Fast computation of power series solutions of systems of differential equations

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joint work with

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Context

find *fast* algorithms for solving (in power series) ordinary differential equations

applications: combinatorics, numerical analysis (control), algebraic complexity

- *fast* means using *few operations* $(\pm, \times, \div)$ in the base field $\mathbb{K}$.

More precisely

Given a linear differential equation with power series coefficients

$$a_r(t)y^{(r)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0$$

compute the first N terms of a (basis of) power(s) series solution(s) $y(t)$.

- naive algorithm (undetermined coefficients), $\mathcal{O}(rN^2)$.
- Best that can be hoped: complexity linear in N and polynomial in r.
- Same problem for linear systems $Y' = AY$ and for non-linear systems.

Fast polynomial (matrix) multiplication

$$\begin{aligned}
\mathsf{M}(N) &= \text{complexity of polynomial multiplication in degree } < N \\
&= \mathcal{O}(N^2) \text{ by the naïve algorithm} \\
&= \mathcal{O}\left(N^{1.58}\right) \text{ by Karatsuba's algorithm} \\
&= \mathcal{O}\left(N^{\log_\alpha (2\alpha-1)}\right) \text{ by Toom-Cook algorithm} \\
&= \mathcal{O}\left(N \log(N) \operatorname{loglog}(N)\right) \text{ by Schönhage–Strassen FFT}
\end{aligned}$$

$$\begin{aligned}
\mathsf{MM}(r, N) &= \text{complexity of polynomial matrix mult., } \deg < N, \text{ size} = r \\
&= \mathcal{O}(r^\omega \, \mathsf{M}(N)) \text{ by the Cantor–Kaltofen algorithm} \\
&= \mathcal{O}(r^\omega N + r^2 \, \mathsf{M}(N)) \text{ by the B.–Schost algorithm}
\end{aligned}$$

Previous results

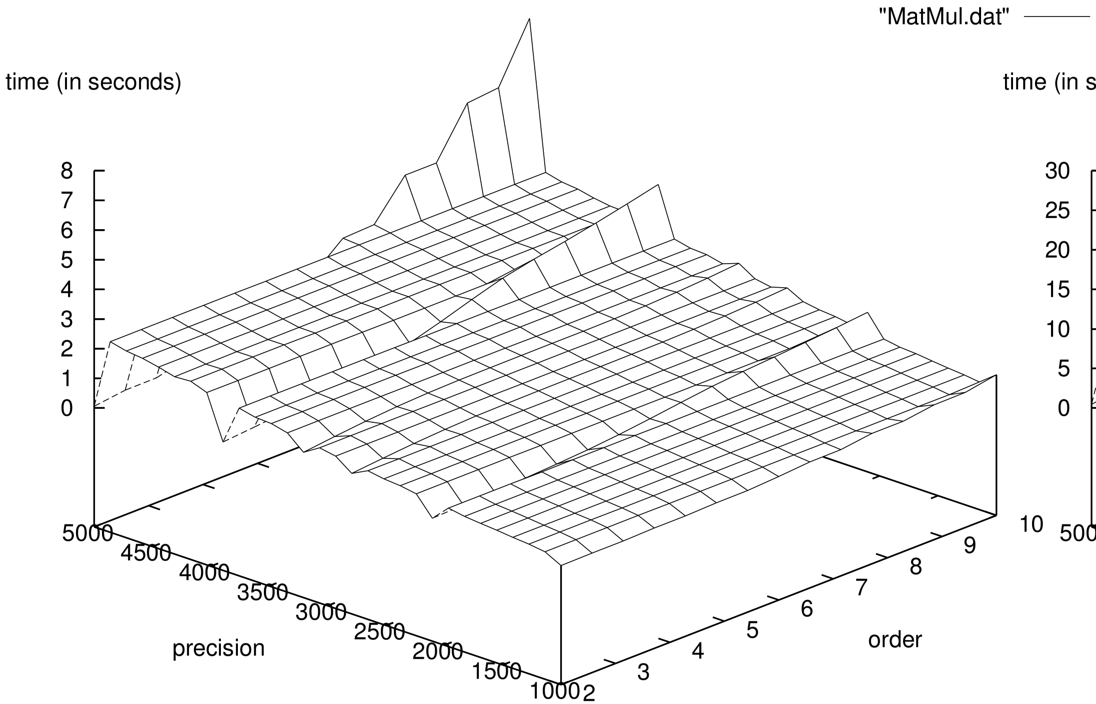
- $r = 1$ Brent & Kung (1978): reduction to exponentials of power series + Newton iteration, $\mathcal{O}(\mathsf{M}(N))$.
- $r = 2$ Brent & Kung (1978): reduction to Riccati non-linear equations + linearization, $\mathcal{O}(\mathsf{M}(N))$.
- $r > 1$ Brent & Kung (1978), van der Hoeven (2002): order reduction + linearization, $\mathcal{O}(r^r \, \mathsf{M}(N))$.

