x + \sum_{i\ge1} \frac{h_i}{x^i}$$

and identifying coefficients of $x^{-i}$, $i \ge 1$, one deduces

$$h_1 = \frac{A - \tilde{A}}{5} \quad \text{and} \quad h_2 = \frac{B - \tilde{B}}{7},$$

and for all $k \ge 3$:

$$h_k = \frac{3}{(k-2)(2k+3)} \sum_{i=1}^{k-2} h_i h_{k-1-i} - \frac{2k-3}{2k+3} A h_{k-2} - \frac{2(k-3)}{2k+3} B h_{k-3}.$$

$\Rightarrow h_3, \ldots, h_{\ell-2}$ using $O(\ell^2)$ operations in $\mathbf{K}$.
Expanding at infinity the right-hand side of Vélu's formulas, one connects the power sums $p_1 = \sigma, p_2, \ldots$ of $D(x)$ to the coefficients $h_i$ for $i \ge 1$:

$$h_i = (2i+1)p_{i+1} + (2i-1)Ap_{i-1} + (2i-2)Bp_{i-2}.$$

$\Rightarrow O(\ell)$ operations.
**Total complexity:** $O(\ell^2)$.

# V. Our fast variant

**Idea:** improve on the computation of the $h_i$'s in Elkies98.

**Remember:** $R(z) = 1/\sqrt{\wp(z)}$ and $\tilde{R}(z) = 1/\sqrt{\tilde{\wp}(z)}$.

There exists $S$ s.t. $\tilde{R} = S \circ R$ from which

$$\frac{N(x)}{D(x)} = \frac{1}{S\left(\frac{1}{\sqrt{x}}\right)^2}.$$

$$S(x) = x + \frac{\tilde{A} - A}{10}x^5 + \frac{\tilde{B} - B}{14}x^7 + O(x^9) \in x + x^3\mathbf{K}[[x^2]]$$

is such that

$$(Bx^6 + Ax^4 + 1)\,S'(x)^2 = 1 + \tilde{A}\,S(x)^4 + \tilde{B}\,S(x)^6.$$



Write:

$$S(x) = xT(x^2) \quad \text{and} \quad U(x) = \frac{1}{T(x)^2} \in 1 + x^2\mathbf{K}[[x]]$$

so that

$$\frac{N(x)}{D(x)} = x\,U\left(\frac{1}{x}\right).$$

We assume $\sigma$ is known:

1. Compute $C(x) = (Bx^6 + Ax^4 + 1)^{-1} \bmod x^{2\ell-1} \in \mathbf{K}[[x]]$;
2. Compute $S(x) \bmod x^{2\ell}$ as usual and deduce $T(x) \bmod x^{\ell}$;
3. Compute $U(x) = 1/T(x)^2 \bmod x^{\ell}$;
4. Compute the coefficients $h_1, \ldots, h_{\ell-2}$ of $N(x)/D(x)$, using $N(x)/D(x) = xU(1/x)$;
5. Compute the power sums $p_2, \ldots, p_{\ell-1}$ of $D(x)$, using the linear recurrence;
6. Recover $D(x)$ from its power sums;
7. Deduce $N(x)$.

**Cost:** Steps (1) and (5) have cost $O(\ell)$. Steps (2), (3), (6) and (7) can be performed in $O(\mathsf{M}(\ell))$ operations, and Step (4) requires no operation. $\Rightarrow O(\mathsf{M}(\ell))$.



# We can do without $\sigma$

(1') Compute $C(x) = (Bx^6 + Ax^4 + 1)^{-1} \bmod x^{8\ell-5} \in \mathbf{K}[[x]]$;
(2') Compute $S(x) \bmod x^{8\ell-4}$ and deduce $T(x) \bmod x^{4\ell-2}$;
(3') Compute $U(x) = 1/T(x)^2 \bmod x^{4\ell-2}$;
(4') Reconstruct the rational function $U(x)$;
(5') Return $N(x)/D(x) = xU(1/x)$.

Using fast rational reconstruction, Step (4') can be performed in $O(\mathsf{M}(\ell)\log\ell)$ operations in $\mathbf{K}$.



# Remarks

1. In the case of odd $\ell$, we can compute $g(x)$ instead of $D(x)$, *mutatis mutandis*.

2. There is a fastAtkin method, which is actually the same fastElkies when looking up the details.

# VI. Benchmarks

The first series of timings concerns the computation of isogenies over a small field, $\mathbf{K} = \mathbb{F}_{10^{19}+51}$, for the curve $E : y^2 = x^3 + 4589x + 91128$.

