Classifying walks in the quarter plane

Marni Mishna

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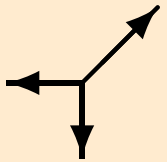
Walks in the quarter plane



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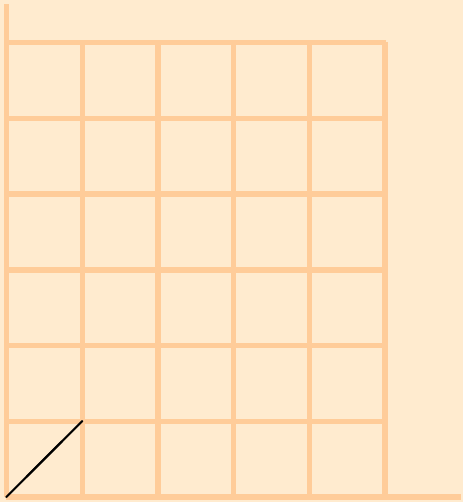

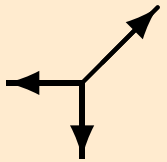

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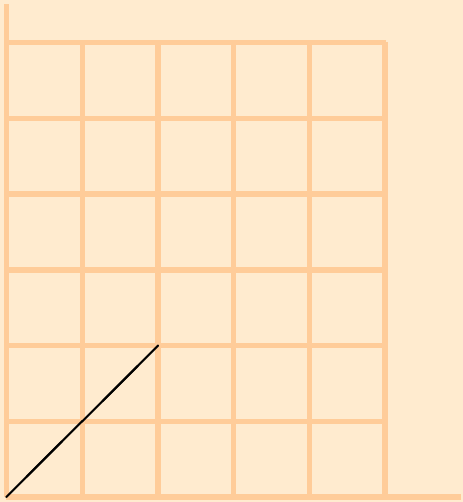

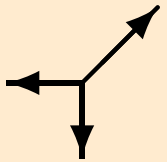

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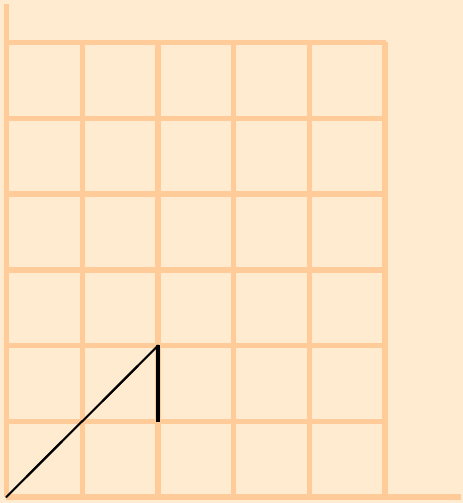

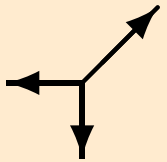

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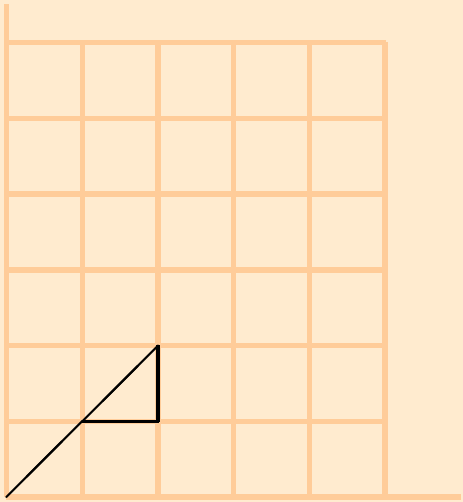

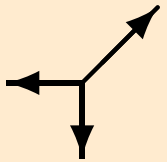

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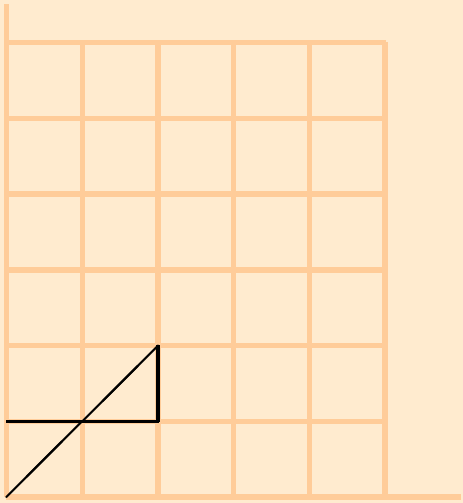

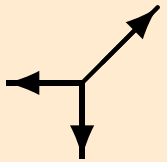

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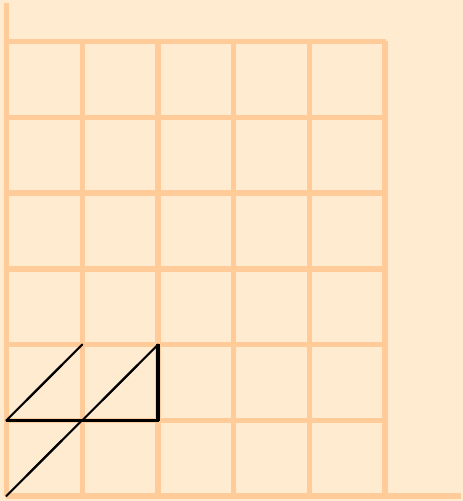

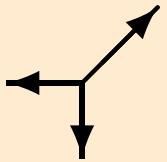

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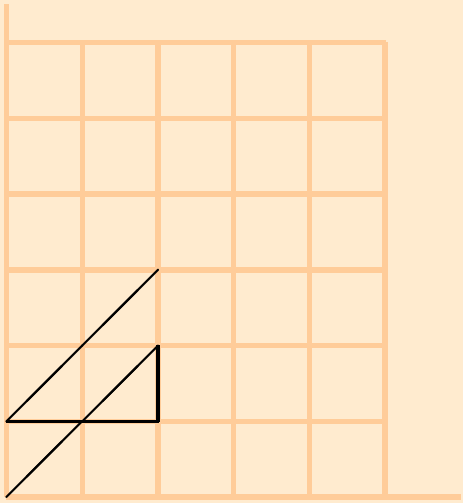

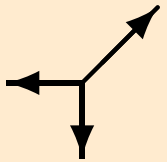

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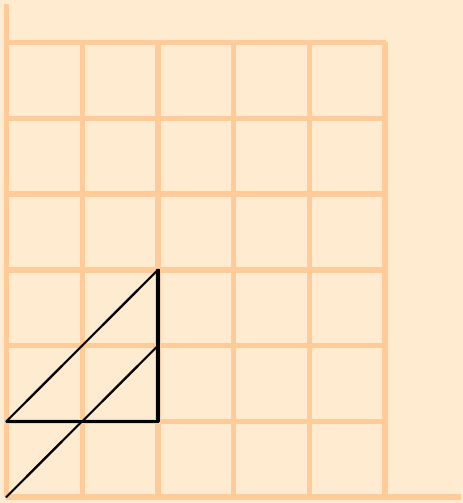
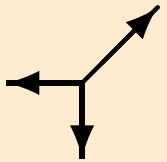




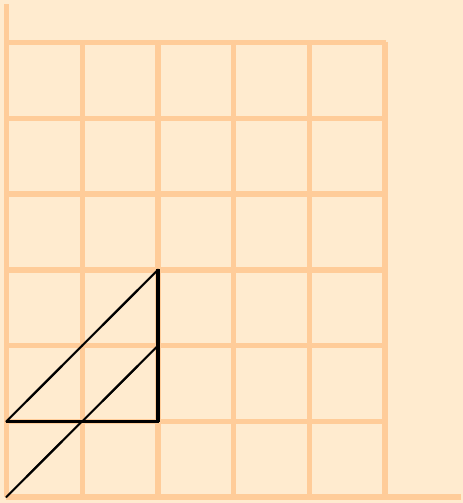
How many walks are there of length n, ending at (i, j), with steps from $\mathcal{Y} \subset \{\mathsf{N}, \mathsf{NW}, \mathsf{W}, \ldots, \mathsf{NE}\}$?

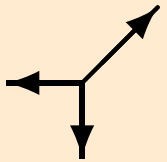
Define: complete generating series $Q_{\mathcal{Y}}(x, y; t)$:

$$Q_{\mathcal{Y}}(x, y; t) = \sum_{i,j,n} c_{ij}(n) x^i y^j t^n$$

where $c_{ij}(n)$= # walks ending at (i, j), of length n

Walks in the quarter plane





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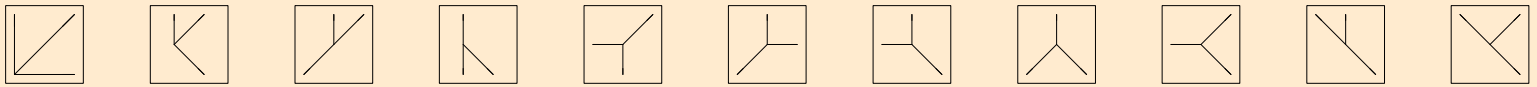
Goal: Classify $\mathcal{Y}$ according to nature of $Q_{\mathcal{Y}}(x, y; t)$

Results

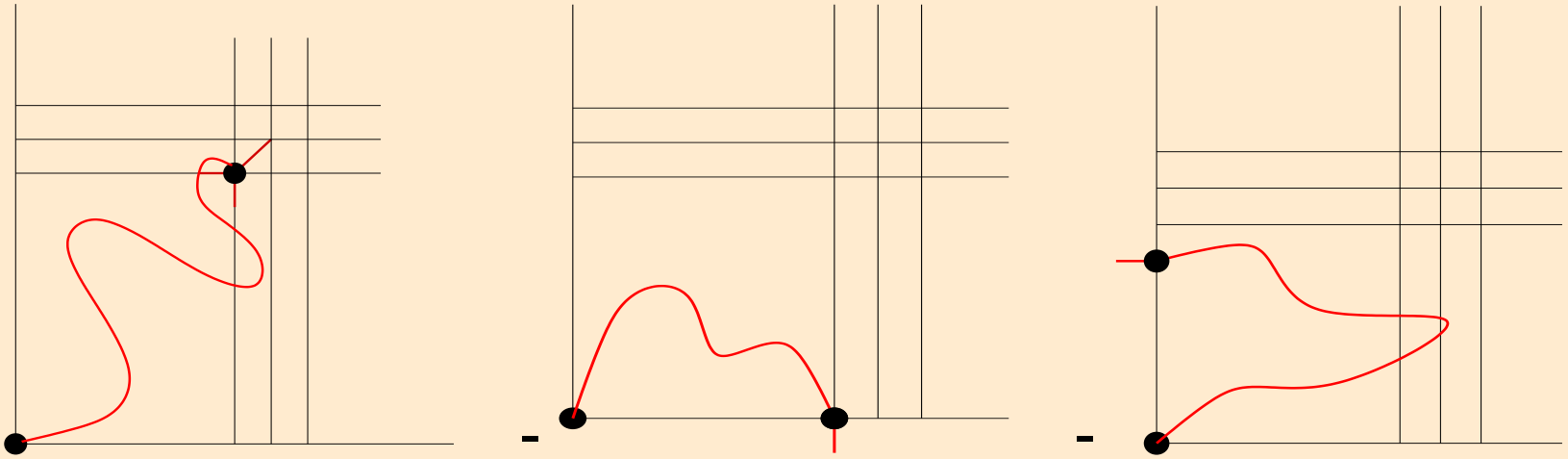
Classification when $|\mathcal{Y}| = 3$

											
Rational	x										
Algebraic	x	x	x	x	x	x	x^+				
D-finite	x	x	x	x	x	x	x	x	x		
Not D-finite										x^*	x^*

The functional equation approach

Functional equation: A walk is a walk plus a step.