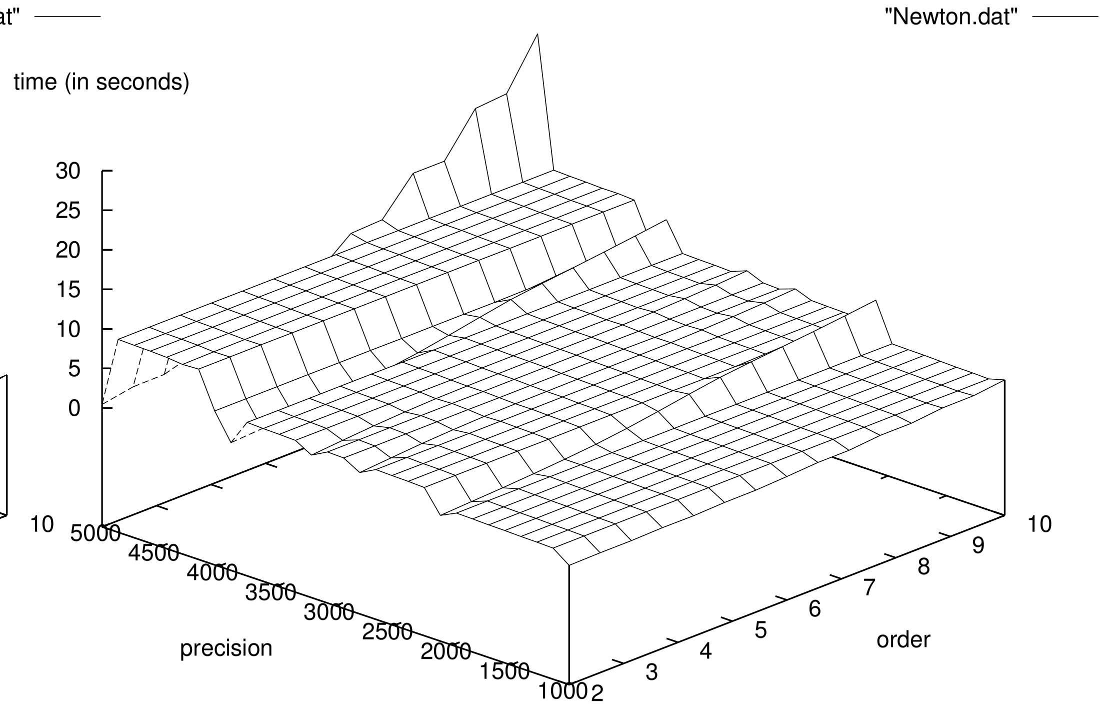
- $r > 1$ van der Hoeven (2002) 1 solution in $\mathcal{O}(r \, \mathsf{M}(N) \log N)$, model of relaxed multiplication

New results

Problem (input, output)	constant coefficients	polynomial coefficients	power series coefficients	output size
(eqn, basis)	$\mathcal{O}(\mathsf{M}(r)N)$	$\mathcal{O}(dr^2N)$	$\mathcal{O}(\mathsf{MM}(r,N))$	$\mathcal{O}(rN)$
(eqn, 1 sol)	$\mathcal{O}(\mathsf{M}(r)N/r)$	$\mathcal{O}(drN)$	$\mathcal{O}(r\,\mathsf{M}(N)\log N)$	$\mathcal{O}(N)$
(sys, basis)	$\mathcal{O}(r\mathsf{M}(r)N)$	$\mathcal{O}(dr^\omega N)$	$\mathcal{O}(\mathsf{MM}(r,N))$	$\mathcal{O}(r^2N)$
(sys, 1 sol)	$\mathcal{O}(\mathsf{M}(r)N)$	$\mathcal{O}(dr^2N)$	$\mathcal{O}(r^2\,\mathsf{M}(N)\log N)$	$\mathcal{O}(rN)$

