

# Recurrence Relations and Inequalities

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# Proving Inequalities by Computer Algebra

(with M. Kauers, ISSAC 2005)

- We want to prove  $a_n > 0, n \geq 0$
- $a_n$  polynomially recursive (not necessarily holonomic)

$$a_{n+s} = P(a_n, \dots, a_{n+s-1}), \quad n \geq 0.$$

Example:  $2^{2^n}$

- induction step

$$a_n > 0, \dots, a_{n+r-1} > 0 \implies a_{n+r} > 0$$

- Sufficient:

$$X_0 > 0, \dots, X_{r-1} > 0 \implies X_r > 0 \quad (*)$$

for all real numbers  $X_0, \dots, X_r$  that satisfy polynomial equations arising from the recurrence of  $a_n$ .

- Increase  $r$ , if formula (\*) does not hold;  
or encode known inequalities/identities as additional inequalities/equations for the  $X_k$ .

## Proving Inequalities by Computer Algebra

- $a_n$  may involve undetermined parameters
- Works on many examples (Cauchy-Schwarz, Bernoulli, Turán, ...)
- But no a priori termination criterion known
- Example gallery (omitting some constraints):

$$(x + 1)^n \geq 1 + nx, \quad n \geq 0, x \geq -2 (!)$$

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0$$

$$\left( \sum_{k=1}^n x_k y_k \right)^2 \leq \sum_{k=1}^n x_k^2 \sum_{k=1}^n y_k^2$$

$$\prod_{k=1}^n (1 - a_k) > 1 - \sum_{k=1}^n a_k$$

$$(n - 1) \left( \sum_{k=1}^n a_k \right)^2 \geq 2n \sum_{k=1}^n a_k \sum_{i=1}^{k-1} a_i$$

$$\sqrt{n + \sqrt{(n-1) + \sqrt{\cdots + \sqrt{2 + \sqrt{1}}}}} < \sqrt{n} + 1$$

## Linear Recurrences with Constant Coefficients

- Is there a subclass over which positivity is decidable?
- C-finite sequences are a natural candidate:

$$c_0 a_n + c_1 a_{n+1} + \cdots + c_d a_{n+d} = 0, \quad n \geq 0.$$

- Question:  $a_n > 0$ ,  $n$  large?
- power sum representation

$$a_n = \sum_{k=1}^d b_k \alpha_k^n,$$

$\alpha_k$  roots of characteristic polynomial (assumed to be simple here).

- dominating roots := roots of largest modulus
- Usually the sign is trivial:

$$\begin{aligned} 3^n + (-2)^n &> 0 \\ (-3)^n + (-2)^n &\leq 0 \end{aligned}$$

## Sequences with no Positive Dominating Root

- What is the sign of  $(2 + i)^n + (2 - i)^n$ ?
- (Side remark: Lower estimate

$$5^{n/2}/n^C < |(2 + i)^n + (2 - i)^n|$$

Schinzel 1967, based on Baker's theorem about linear forms in logarithms)

- Conjecture: infinitely many sign changes, if no dominating root is positive.
- 1 dominating root: trivial (root has to be real negative)
- 2 dominating roots: Burke, Webb 1981

Example:

$$(2 + i)^n + (2 - i)^n \leq 0$$

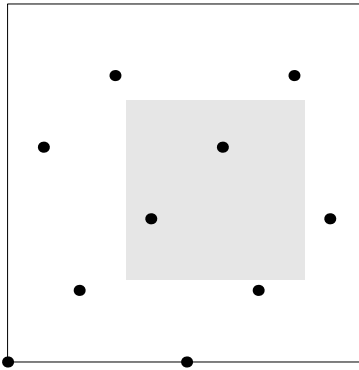
- 3 or 4 dominating roots: SG 2005 (By Diophantine geometry)

Example:

$$(-\sqrt{5})^n + (2 + i)^n + (2 - i)^n \leq 0$$

## Sequences with no Positive Dominating Root: Two Pairs of Conjugated Roots

- $a_n = w_1 \sin(2\pi n\theta_1 + \varphi_1) + w_2 \sin(2\pi n\theta_2 + \varphi_2)$
- Example:  $(\theta_1, \theta_2) = (\frac{7}{10}, \frac{1}{5})$
- The set  $\{(7n/10, n/5) \bmod 1 : n \in \mathbb{N}\}$



- Every square with side length  $1/2$ , parallel to the axes, contains a point (Minkowski's theorem from Diophantine geometry)
- Hence  $(\sin(2\pi n\theta_1 + \varphi_1), \sin(2\pi n\theta_2 + \varphi_2))$  assumes all four sign combinations

$$(+1, +1), (+1, -1), (-1, +1), (-1, -1).$$

- Hence  $a_n$  oscillates for all  $w_1, w_2$  (not both zero).

## Sequences with no Positive Dominating Root: General Case

- **Theorem** (Bell, SG 2005): Let  $a_n$  be a C-finite sequence, not identically zero, with no positive dominating root. Then the sets  $\{n : a_n > 0\}$  and  $\{n : a_n < 0\}$  have positive density.
- **Theorem** (Kronecker, Weyl): The sequence  $(n\theta_1, \dots, n\theta_m)$  is uniformly distributed modulo 1, if the numbers  $1, \theta_1, \dots, \theta_m$  are linearly independent over  $\mathbb{Q}$ .
- Idea: Study the function of  $\mathbf{t}$  resulting from  $a_n$  upon replacing each  $n\theta_k$  by a real  $t_k$ .
- The other “extreme case”: all  $\theta_k$  are rational. Let  $q$  be a common denominator.

$$\begin{aligned}\sum_{j=0}^{q-1} a_j &= \sum_{j=0}^{q-1} \sum_{i=1}^m w_i \sin(2\pi j\theta_i + \varphi_i) \\ &= \sum_{i=1}^m w_i \sum_{j=0}^{q-1} (\cos \varphi_i \cdot \sin 2\pi j\theta_i + \sin \varphi_i \cdot \cos 2\pi j\theta_i) \\ &= 0.\end{aligned}$$

## Which Numbers Occur as Density of the Positivity Set?

- No positive dominating root: Numerical experiments usually yield approximations  $\approx 1/2$ .
- **Theorem:** For every  $\kappa \in ]0, 1[$ , there is a C-finite sequence  $a_n$  with no positive dominating root and

$$\text{density}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa.$$

- With positive dominating root: all numbers from  $[0, 1]$  occur.
- **Theorem:**  $r$  rational,  $0 \leq \kappa, r \leq 1$ ,  $\kappa + r \leq 1$ . Then there is a C-finite sequence  $a_n$  with

$$\text{density}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa,$$

$$\text{density}(\{n \in \mathbb{N} : a_n = 0\}) = r.$$

- $r$  must be rational by the Skolem-Mahler-Lech theorem (zero set is periodic).

## Conclusion

- Inequalities are more difficult than identities
- Still, many examples can be done by the method we presented
- But even positivity of C-finite sequences is not known to be decidable
- Diophantine methods reveal oscillating behaviour
- Problem: Do we have

$$1 + \sin(2\pi\theta n) + (-1/2)^n > 0, \quad \text{where } \theta = \sqrt{2}?$$

(Remark: for almost all  $\theta$ , this sequence is eventually positive.)