$\mathcal{Y} = \{\mathsf{S}, \mathsf{W}, \mathsf{NE}\} =$ 



$$Q_{\mathcal{Y}}(x,y;t) = 1 + t(\tfrac{1}{x} + \tfrac{1}{y} + xy)Q_{\mathcal{Y}}(x,y;t) - \tfrac{t}{y}Q_{\mathcal{Y}}(x,0;t) - \tfrac{t}{x}Q_{\mathcal{Y}}(0,y;t)$$

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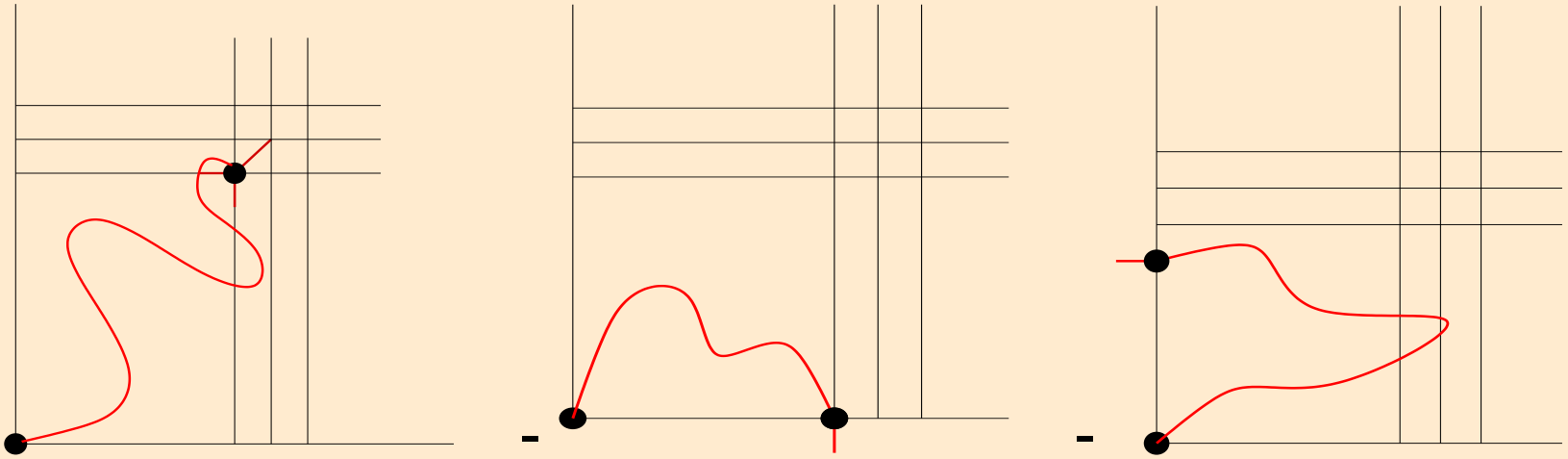

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Rewrite:

$$(1 - t(\bar{y} + \bar{x} + xy))Q(x,y;t) = 1 - \tfrac{t}{y}Q_0(x;t) - \tfrac{t}{x}Q_0(y;t)$$

$$(xy - t(x + y + x^2y^2))Q(x,y;t) = xy - txQ_0(x;t) - tQ_0(y;t)$$

Classification of power series

Rational

Algebraic

D-finite

Differentiably algebraic

Algebraic functions

$F(x,y,z)$ satisfi es $P(F(x,y,z),x,y,z)=0$ for some polynomial P.
Useful properties:

- Singularities and coeff asymptotics:
  singularity: $(x-\zeta)^{-\alpha}$ coeff: $n^{\alpha-1}\zeta^{-n}$
- The Hadamard product of two rational series is algebraic.
  $\sum_n f_n x^n \times \sum_n g_n x^n = \sum_n f_n g_n x^n$

D-finite functions

$f(x_1, x_2, \ldots, x_n)$ is Differentiably finite (D-finite) with respect to $X = x_1, \ldots, x_n$ if

- For $1 \leq j \leq n$, f satisfies n linear differential equation s with polynomial coefficients:

$$\phi_0(X)f(X) + \phi_1(X)\frac{\partial f(X)}{\partial x_j} + \ldots + \phi_k(X)\frac{\partial^k f(X)}{\partial x_j^k} = 0$$

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Useful properties:

- The Hadamard product wrt any subset of variables is D-finite; $(\sum f_n x^n \times \sum g_n x^n = \sum f_n g_n x^n)$
- algebraic substitution $\Longrightarrow$ still D-finite

Context of our study

- 1-Dimensional Lattice Paths
  - Flajolet, Banderier (2002)
  - Always algebraic

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- Walks in the quarter plane
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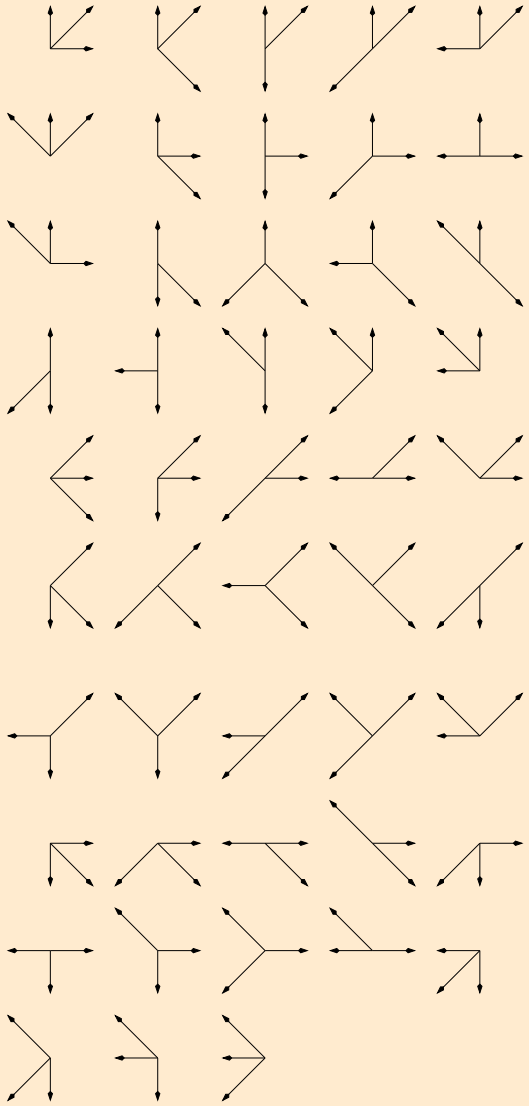
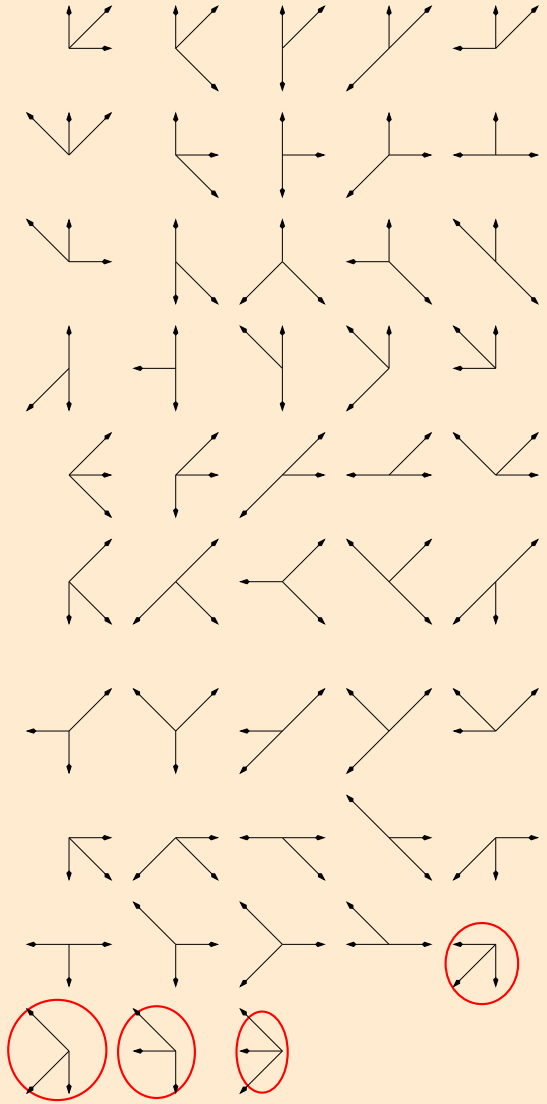
- Walks in the Slit-Plane
  -GF approach: Bousquet-Mélou, Schaeffer (2002)
  -small height variations= Algebraic