Experimental results



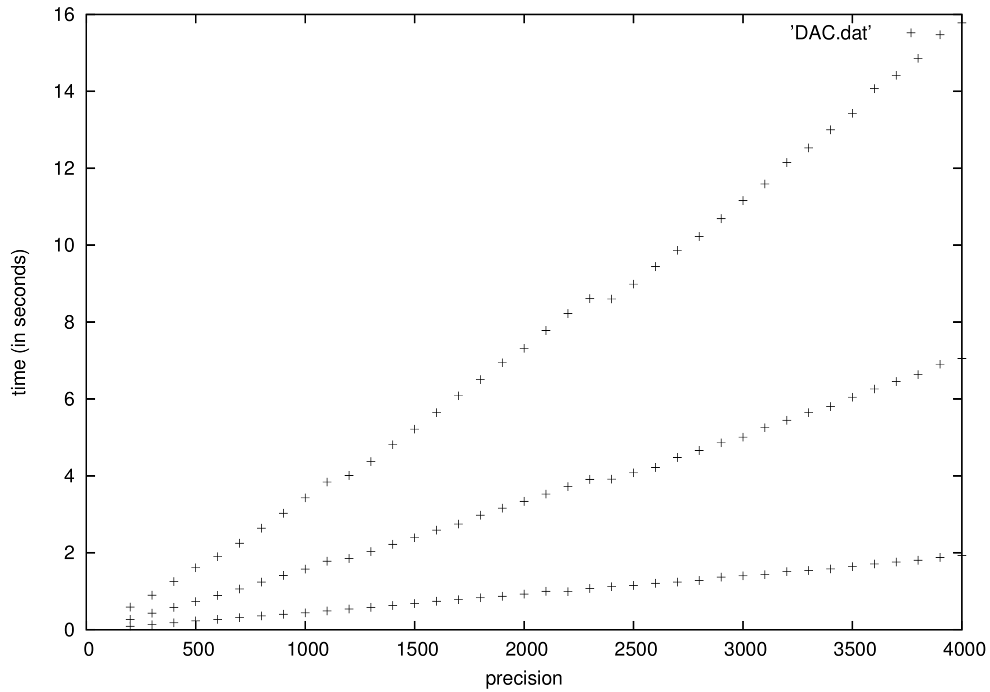


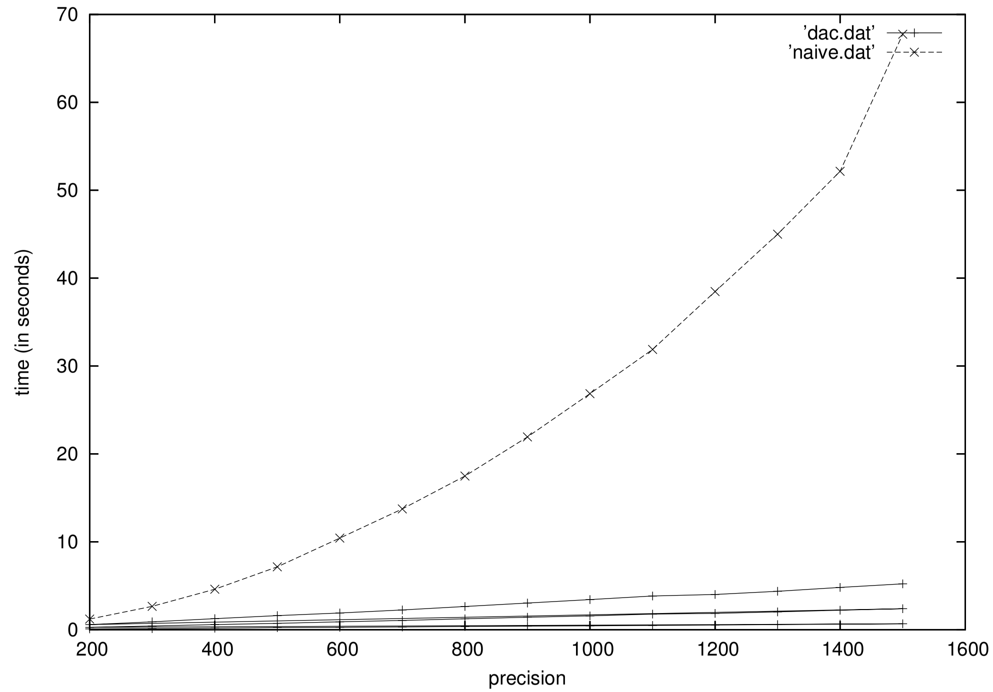
Experimental results

$N \cdot^{\cdot} \cdot r$	2	4	8	16
256	0.02 vs. 2.09	0.08 vs. 6	0.44 vs. 28	3 vs. 169
512	0.04 vs. 8	0.17 vs. 25	1 vs. 113	6.41 vs. 688
1024	0.08 vs. 32	0.39 vs. 104	2.30 vs. 484	15 vs. 2795
2048	0.18 vs. 128	0.94 vs. 424	5.54 vs. 2025	36 vs. $> \mathbf{3}$h *
4096	0.42 vs. 503	2.26 vs. 1686	13.69 vs. 8348	92 vs. $> \mathbf{1/2}$ day*

Basis, system with r equations, precision N: new vs. naive

Experimental results





Left: DAC computation of one solution for LDE of orders 2, 4, and 8

Right: DAC vs. naive, one solution of a LDE of order 2

The divide-and-conquer algorithm

Problem: solve $\mathcal{L}y = 0$, where $\mathcal{L} = \sum_i a_i(t)\delta^i$, with $\delta = t\frac{d}{dt}$.

Idea: the lowest terms of $y(t)$ only depend on the lowest terms of a_i.

