Computing Multisections of a Given Series Solution of a Linear Differential Operator

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Statement of the Problem

The r-multisections of a power series $\hat{f} = \sum_{m\geq 0} a_m x^m$ are the

$$\hat{f}^j = a_{0r+j}x^{0r+j} + a_{1r+j}x^{1r+j} + a_{2r+j}x^{2r+j} + \cdots = \sum_{m\geq 0} a_{rm+j}x^{rm+j}.$$

Problem: Given a linear differential operator $L = L(x, \partial)$, compute a common annihilator of the $\hat{f}^j$ for any $0 \leq j < r$ and any solution $\hat{f}$ of L.

Dual Problem: Given a linear recurrence operator $L^* = L^*(m, \tau)$, compute a common annihilator of the $a^j = (a_{rm+j})_{m\geq 0}$ for any $0 \leq j < r$ and any solution $(a_m)_{m\geq 0}$ of L^*.

Multisections Computation Also Known as Decimation

Decimation of linear recurrent sequences, i.e., solutions of recurrences with constant coefficients, related to a generalization of the Graeffe polynomial:

- automatic series and number theory
- cryptography
- Fibonacci Quarterly
- "Fast computation of special resultants" (BoFlSaSc06)

Summability of Divergent Series

For $k > 0$, the series $\hat{f} \in \mathbb{C}[[x]]$ is k-summable in the direction $d \in \mathbb{R}$ if there exist a sector $S(d, \alpha, \rho)$ of aperture $\alpha > \pi/k$ and a function f admitting $\hat{f}$ as asymptotic expansion of order k, that is:

$$\forall \bar{T} \subset S,\ \exists C, K > 0,\ \forall n \in \mathbb{N},\ \forall x \in \bar{T},$$

$$\left| f(x) - \sum_{j=0}^{n-1} a_j x^j \right| \leq C K^n |x|^n \Gamma(1 + n/k).$$

The function f is then unique and called “sum of f.” (Ramis)

Summation of k-Summable Series

Borel transform: $\hat{B}_1(x^{n+1}) = t^n/n!, \quad \hat{B}_k(x^n) = t^{n-k}/\Gamma(n/k).$

Laplace transform: $\mathfrak{L}_k(\phi)(x) = \int_d \phi(t)\exp(-t^k/x^k)\,d(t^k).$

$$\mathfrak{L}_k\hat{B}_k(x^n) = x \mapsto x^n$$

$$\begin{array}{ccc} k\text{-summable DV series } \hat{f} & \xrightarrow{\hat{B}_k} & \text{CV series} \\ \text{asymptotic}\uparrow\text{expansion} & & \downarrow\text{sum} \\ \text{sum function } f & \xleftarrow{\mathfrak{L}_k} & \text{function} \end{array}$$

$\hat{f}$ is k-summable $\Longrightarrow$ the k-multisections $\hat{f}^j$ are 1-summable.

Summation of 1-summable series is numerically more stable.

Summing Euler's Divergent Series

$$\hat{g} = \sum_{n\geq 0} (-1)^n n!\, x^{n+1}$$

$$\hat{B}_1 \hat{g} = \sum_{n\geq 0} (-1)^n t^n \text{ converges to the function } t \mapsto \frac{1}{1+t}$$

$$g = \mathcal{L}_1 \hat{B}_1 \hat{g} = \left(x \mapsto \int_0^\infty \frac{\exp(-t/x)}{1+t} dt \right)$$

Note: $(x^2\partial + 1)\hat{g} = x$, so that $(x\partial - 1)(x^2\partial + 1)\hat{g} = 0$.

Multisummability

For $k_1 > \cdots > k_r > 0$, the series $\hat{f} \in \mathbb{C}[[x]]$ is $(k_1, \ldots, k_r)$-summable in the direction $d \in \mathbb{R}$ if $\hat{f} = \hat{f}_1 + \cdots + \hat{f}_r$ where $\hat{f}_i$ is k_i-summable. The sum of $\hat{f}$ is then given as $f = f_1 + \cdots + f_r$ where f_i is the sum of $\hat{f}_i$. (Écalle, Malgrange, Ramis)

Theoretical approaches to multisummation: by iterated Laplace and Borel transforms (Balser); by accélératrices (Écalle).