When $|\mathcal{Y}| = 3$



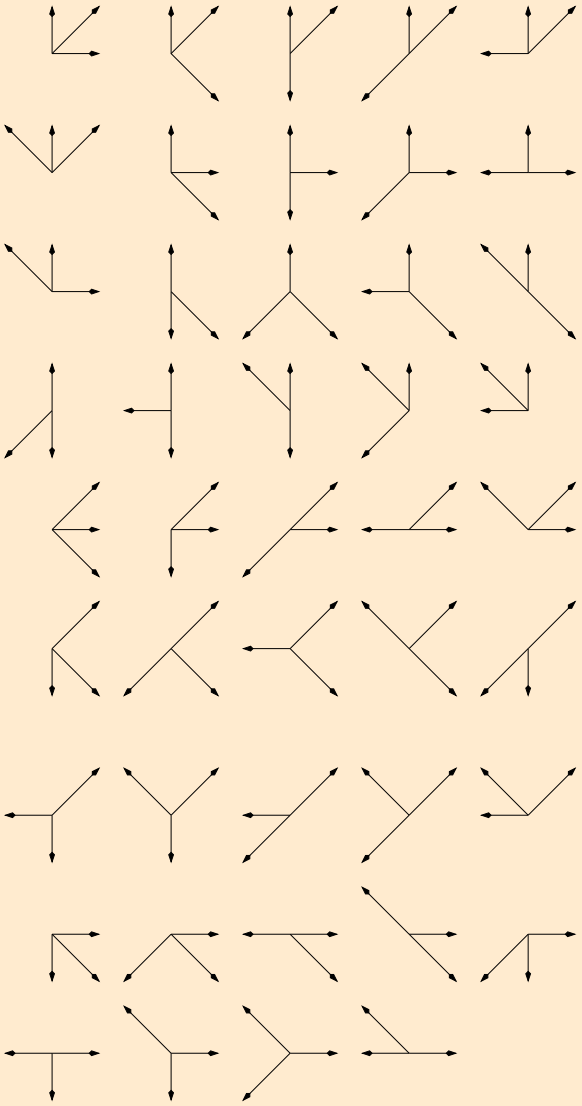
all of the possibilities

When $|\mathcal{Y}| = 3$



no walks in quarter plane

When $|\mathcal{Y}| = 3$



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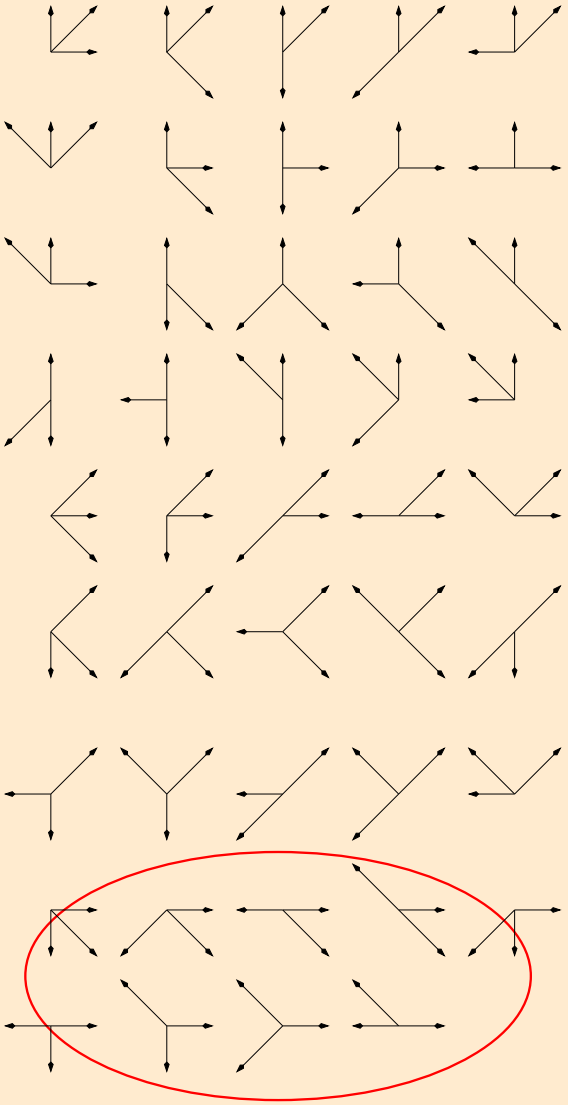
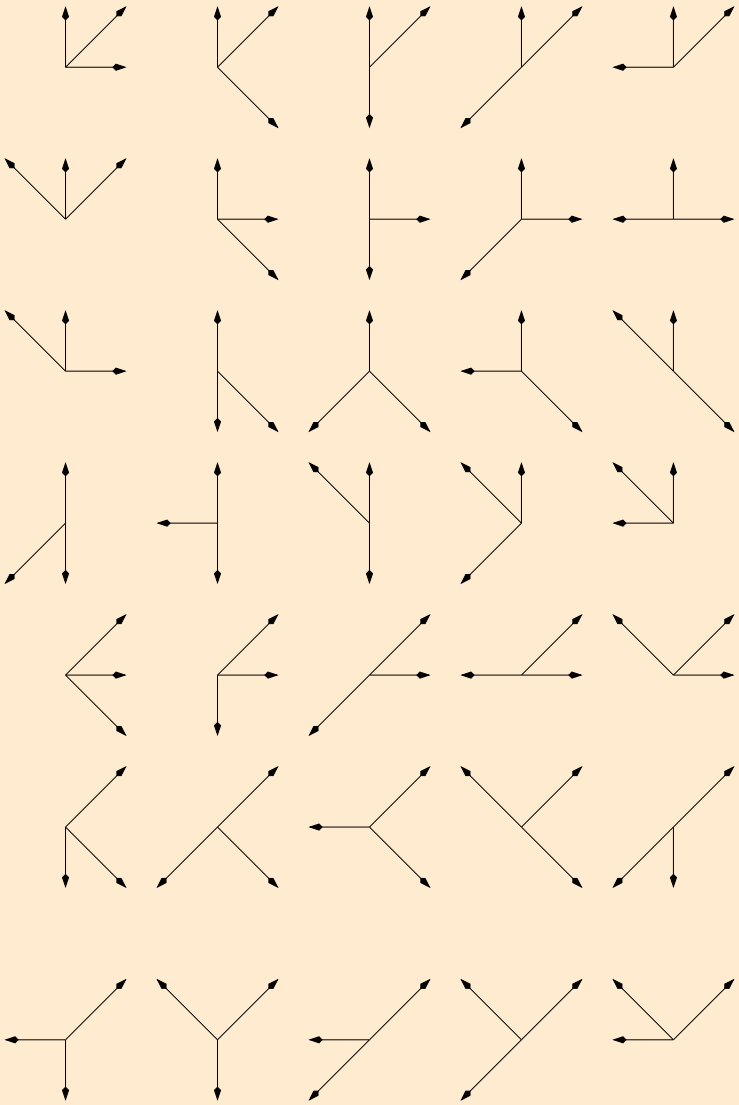


identical to others reflected in $x = y$

When $|\mathcal{Y}| = 3$



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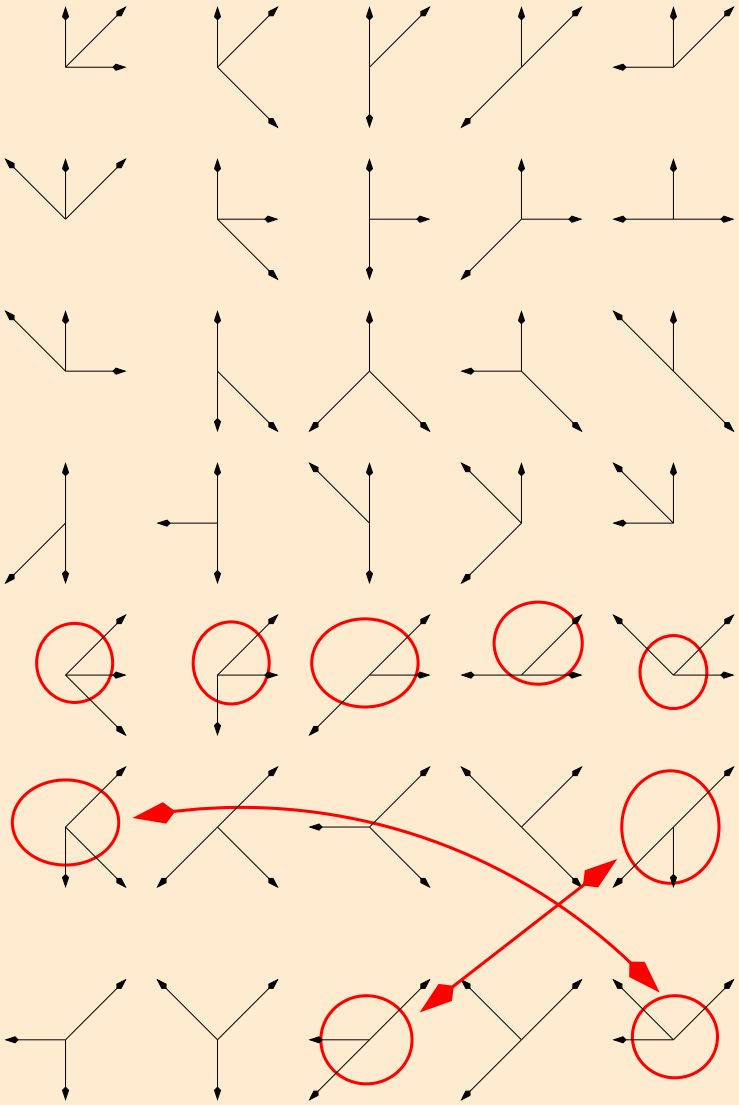
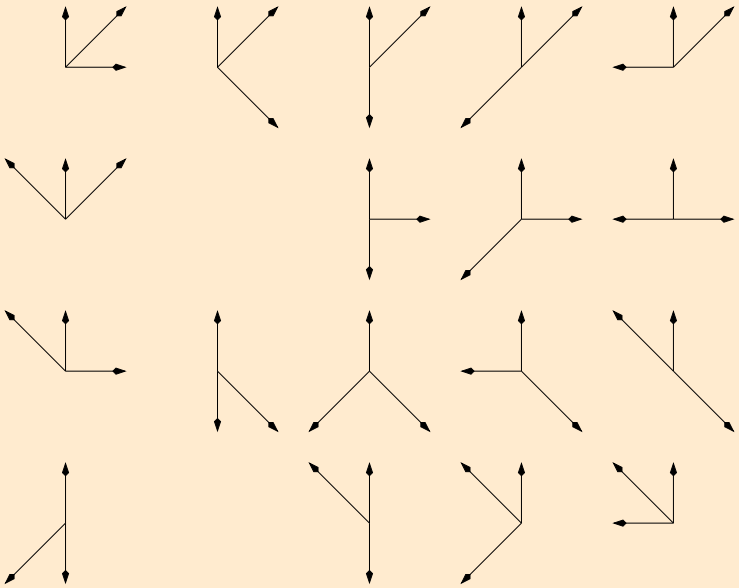


more identical pairs...

When $|\mathcal{Y}| = 3$



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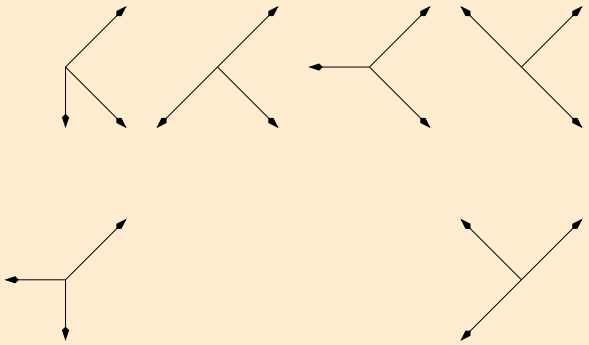

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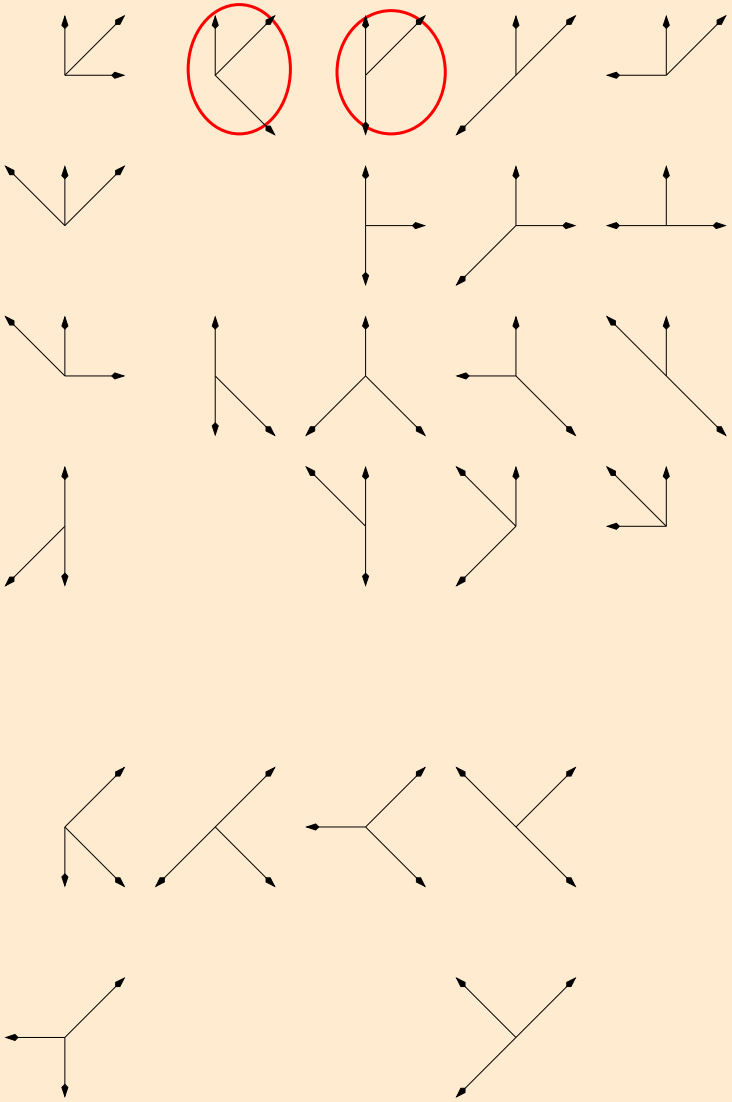
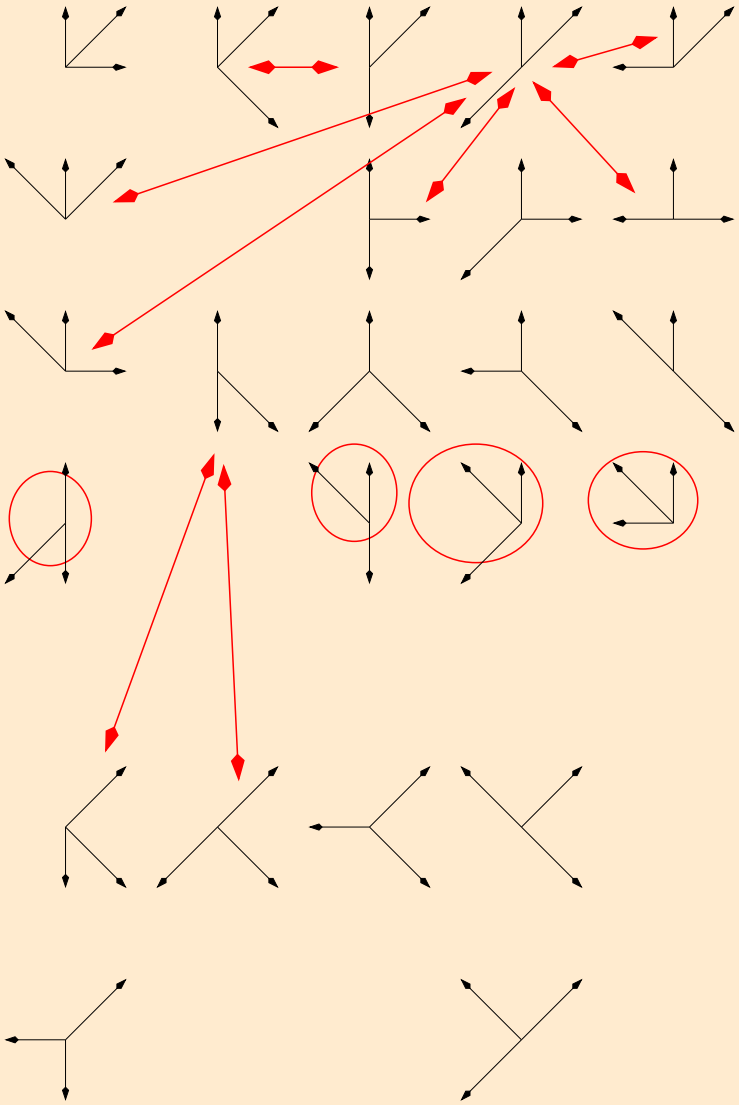