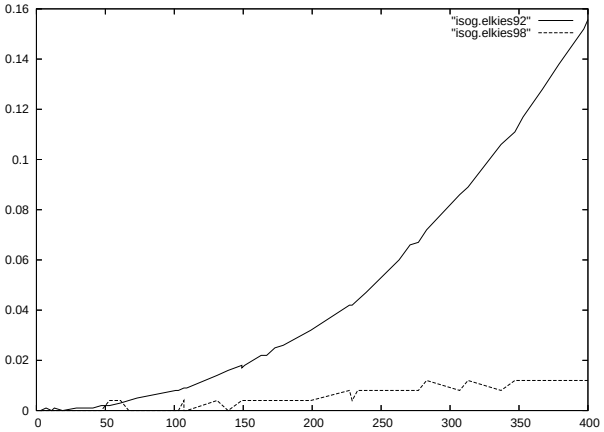



Figure: Elkies1992 vs. Elkies1998.



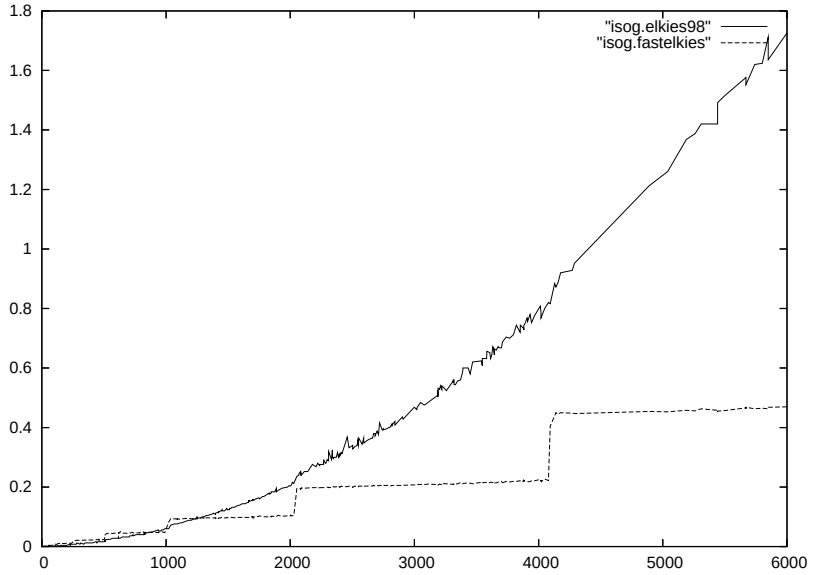



Figure: Elkies1998 vs. fastElkies.



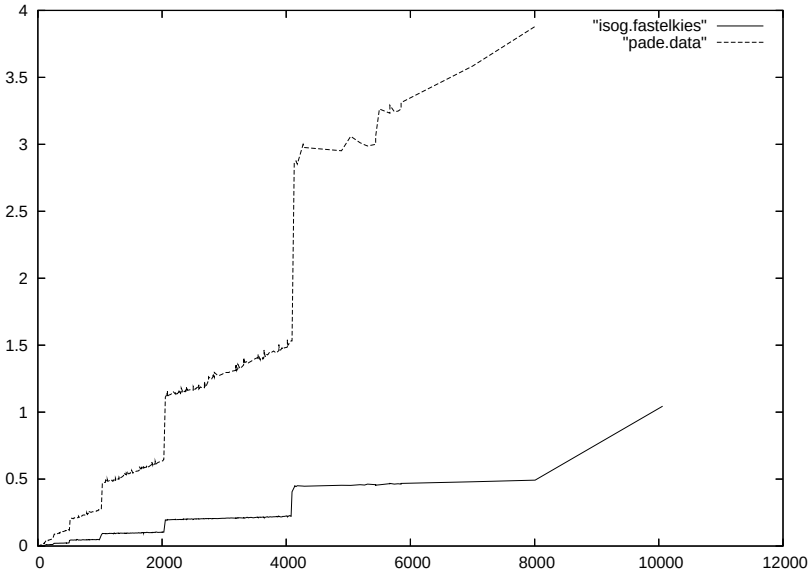



Figure: FastElkies vs. FastElkies′



Computing $\ell$-isogenies for the curve $E : y^2 = x^3 + Ax + B$ where

$$A = \lfloor 10^{1990}\pi \rfloor = 31415926\ldots58133904,$$

$$B = \lfloor 10^{1990}e \rfloor = 27182818\ldots94787610,$$

over $\mathbb{F}_{10^{2004}+4683}$.