Proof: if $y = y_0 + t^m y_1$, then $\mathcal{L}(\delta)y = \mathcal{L}(\delta)y_0 + t^m\mathcal{L}(\delta + m)y_1$

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DAC algorithm to solve $\mathcal{L}(\delta)y = 0 \mod t^{2m}$:

1. determine y_0, by recursively solving $\mathcal{L}(\delta)y_0 = 0 \mod t^m$;
2. compute the "error" R, such that $\mathcal{L}(\delta)y_0 = t^m R \mod t^{2m}$;
3. compute y_1, by recursively solving $\mathcal{L}(\delta + m)y_1 = -R \mod t^m$.

$$\mathsf{C}(N) = 2\,\mathsf{C}(N/2) + \mathcal{O}(r\mathsf{M}(N)) \quad ➥ \quad \mathsf{C}(N) = \mathcal{O}(r\,\mathsf{M}(N)\log N)$$

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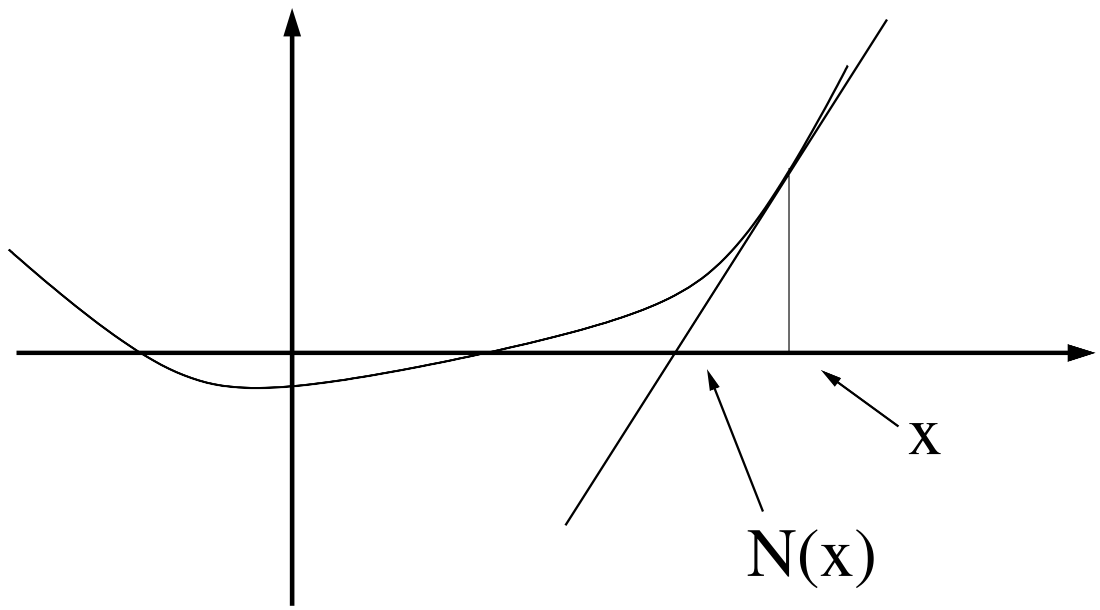
► Newton method: computing a *whole basis of solutions* in 1. will allow to determine y_1 in 3. from y_0 and R alone, without a second recursive call:

$$\tilde{\mathsf{C}}(N) = \tilde{\mathsf{C}}(N/2) + \mathcal{O}(\mathsf{M}(r, N)) \quad ➥ \quad \tilde{\mathsf{C}}(N) = \mathcal{O}(\mathsf{M}(r, N))$$

Newton iteration: real case



$$x_{\kappa+1} = x_\kappa - (x_\kappa^2 - 2)/(2x_\kappa), \quad x_0 = 1.5$$

$$x_1 = 1.41666666666666666666666666667$$

$$x_2 = 1.41421568627450980392156862745 10$$

$$x_3 = 1.41421356237468991062629557889 01$$

$$x_4 = 1.41421356237309504880168962350 25$$

Newton iteration: power series case

Let $\varphi : \mathbb{K}[[t]] \longrightarrow \mathbb{K}[[t]]$. To solve $\varphi(g) = 0$ in $\mathbb{K}[[t]]$, one can apply **Newton's tangent method**:

$$g_{\kappa+1} = g_\kappa - \frac{\varphi(g_\kappa)}{\varphi'(g_\kappa)} \mod t^{2^{\kappa+1}}.$$

- The number of **correct** coefficients doubles after each iteration.
- **Total cost** $=$ $2 \times$ $\Big($ the cost of the *last* iteration $\Big)$.

Theorem [Cook (1966), Sieveking (1972) & Kung (1974), Brent 1975]
Division, logarithm and exponential of a power series in $\mathbb{K}[[t]]$ *can be computed at precision* N *using* $\mathcal{O}(\mathsf{M}(\mathrm{N}))$ operations in $\mathbb{K}$.

Division and logarithm of power series

- To compute the reciprocal of $f \in \mathbb{K}[[t]]$, choose $\varphi(g) = 1/g - f$:

$$g_0 = \frac{1}{f_0} \quad \text{and} \quad g_{\kappa+1} = 2\,g_\kappa - f g_\kappa^2 \mod t^{2^{\kappa+1}} \quad \text{for } \kappa \geq 0.$$

- division of power series at precision N in $\mathcal{O}(\mathsf{M}(N))$.

- $\log(f) = -\sum_{i\geq 1} \frac{1}{i}(1-f)^i$ of $f \in 1 + t\mathbb{K}[[t]]$ in $\mathcal{O}(\mathsf{M}(N))$ by:

1. computing the Taylor expansion of $h = f'/f$ modulo t^{N-1};
2. taking the antiderivative of h.

Exponentials of power series

- To compute the first N terms of the exponential $\exp(f) = \sum_{i \geq 0} \frac{1}{i!} f^i$, choose $\varphi(g) = \log(g) - f$. Iteration:

$$g_0 = 1 \quad \text{and} \quad g_{\kappa+1} = g_\kappa - g_\kappa \left(\log(g_\kappa) - f\right) \quad \bmod t^{2^{\kappa+1}} \quad \text{for } \kappa \geq 0.$$

- First order differential equations: compute the first N terms of $f \in \mathbb{K}[[t]]$ such that $af' + bf = c$.
  - if $c = 0$ then the solution is $f_0 = \exp(-\int b/a)$
  - else, *variation of constants*: $f = f_0 g$, where $g' = c/af_0$.

Intermezzo: constant coefficients case

Problem: solve $y'(t) = A\,y(t),\ y(0) = v$, where A is a scalar matrix.

Equivalent question: compute $y(t) = \exp(\int Av)$.

Warning: this equivalence is no longer true in the non-scalar case!

Idea: the Laplace transform $z_N = \sum_{i=0}^{N-1} A^i v t^i$ of $y(t)$ is the truncation at order N of $z = \sum_{i\geq 0} A^i v t^i = (I - tA)^{-1} v$.