These two are governed by the same inequality:

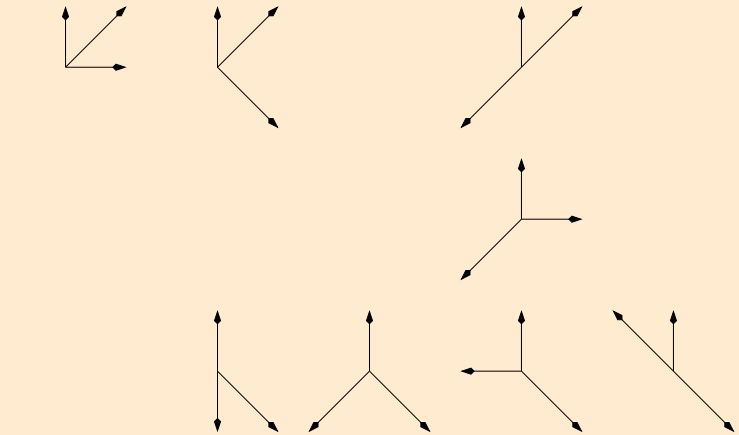
$$\#a + \#b \geq \#c$$

When $|\mathcal{Y}| = 3$



More cases of this
+ 1 dimensional
cases

When $|\mathcal{Y}| = 3$



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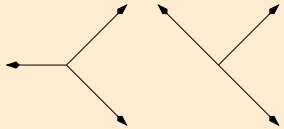

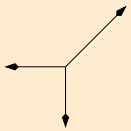

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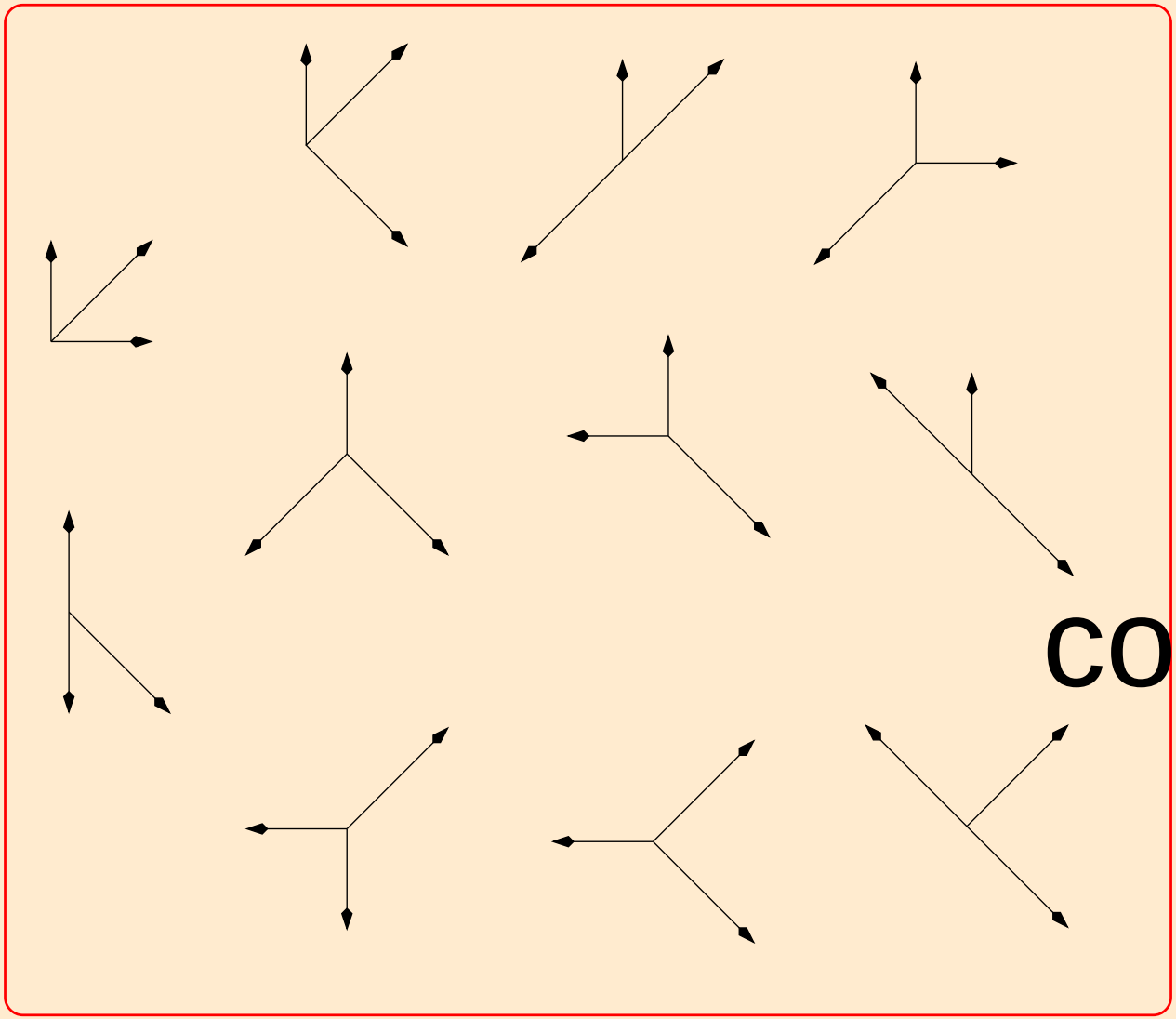
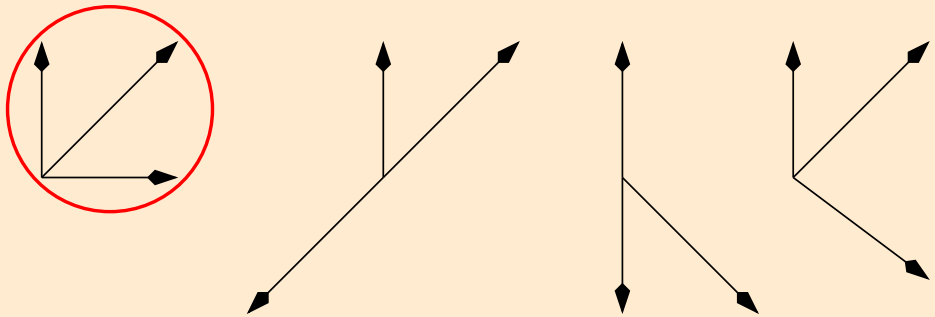


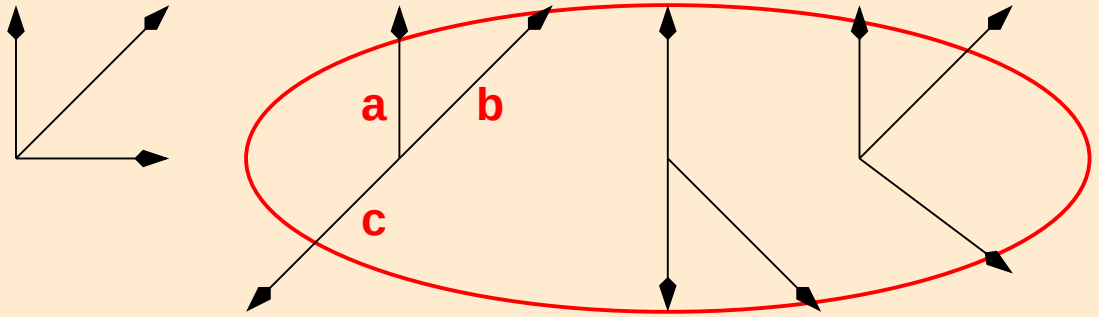
Final contenders!

Easy cases



- Rational: Only one, unrestricted case

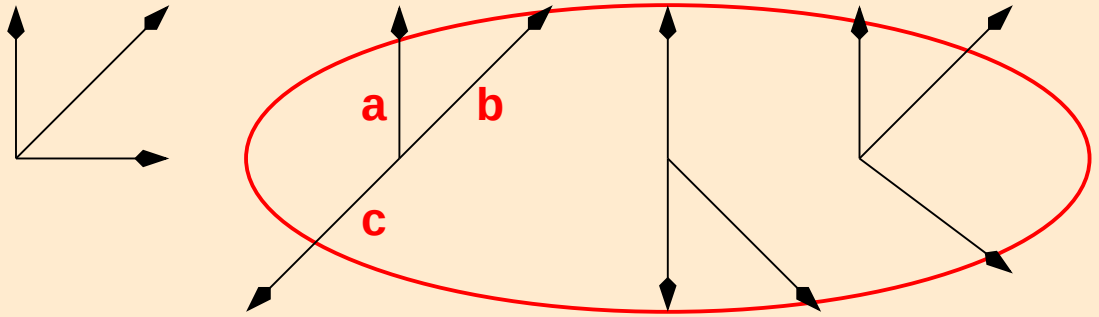
Easy cases



- Rational: Only one, unrestricted case
- "Easy algebraic": Only one true condition.
  - Reduces to half plane condition $\Longrightarrow$ algebraic
  - One condition $\Longrightarrow$ can be recognized by a pushdown automaton (CFL)