Numerically, multisections to reduce to 1-summable series is a good, stable approach (Thomann, Jung, Naegelé).

$$\hat{f} = \hat{g}(x) + \hat{g}(x^2) \quad \rightarrow \quad \text{is } (2,1)\text{-summable (Ramis–Sibuya).}$$

Formal Mellin Transform

$$\left(\sum_{m\in\mathbb{Z}} a_m x^m\right)' = \sum_{m\in\mathbb{Z}} (m+1)a_{m+1}x^m, \quad x\sum_{m\in\mathbb{Z}} a_m x^m = \sum_{m\in\mathbb{Z}} a_{m-1}x^m.$$

∂ = differential operator d/dx: $\partial x - x\partial = 1$

δ = Eulerian operator $x\,d/dx$: $\delta x - x\delta = x$

Algebra isomorphism $\mathbb{C}\langle x, x^{-1}, \partial\rangle \simeq \mathbb{C}\langle m, \sigma, \tau\rangle$.

σ = forward shift operator with respect to m: $\sigma m = (m+1)\sigma$

τ = backward shift operator with respect to m: $\tau m = (m-1)\tau$

For Euler's series: $(x\partial - 1)(x^2\partial + 1) \leftrightarrow (m-1)\big((m-1)\tau + 1\big)$.

Saturation Under the Galois Group of $\omega^r = 1$

$$r\hat{f}^j = \hat{f}(x) + \omega^{-j}\hat{f}(\omega^j x) + \cdots + \omega^{-(r-1)j}\hat{f}(\omega^{j(r-1)}x)$$

$$\mathbb{C}\hat{f}^0 \oplus \cdots \oplus \mathbb{C}\hat{f}^{r-1} = \mathbb{C}\hat{f}(x) \oplus \mathbb{C}\hat{f}(\omega x) \oplus \cdots \oplus \mathbb{C}\hat{f}(\omega^{r-1}x)$$

$$\operatorname{ann}(\hat{f}^0, \ldots, \hat{f}^{r-1}) = \operatorname{lclm}\big(\operatorname{ann}\hat{f}(x), \operatorname{ann}\hat{f}(\omega x), \ldots, \operatorname{ann}\hat{f}(\omega^{r-1}x)\big)$$

$$x \to \omega x,\ \partial \to \omega^{-1}\partial,\ \delta \to \delta,\ m \to m,\ \tau \to \omega\tau,\ \sigma \to \omega^{-1}\sigma$$

Computations in $\mathbb{Q}(\omega, x)\langle\partial\rangle$, resp. $\mathbb{Q}(\omega, m)\langle\sigma\rangle$.

Variant Elimination

Replace $\omega \in \mathbb{C}$ with an indeterminate w:

$$x \to wx,\ \partial \to w^{-1}\partial,\ \tau \to w\tau,\ \sigma \to w^{-1}\sigma.$$

$L \in \mathbb{Q}\langle x, \partial\rangle \quad \to \quad$ Bézout relation in $\mathrm{Frac}(\mathbb{Q}\langle x, \partial\rangle)[w]$:

$$S(x, \partial, w)L(wx, w^{-1}\partial) + T(x, \partial, w)(w^r - 1) = u(x)M(x, \partial),$$

for $S, L, M \in \mathbb{Q}\langle x, \partial\rangle$.

$L \in \mathbb{Q}\langle m, \tau\rangle \quad \to \quad$ Bézout relation in $\mathrm{Frac}(\mathbb{Q}\langle m, \tau\rangle)[w]$:

$$S(m, \tau, w)L(m, w\tau) + T(m, \tau, w)(w^r - 1) = u(m)M(m, \tau),$$

for $S, L, M \in \mathbb{Q}\langle m, \tau\rangle$.

Computation by Hadamard Product

Bruno Salvy's first instinct: $\hat{f}^j = \hat{f} \odot \frac{x^j}{1-x^r}$, where

$$\sum_{m\geq 0} a_m x^m \odot \sum_{m\geq 0} b_m x^m = \sum_{m\geq 0} a_m b_m x^m.$$

Hadamard product algorithm (e.g., in *gfun*) specializes to:

W.l.o.g., $L \in \mathbb{Q}(m)\langle\tau\rangle$ is monic of some order, o, so for any $k \geq 0$,

$$\tau^{kr} \in \bigoplus_{\ell=0}^{o-1} \mathbb{Q}(m)\tau^\ell.$$

$\mathbb{Q}(m)$-dependency between the τ^{kr} by Gaussian elimination.

Direct Computation of Differential Invariants

Really a dual of the Hadamard product approach.

$$\mathbb{Q}(x) = \bigoplus_{i=0}^{r-1} \mathbb{Q}(t)x^i \qquad \text{for } t = x^r, \text{ so}$$

$$L(x,\delta) = L_0(t,\delta) + xL_1(t,\delta) + \cdots + x^{r-1}L_{r-1}(t,\delta).$$

W.l.o.g., $L \in \mathbb{Q}(x)\langle\delta\rangle$ is monic of some order, o, so for any $k \geq 0$,

$$\delta^k \in \bigoplus_{i=0}^{r-1} \bigoplus_{j=0}^{o-1} \mathbb{Q}(t)x^i\delta^j.$$

$\mathbb{Q}(t)$-dependency between the δ^k by Gaussian elimination.