Algorithm (sketch):

1. compute z as a rational function of degre $\leq r$ (indep. on N);
2. deduce its Taylor expansion modulo t^N: $\mathcal{O}(N/r\,\mathsf{M}(r))$ to expand each coordinate of z.

$$\frac{a}{b} = c + \frac{d}{b} = c + \left(e + \frac{f}{b}\right)$$

Brent & Kung's algorithm for non-linear equations

1. reduce $y''(t) + a(t)y'(t) + b(t)y(t) = 0$ to a 1st-order equation
   - factor $D^2 + a(t)D + b(t)$ as $(D + S(t))(D + T(t))$:

     $$S + T = a, T' + ST = b, \quad \text{thus} \quad T' + aT - T^2 - b = 0$$

2. reduce the Riccati equation to a linear equation (by linearization)
3. find S, T and solve two linear 1st order equations to get $y(t)$

- generalizes to arbitrary orders:

$$\mathsf{Lin}(r, N) = \mathsf{NonLin}(r-1, N) + \mathcal{O}(r\,\mathsf{M}(N)) + \mathsf{Lin}(r-1, N)$$
$$\mathsf{NonLin}(r-1, N) = \mathcal{O}(\mathsf{Lin}(r-1, N))$$

➥ $\mathsf{Lin}(r, N) = \mathcal{O}(r^r\,\mathsf{M}(N))$.

Newton iteration hits again

Suppose we have to solve a "functional" equation $\phi(Y) = 0$, where $\phi : \mathcal{M}_{r\times r}(\mathbb{K}[[t]]) \to \mathcal{M}_{r\times r}(\mathbb{K}[[t]])$ is differentiable.

Define the sequence $Y_{\kappa+1} = Y_\kappa - U_{\kappa+1}$, where

- $U_{\kappa+1}$ is a solution of valuation $\geq 2^{\kappa+1}$ of the *linearized equation*

$$D\phi|_{Y_\kappa} \cdot U = \phi(Y_\kappa),$$

- $D\phi|_{Y_\kappa}$ is the differential of ϕ at Y_κ.

Then, the sequence Y_κ converges quadratically to the solution Y.

First application: matrix inversion

To compute the inverse Z of a matrix Y of power series:

- choose the map $\phi : Z \mapsto I - YZ$ with differential $Z \mapsto -YZ$
- the equation for U becomes $\quad -YU = I - YZ_\kappa \mod t^{2^{\kappa+1}}$
- solution $\ U = -Y^{-1}(I - YZ_\kappa) = -Z_\kappa(I - YZ_\kappa) \mod t^{2^{\kappa+1}}$

This yields the Newton–Schulz iteration for Y^{-1} [Schulz, 1933]

$$Z_{\kappa+1} = Z_\kappa + Z_\kappa(I_r - YZ_\kappa) \mod t^{2^{\kappa+1}}.$$

$$\mathsf{C}_{\text{inv}}(N) = \mathsf{C}_{\text{inv}}(N/2) + \mathcal{O}(\mathsf{M}(r, N)) \quad ➡ \quad \mathsf{C}_{\text{inv}}(N) = \mathcal{O}(\mathsf{M}(r, N))$$

Second application: solving differential equations

To compute the solution Y of the system $Y' = AY$

- choose the map $\phi : Y \mapsto Y' - AY$, with differential ϕ.
- the equation for U is $\quad U' - AU = Y_\kappa' - AY_\kappa \mod t^{2^{\kappa+1}}$
- using Lagrange's method of variation of parameters, solution
  $U = Y_\kappa V_\kappa \mod t^{2^{\kappa+1}}, \quad Y_\kappa' - AY_\kappa = Y_\kappa V_\kappa' \mod t^{2^{\kappa+1}}.$

This yields the BCOSSS iteration for Y:

$$Y_{\kappa+1} = Y_\kappa - Y_\kappa \int Y_\kappa^{-1} \left(Y_\kappa' - AY_\kappa\right) \mod t^{2^{\kappa+1}}.$$

$$\mathsf{C}_{\text{solve}}(N) = \mathsf{C}_{\text{solve}}(N/2) + \mathcal{O}(\mathsf{M}(r, N)) \quad ➥ \quad \mathsf{C}_{\text{solve}}(N) = \mathcal{O}(\mathsf{M}(r, N))$$

Further questions

constant factor improvements: middle products of polynomial matrices

small characteristic case: Padé approximants? p-adic lifting?

faster Newton for the case of a single equation: exploit companion form

bit complexity analysis