$$V : a + b \geq c \quad H : b \geq c \quad H \Longrightarrow V$$

Easy cases



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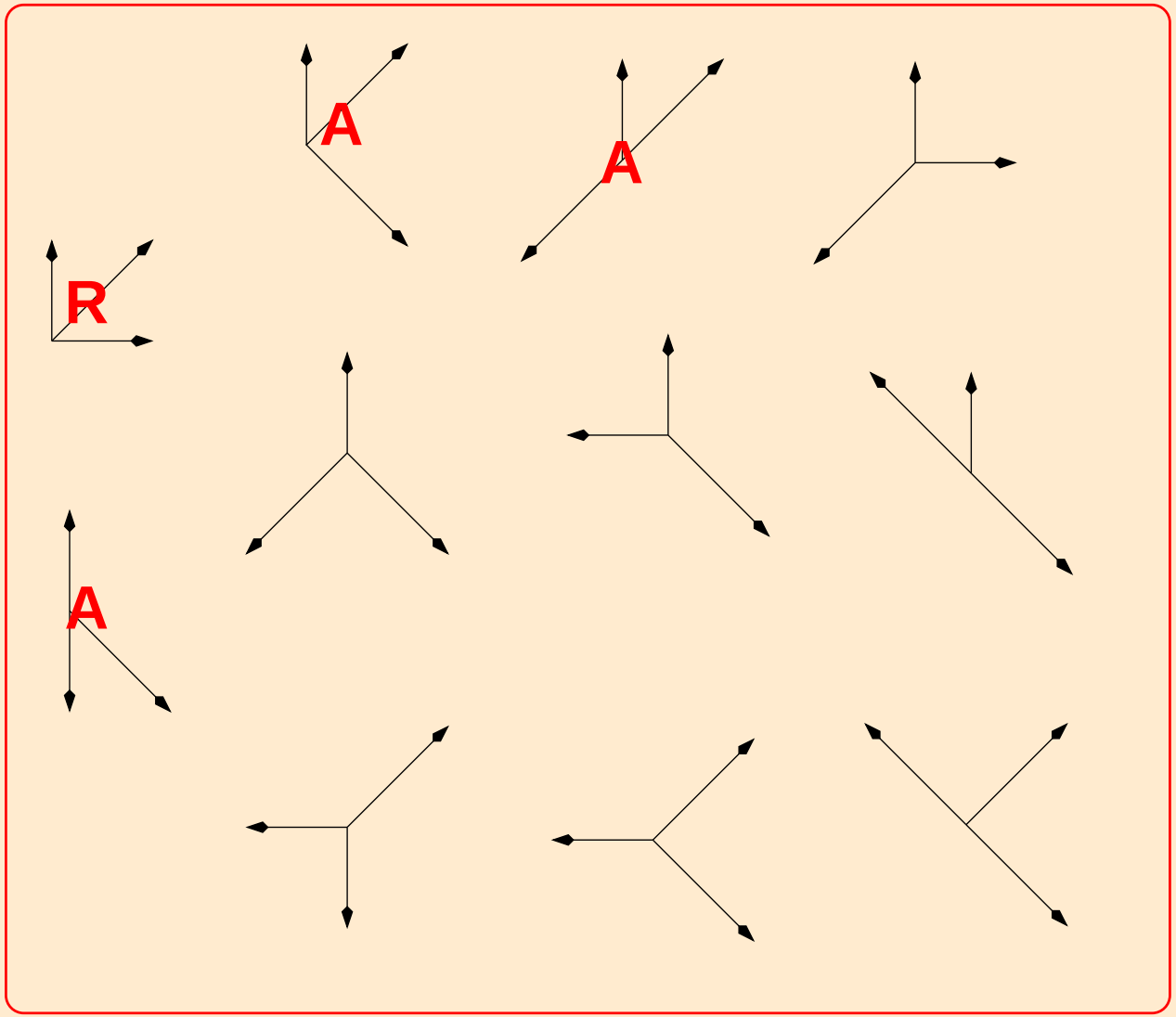
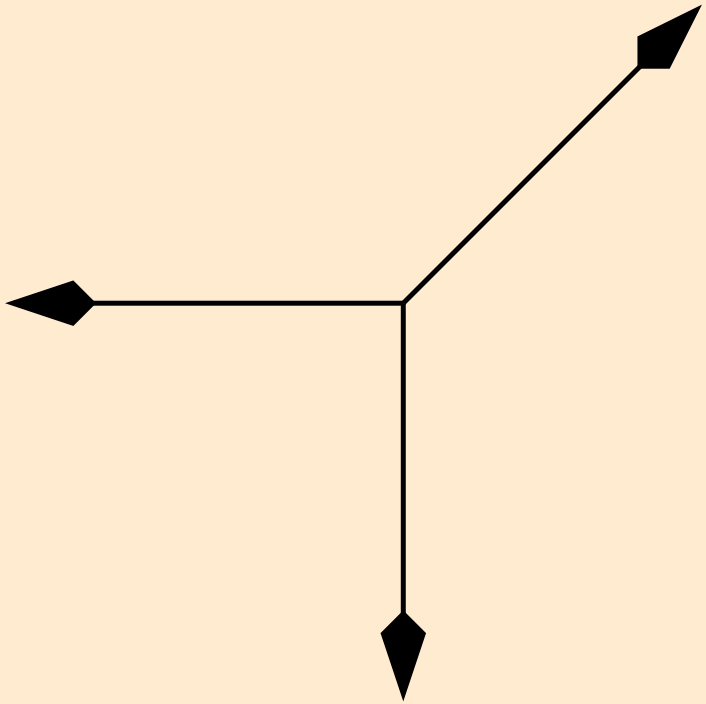
Later, we will call these cases singular

Update



The novelty of Kreweras' model



- Well studied: Kreweras (60s) – MBM(04)
- NOT "easy algebraic" (Not context-free!)
- Algebraic, with explicit description of GF

Algebraic Kernel Method

Recall a general form of the equation satisfied by $Q(x, y; t)$

Rational Kernel: K_r

The coefficient of $Q(x, y; t)$: $K_r = 1 - t \sum_{(i,j) \in \mathcal{Y}} x^i y^j$

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Clear the denominator of K_r

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The group of the walk: G

Write K as a function of y : $K = a(x,t)y^2 + b(x,t)y + c(x,t) \implies$ roots: Y_0, Y_1. Define $\tau_y : (x, y) \mapsto (x, Y_0 Y_1/y)$ and similarly τ_x.

The group of the walk $G(\mathcal{Y})$:= The group generated by τ_x, τ_y.

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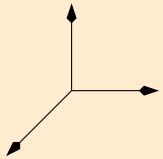
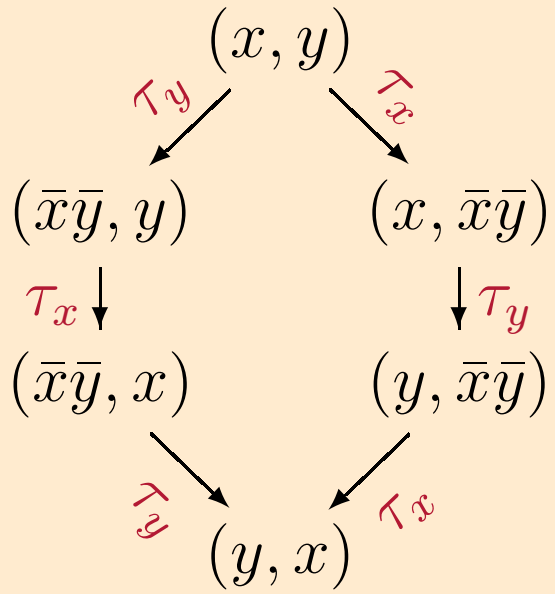
Lemma: $K_r(x, y) = K_r(\rho(x, y))$ for $\rho \in G(\mathcal{Y})$

Reverse Kreweras walks



$$K_r = 1 - t(x + y + \bar{x}\bar{y}) \quad Y_0 Y_1 = 1/x = \bar{x}$$



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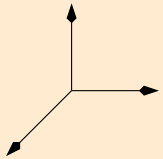

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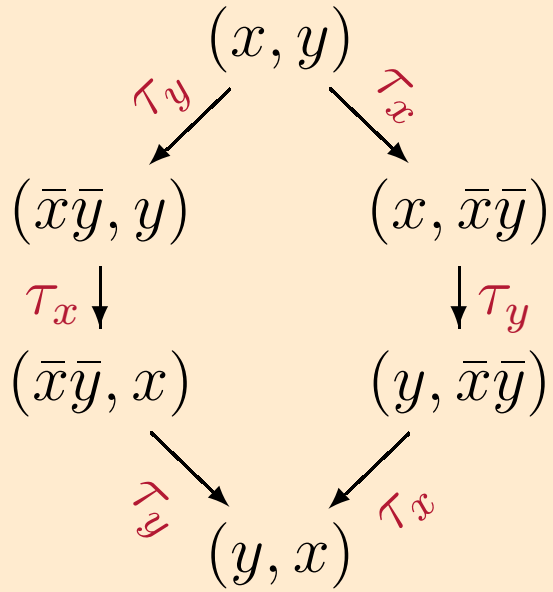

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Apply to fundamental equation

$$\begin{aligned}
(1) \quad & \bar{x}\bar{y}\bar{K}_r Q(\bar{x}, \bar{y}; t) &&= \bar{x}\bar{y} - tQ_0(\bar{x}) - tQ_0(\bar{y}) + tQ_{00}(t) \\
(2) \quad & y\bar{K}_r Q(\bar{x}, xy; t) &&= y - tQ_0(\bar{x}) - tQ_0(xy) + tQ_{00}(t) \\
(3) \quad & x\bar{K}_r Q(xy, \bar{y}; t) &&= x - tQ_0(xy) - tQ_0(\bar{y}) + tQ_{00}(t)
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New equation: (1)+(2)-(3)

$$\begin{aligned}
&\bar{x}\bar{y}Q(\bar{x}, \bar{y}; t) + yQ(\bar{x}, xy; t) - xQ(xy, \bar{y}; t) \\
&= \frac{1}{\bar{K}_r}\left(\bar{x}\bar{y} + y - x - 2tQ_0(\bar{x}) + tQ_{00}(t)\right)
\end{aligned}$$

Reverse Kreweras walks

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Extract constant term wrt y:

$$xQ_d(x) = \frac{1}{\sqrt{\Delta(x)}}\left(2Y_0 - x - 2tQ_0(\bar{x}) + tQ_{00}(t)\right)$$

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Canonical factorization of $f \in C[x,\bar{x}][[t]]$ into $f^+(x,t)f(t)f(\bar{x},t)$.

Extract non-positive powers of x to get $Q_0(\bar{x})$.

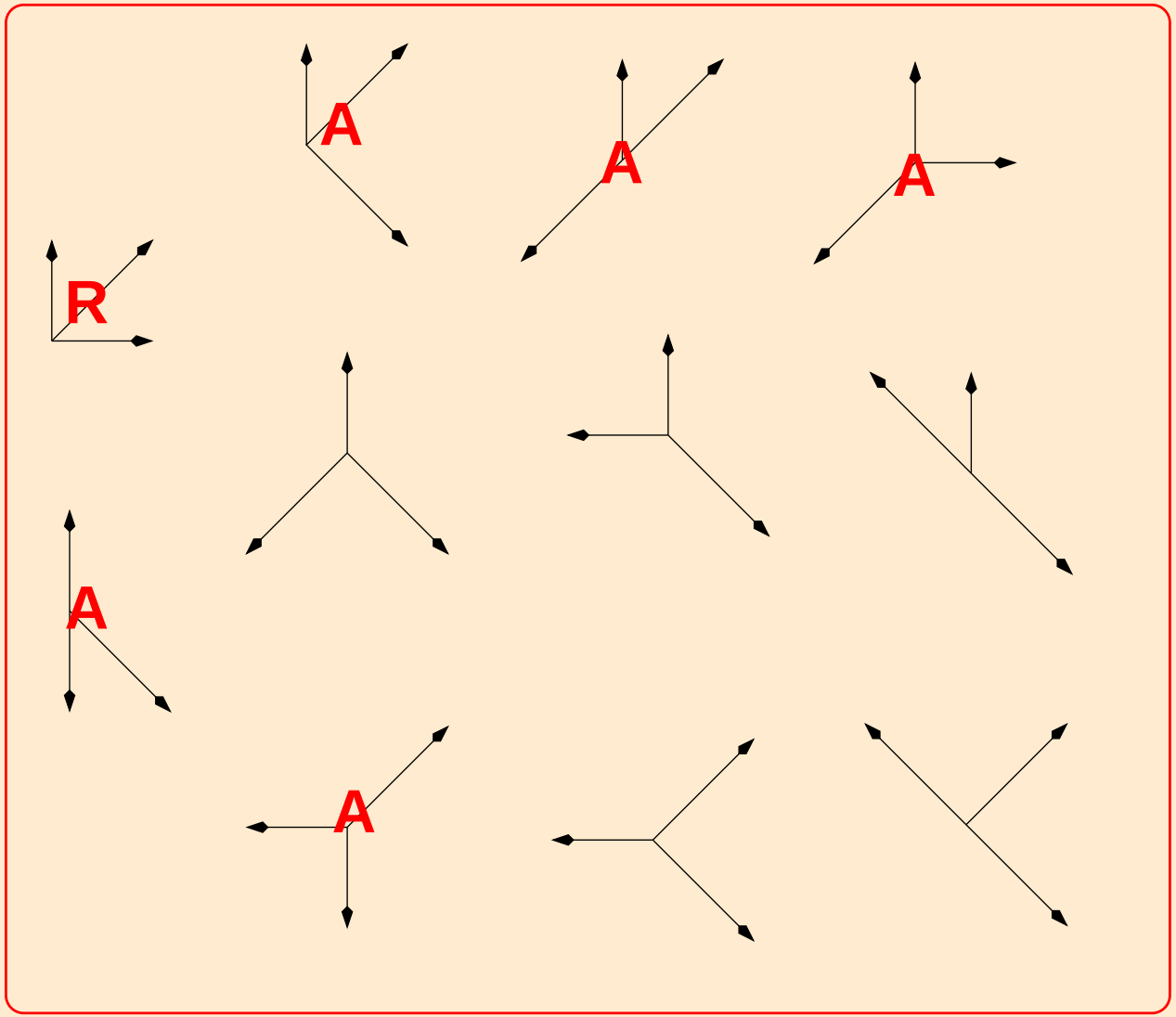
Reverse Kreweras walks

Theorem:
Let $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{SW}\}$. Then:

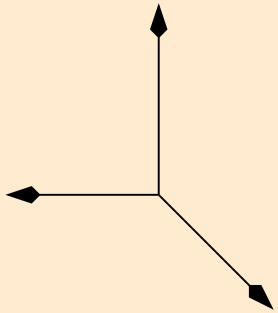
$$2Q_{\mathcal{Y}}(x,0;t) = \frac{4T - T^2}{8t} + \left(\frac{-2x}{Tt}\left(1 - \frac{T^2}{2x}\right) + \frac{1}{tx}\right)\sqrt{U} + \left(1 - tx - \frac{t}{x^2}\right) xt^{-2}.$$

$T = t(2 + T^3)$, $U = 1 - xT(1 + T^3/4) + x^2T^2/4$

Update

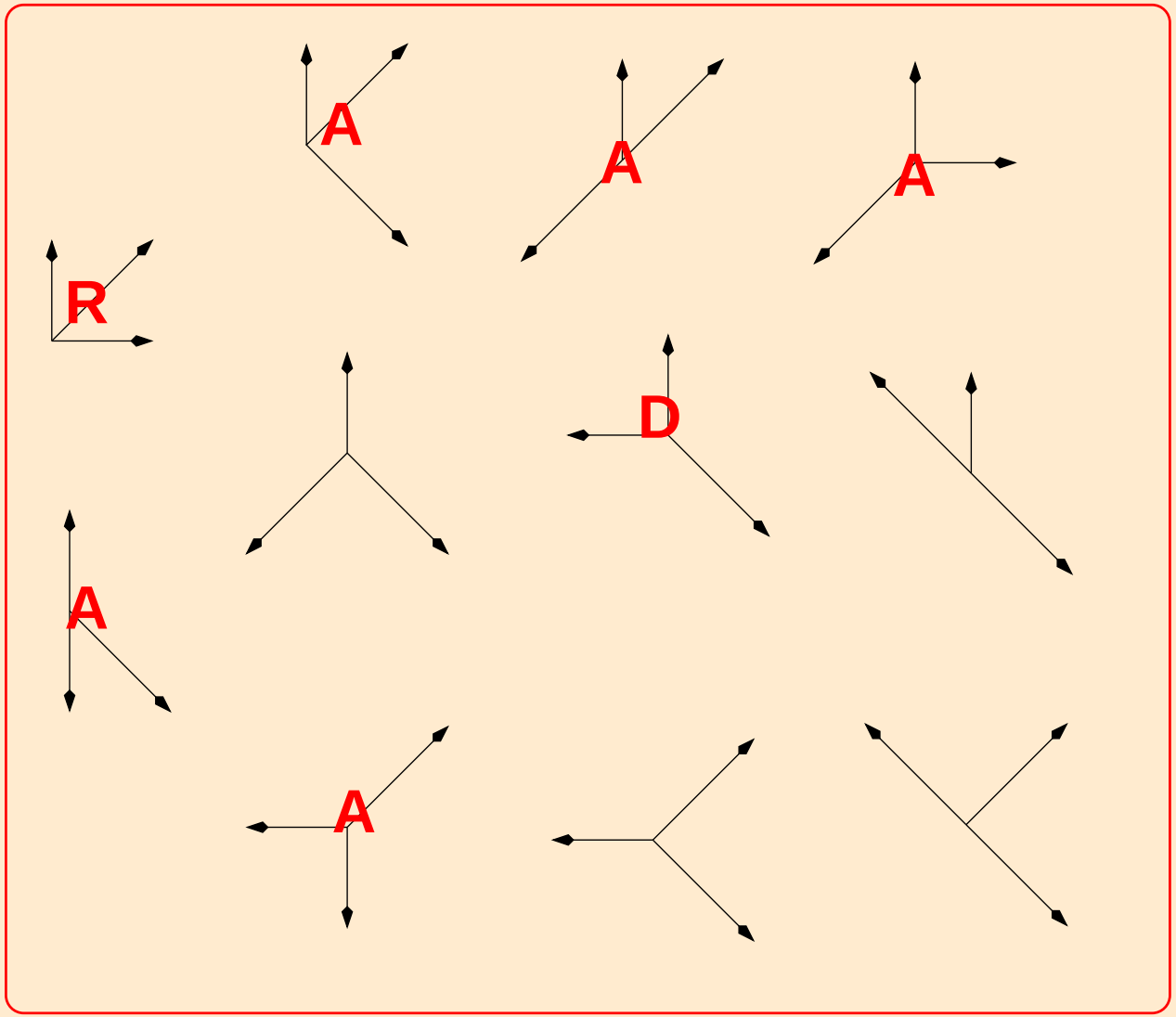


An interesting case



- In bijection with Young tableaux of height 3 ($\#a \geq \#b \geq \#c$) $\Longrightarrow$ explicit formula for complete gf. (D-finite; non-algebraic)
- Regev (1981) gives explicit form for counting gf: algebraic!
- Motzkin!

Update



A D-finiteness criterion

Theorem. [Bousquet-Mélou (Petkovsěk)]
Let $\mathcal{Y}$ be a finite subset of $\{\pm 1, 0\} \times \mathbb{Z} \setminus \{(0,0)\}$ that is symmetric with respect to the y-axis. Then the complete generating function $Q(x, y; t)$ for walks that start from $(0,0)$, take their steps in $\mathcal{Y}$ and stay in the first quadrant is D-finite.

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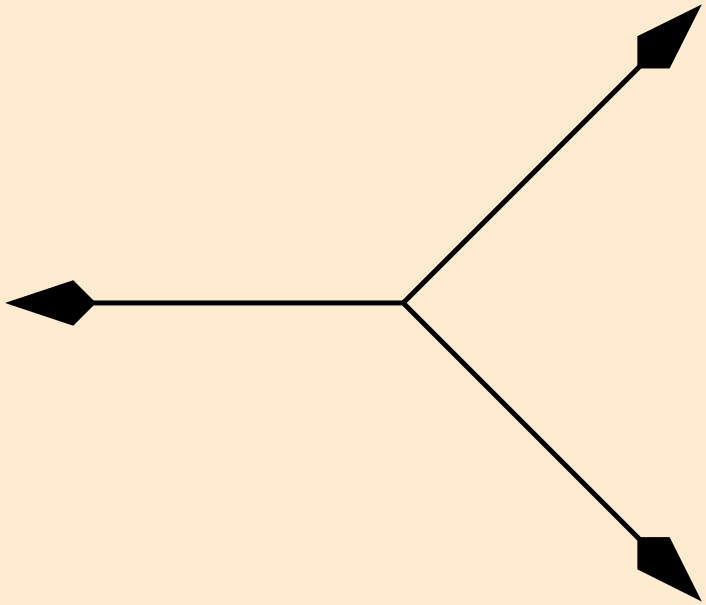
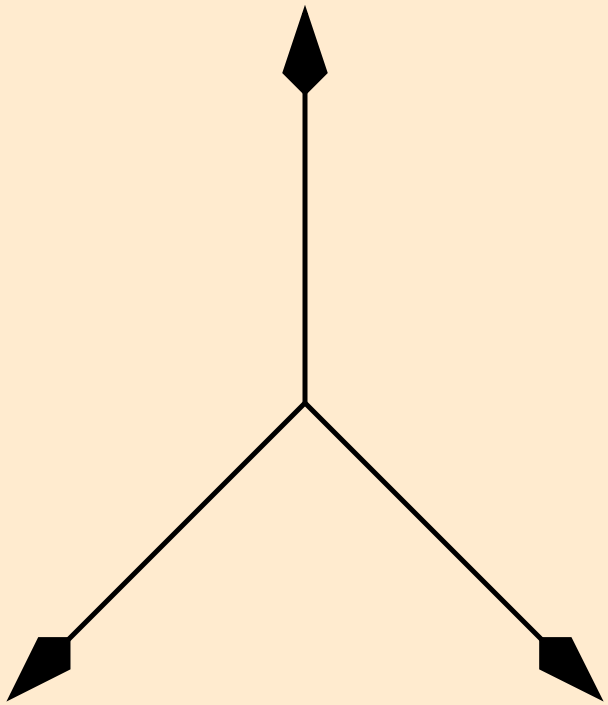
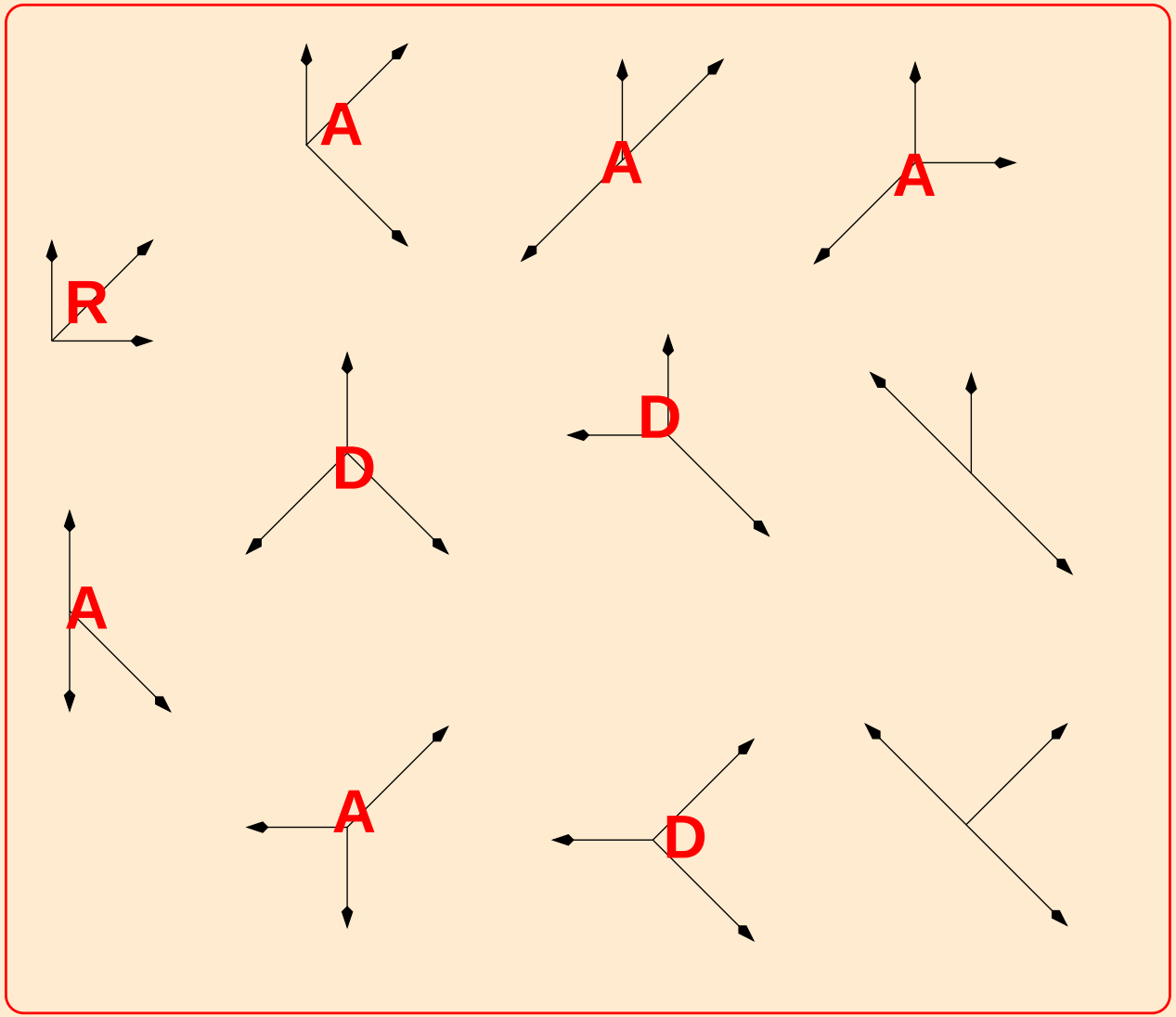
Example:



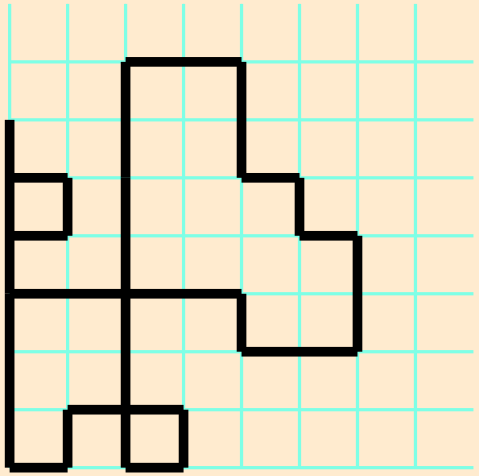


Update



Another D-finite example

Ex: $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}\}$



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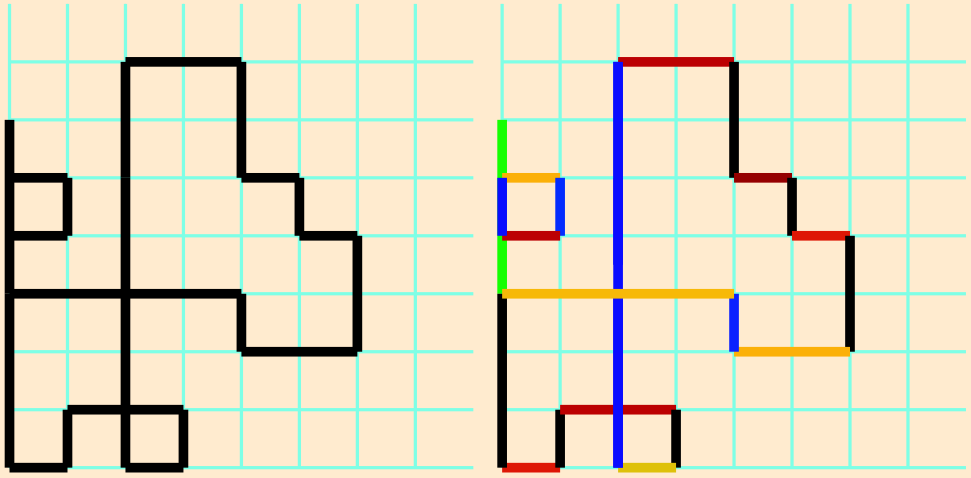

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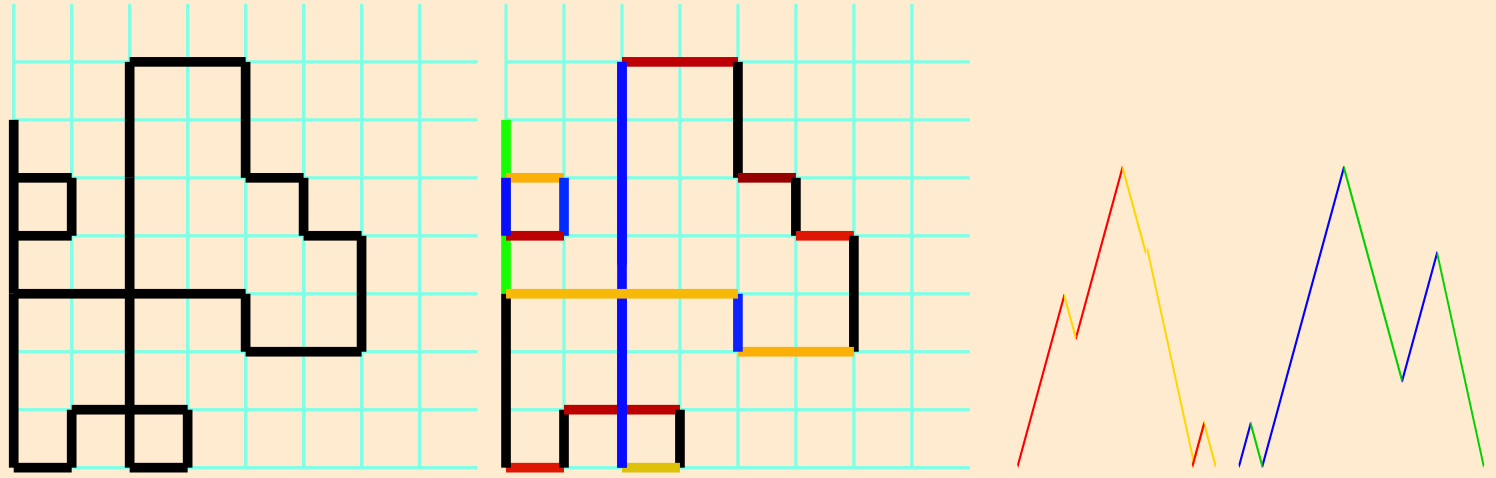

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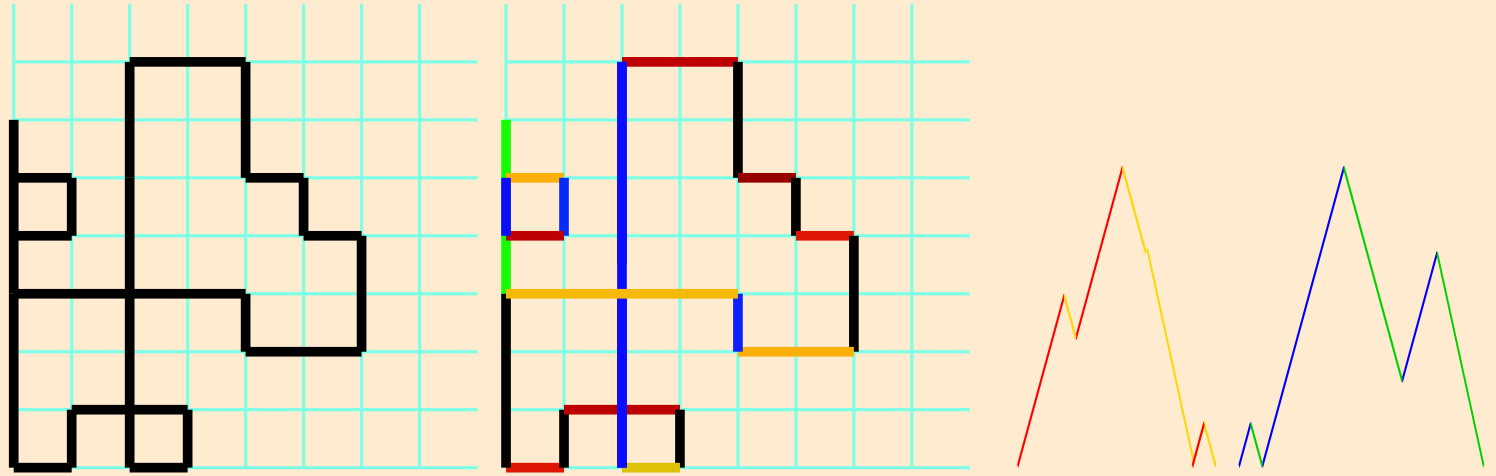
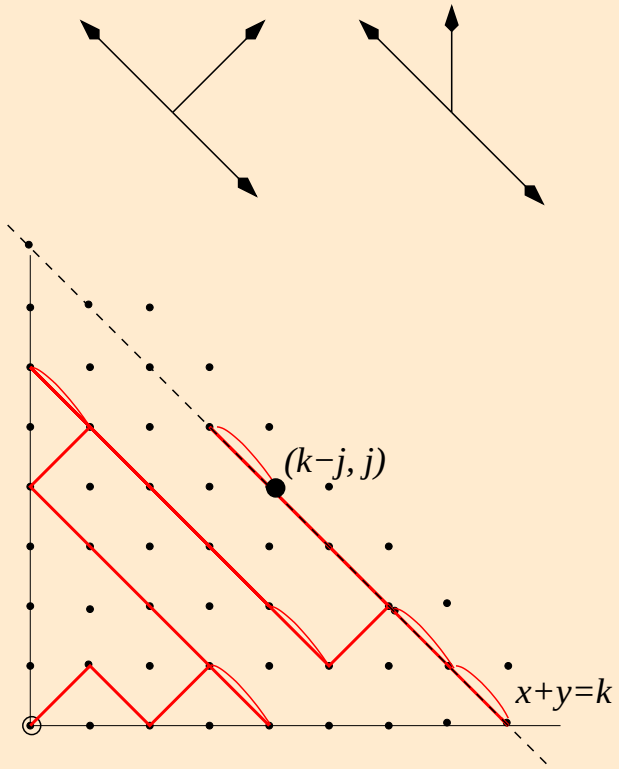


A walk = A pair of Dyck path prefixes implies

$$Q_{\{\mathsf{N},\mathsf{E},\mathsf{S},\mathsf{W}\}}(t) = \sum_{m,n\geq 0} \binom{m+n}{m} \binom{m}{\lfloor m/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor} t^{m+n}$$

which is *not* algebraic, but *is* D-finite.

A non-D-finite example...