Companion Systems and Saturation Again

$$L\hat{f} = 0 \quad \leftrightarrow \quad x^{r+1}Y' = A(x)Y, \quad Y = (\hat{f}, \dots, \hat{f}^{(r-1)})^T$$

$$(CY)' = \big(C' + x^{-(r+1)}CA\big)Y, \qquad C = (c_0, \dots, c_{r-1})$$

$\rightarrow$ Cyclic-vector method on $C = (1, 0, \dots, 0)$ to recover L.

$$x^{r+1}\tilde{Y}' = \left(\bigoplus_{j=0}^{r-1} A(\omega^j x)\right)\tilde{Y}, \quad \tilde{Y} = \big(Y(x), Y(\omega x), \dots, Y(\omega^{r-1}x)\big)^T$$

$\rightarrow$ Cyclic-vector method on $\tilde{C} = (C, \dots, C)$ to compute ann $\hat{f}^0$.

Recurrence analogue of the form $\tau\tilde{Y} = \left(\bigoplus_{j=0}^{r-1} \omega^j A(m)\right)\tilde{Y}$.

Algorithm Inspired by Turrittin's Rank Reduction

In terms of multisections of Y and A, $x^{r+1}Y' = AY$ becomes

$$x^{r+1}(Y^0 + \cdots + Y^{r-1})' = (A^0 + \cdots + A^{r-1})(Y^0 + \cdots + Y^{r-1}).$$

Thus, we consider $x^{r+1}\tilde{Y}' = \tilde{A}\tilde{Y}$, for (Loday-Richaud 2001)

$$\tilde{A} = (A^{i-j})_{i,j=0}^{r} = \begin{pmatrix} A^0 & A^{r-1} & \dots & A^1 \\ A^1 & A^0 & \dots & A^2 \\ \vdots & \vdots & \ddots & \vdots \\ A^{r-1} & A^{r-2} & \dots & A^0 \end{pmatrix} \quad \text{and} \quad \tilde{Y} = \begin{pmatrix} Y^0 \\ Y^1 \\ \vdots \\ Y^{r-1} \end{pmatrix}.$$

Cyclic-vector method with $(1, 0, \dots, 0)$.

Recurrence analogue: $\tau^r \tilde{Y} = A(m - r + 1) \dots A(m + 1)A(m)\tilde{Y}$.

Comparing Outputs of Dual Methods

$$L\hat{f} = 0 \quad \leftrightarrow \quad L^{*}a = 0$$

$\mathbb{C}\langle x, x^{-1}, \partial\rangle \simeq \mathbb{C}\langle m, \sigma, \tau\rangle$ but calculations in $\mathbb{Q}(x)\langle\partial\rangle$ and $\mathbb{Q}(m)\langle\tau\rangle$.

$$\Lambda_1(x, \partial) = \operatorname{ann}\left(\hat{f}^0, \ldots, \hat{f}^{r-1}\right), \qquad \Lambda_2(m, \tau) = \operatorname{ann}\left(a^0, \ldots, a^{r-1}\right).$$

$$\Lambda_1 \in \mathbb{Q}\langle x^r, \delta\rangle, \qquad \Lambda_2 \in \mathbb{Q}\langle m, \tau^r\rangle.$$

$$Q(m, \tau^r)\Lambda_1(x, \partial)^{*} = u(\tau^r)v(m)\Lambda_2(m, \tau).$$

Timings

Ramis–Sibuya example with Maple on an Alpha EV6.7 at 667 MHz:
(† means ≥ 1400s; ‡ means ≥ 128MB.)

r	2	3	4	5	6	7	8	9	10
A1	0.9	51	151	†					
A1*	1.5	‡							
A2	2.4	†							
A2*	2.0	‡							
A3	.6	2.3	3.0	9.0	8.9	28.7	21.5	196.9	50.0
A3*	.5	2.4	2.4	12.5	10.2	43.0	30.2	123.0	71.0
A4	.4	45.2	26.0	‡					
A4*	.8	121.2	1050.0	†					
A5(1)	.5	1.9	2.8	9.6	9.8	41.0	28.0	231.0	71.3
A5*(1)	.3	1.6	1.5	6.4	4.7	19.1	11.6	47.1	25.0
A5(2)	.5	4.0	6.6	89.0	56.0	†			
A5*(2)	.9	4.6	6.0	30.1	25.3	150	88	642.0	341.3

→ Use systems and avoid introducing algebraic numbers.

→ Complexity analysis missing. Resultants? Padé–Hermite?