| $\ell$ | Computing $\wp$ and $\tilde{\wp}$ | | | Recovering $g$ | |
|---|---|---|---|---|---|
| | order | quadratic | fast | quadratic | fast |
| 1013 | 511 | 8.6 | 7.0 | 4.2 | 1.1 |
| 2039 | 1024 | 34.6 | 29.9 | 17.4 | 2.5 |
| 3019 | 1514 | 75.7 | 30.3 | 38.2 | 5.1 |
| 4001 | 2005 | 132.7 | 31 | 66.9 | 5.5 |
| 5021 | 2515 | 209.3 | 64.4 | 106.2 | 11.2 |

| $\ell$ | $\wp, \tilde{\wp}$ | Inverses | | $q_n$ |
|---|---|---|---|---|
| | | quadratic | fast | |
| 1013 | | 23542 | 1222.7 | 28.0 |
| 2039 | See | ?? | 5113.4 | 116.9 |
| 3019 | Table | ?? | 12182 | 258 |
| 4001 | | ?? | 20388 | 418.6 |
| 5021 | | ?? | 38910 | 663.1 |

Table: Stark's algorithm



| $\ell$ | $\wp, \tilde{\wp}$ | Algorithm Aktin1992 | | | |
|---|---|---|---|---|---|
| | | exponential | | $\wp^k$ | $g$ |
| | | naive | fast | | |
| 1013 | | 88.4 | 1.2 | 72.3 | 4.4 |
| 2039 | See | 370.1 | 4.9 | 304.9 | 17.7 |
| 3019 | Table | 955.9 | 5.1 | 755.8 | 38.9 |
| 4001 | | 1503 | 5.2 | 1218.9 | 67.6 |
| 5021 | | 3180 | 10.8 | 2506.4 | 108.7 |

| $\ell$ | $\wp, \tilde{\wp}$ | Algorithm AtkinModComp | | | | |
|---|---|---|---|---|---|---|
| | | $\exp(F)$ | $\mathcal{I}^{1-\ell}$ | modular composition | | $g$ |
| | | | | ModComp1 | ModComp2 | |
| 1013 | | 1.2 | 2.7 | 14.3 | 35.6 | 0.2 |
| 2039 | See | 2.5 | 6.6 | 45.8 | 111.9 | 0.4 |
| 3019 | Table | 5.1 | 10.4 | 95.3 | 241 | 0.7 |
| 4001 | | 5.2 | 11.6 | 143.2 | 338 | 0.9 |
| 5021 | | 10.9 | 20.9 | 240 | 642 | 1.4 |



| $\ell$ | $\wp, \tilde{\wp}$ | Elkies1992 | | |
|---|---|---|---|---|
| | | $\mu$ | $p_i$ | $g$ |
| 1013 | | 10.4 | 4.4 | |
| 2039 | See | 49.1 | 17.9 | See |
| 3019 | Table | 130.6 | 38.9 | Table |
| 4001 | | 263 | 68.4 | |
| 5021 | | 496.5 | 106.6 | |

| $\ell$ | Elkies1998 and fastElkies | | | |
|---|---|---|---|---|
| | $h_i$ | | $p_i$ | $g$ |
| | quadratic | fast | | |
| 1013 | 4.4 | 4.5 | 0.05 | |
| 2039 | 17.3 | 9.6 | 0.1 | See |
| 3019 | 38.0 | 19.5 | 0.16 | Table |
| 4001 | 67.2 | 20.0 | 0.21 | |
| 5021 | 105.0 | 40.7 | 0.27 | |



# Conclusions

**Cryptographic sizes:** Elkies98 the best algorithm by far ($\ell \leq 300$).

**What's left to be done?** the case $p$ small, not for SEA.

Fast versions required for

- CouveignesI (formal groups)?
- Lercier ($p = 2$)?
- Couveignes II (Artin-Schreier towers)?

**Maybe:** use a $p$-adic approach à la Joux/Lercier (2006).