A walk to $(k-j, j)$
= walk to $x+y=k-2$ + step
+ walk along $x+y=k$

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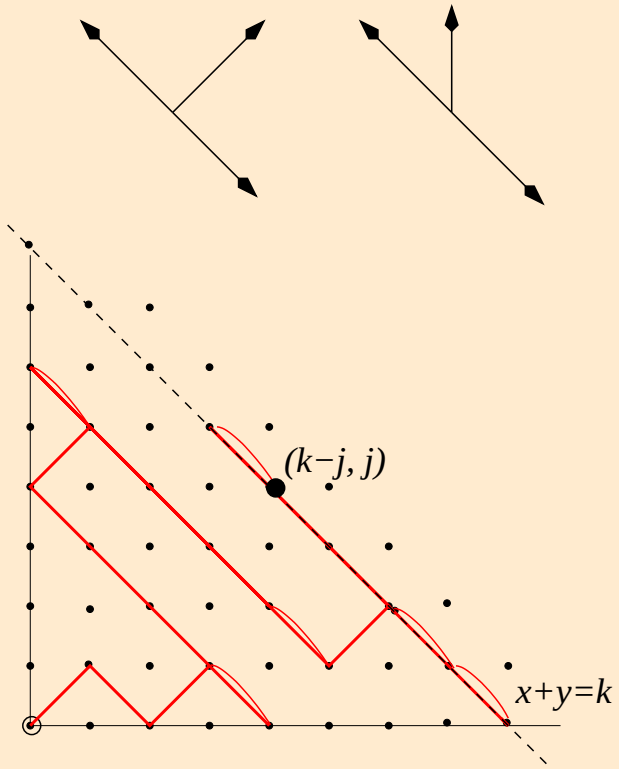

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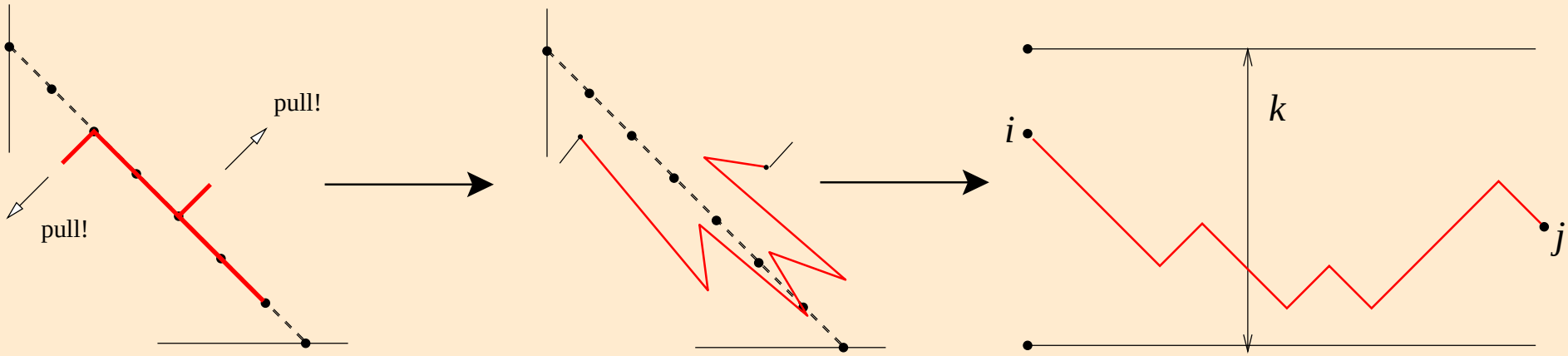
Bounded Dyck paths



Generating function:

$$H_{ij}^{k}(t) = t^{i-j}\,\frac{f_{j+1}f_{k-i+1}}{f_{k+2}} \qquad (i \geq j)$$

Fibonacci polynomials:

$$f_{n+1} = f_n - t^2 f_{n-1}, \qquad f_0 = 0, \qquad f_1 = 1$$

OGF for walks ending on $x + y = k$

$S_{k,j}$:= ogf walks that end at $(k - j, j)$

$$B_k(y) := \sum_{j=0}^{k} S_{k,j} y^j$$

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$$\hat{B}_k(y) = \frac{q^3 \hat{B}_{k-2}(q)(y^{k+2}+1) - qy^2 \hat{B}_{k-2}(y)(q^{k+2}+1)}{(q^{k+2}+1)(yq-1)(y-q)}.$$

These walks are not D-finite...

$$R(s,y,q) := \sum \hat{B}_k(y,q)s^k$$

Lemma: $\hat{B}_k(y)$ is rational, with poles at $q^{k+2} = -1$ ($q^2 \neq 1$)

Proof: Can show $\hat{B}_k(q) \neq 0$ when $q^{k+2} = -1$ and $q^2 \neq -1$.

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Corollary: $R(s,y,q)$ is not D-finite.

Proof: $F(x,y) = \sum_n x^n c_n(y)$ D-finite in x, with $c_n(y)$ rational. For $n \geq 0$ let S_n = poles of $c_n(y)$, and let $S = \bigcup S_n$. Then

S has only a finite number of accumulation points.

In this case, set of poles $\Longrightarrow$ unit circle.

These walks are not D-finite...

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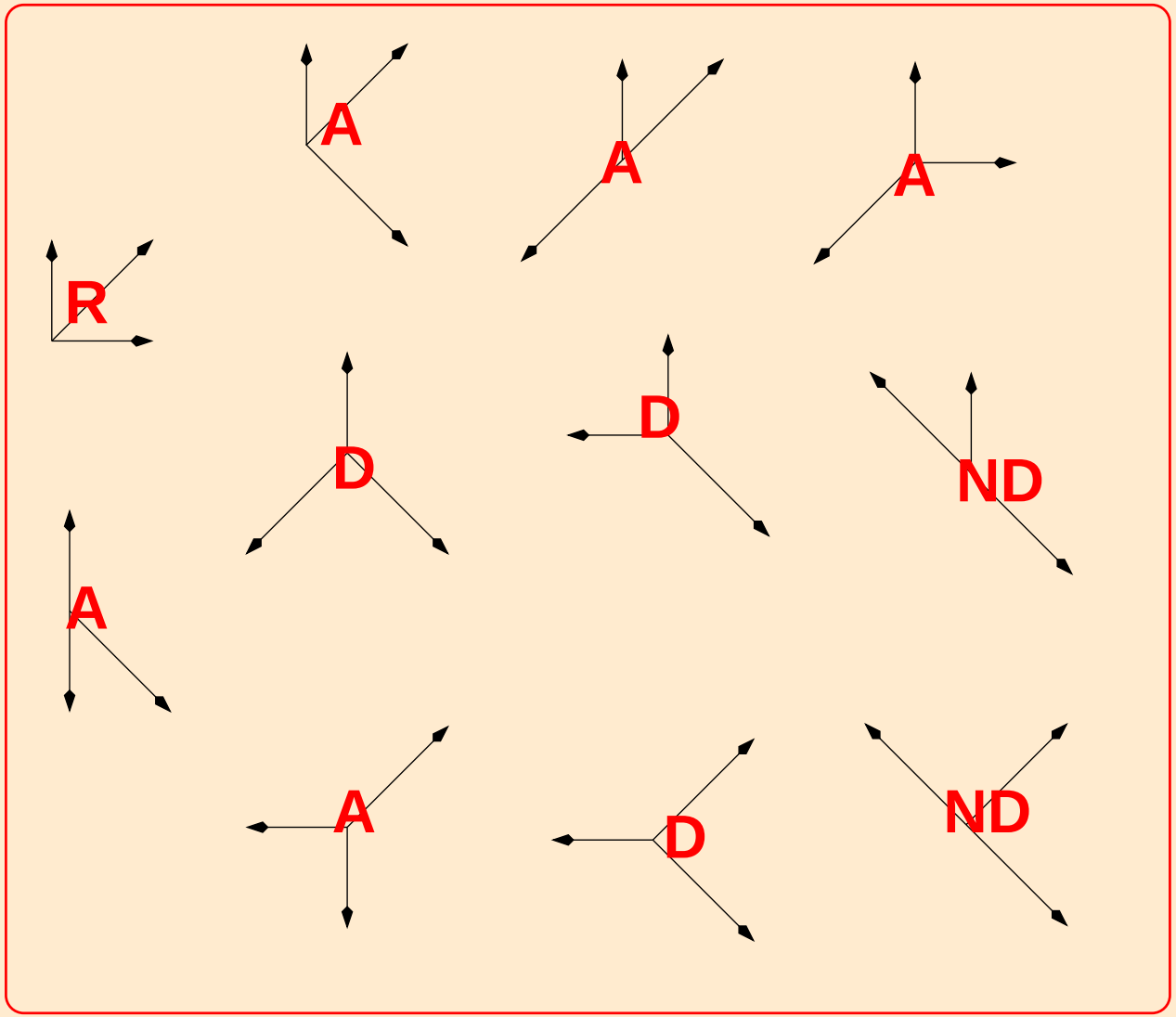
Corollary: $R(s,y,q)$ is not D-finite.

Theorem: $Q(x,y;t)$ is not D-finite.

Proof: $Q(x,y;t) = R(x, y/x, \frac{1-\sqrt{1-4t^2}}{2t})$ and

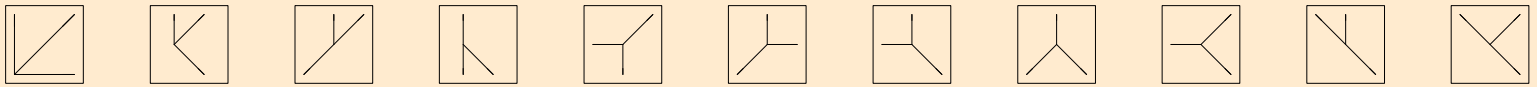
$R(s,y,q) = Q(s, ys; \frac{q}{1+q^2})$.

Update



Results

Classification when $|\mathcal{Y}| = 3$

											
Rational	x										
Algebraic	x	x	x	x	x	x	x^+				
D-finite	x	x	x	x	x	x	x	x	x		
Not D-finite										x^*	x^*

A conjectured criteria for D-finiteness

Conjecture 1.
If $Q_{\mathcal{Y}}$ is transcendental, then $Q_{\mathcal{Y}}$ is D-finite if and only if at least one of the following holds

- $\mathcal{Y}$ is x– or y– axis symmetric;

$\mathsf{rev}(\mathcal{Y})$: each step of $\mathcal{Y}$ in reverse. e.g. $\mathsf{rev}(\mathsf{N}) = \mathsf{S}, \mathsf{rev}(\mathsf{SE}) = \mathsf{NW}$.
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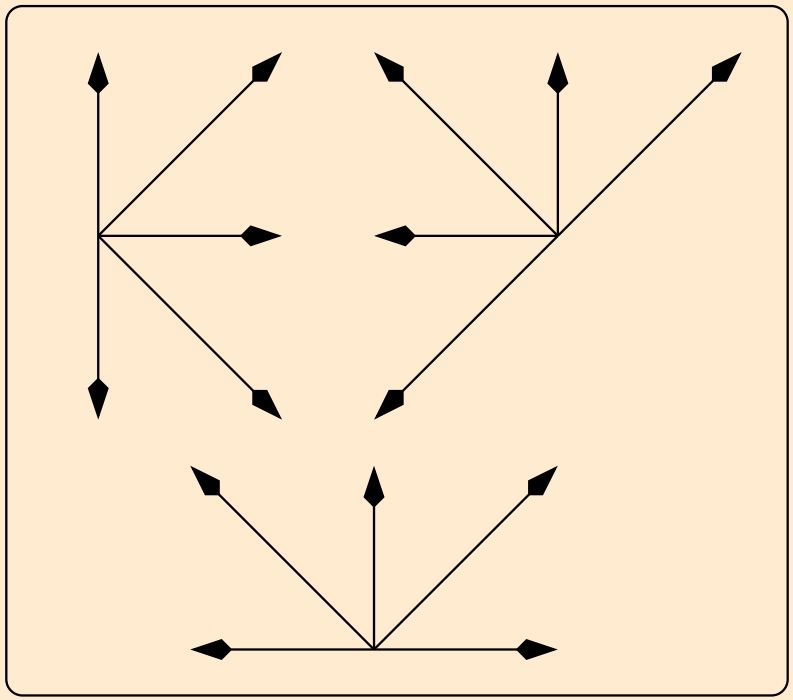
Is the answer in the group?

Conjecture 2.
Suppose the step set group $\mathcal{Y}$ is not singular. Then $Q_{\mathcal{Y}}$ is D-finite if and only if the group of its step set is finite.

Singular Walks



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Possible route?? FIM: *The group is finite if there exists an abelian integral of the third kind, having logarithmic singularities at two points, and represented as the logarithm of an algebraic function ϕ, which belongs to ... .*

Reducing one conj. to the other

Lemma:

In the case of restricted amplitude walks in the quarter plane, suppose $\mathcal{Y}$ is not singular. Then $G(\mathcal{Y})$ is finite if one of the following

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Challenge: Generalize to non-restricted amplitude case!

Other related questions

- Combinatorial actions that preserve either algebraicity or D-finiteness. e.g. reversal?
- Walks of greater amplitude. How generalizable are these arguments?
- General (automatic) asymptotics.

fin.