Classifying walks in the quarter plane

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in collaboration with Mireille Bousquet-Mélou

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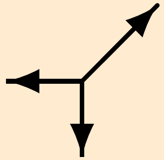
Walks in the quarter plane



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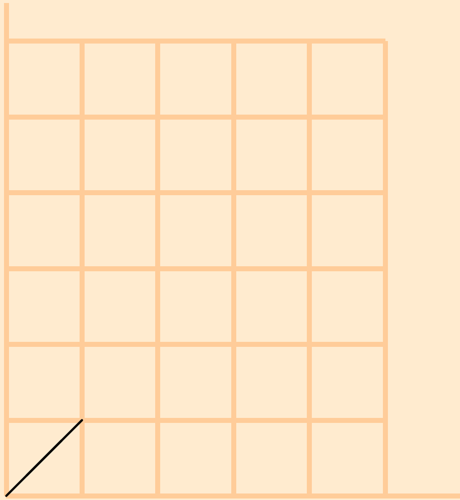

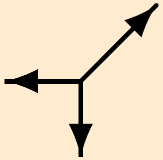

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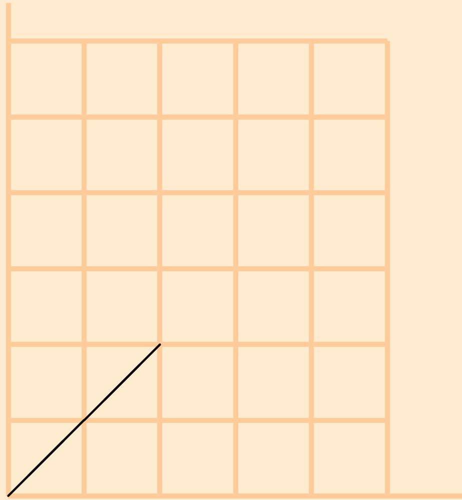

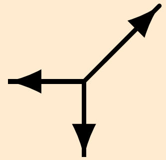

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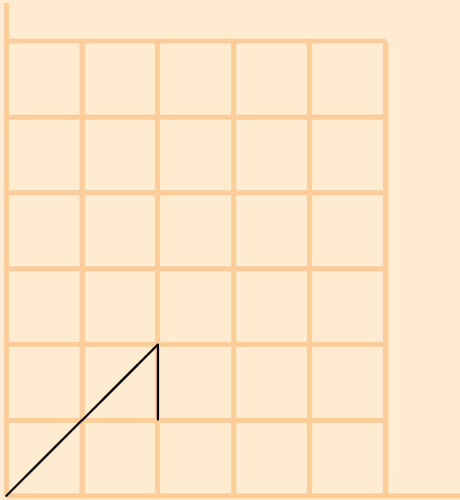

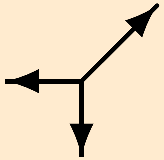

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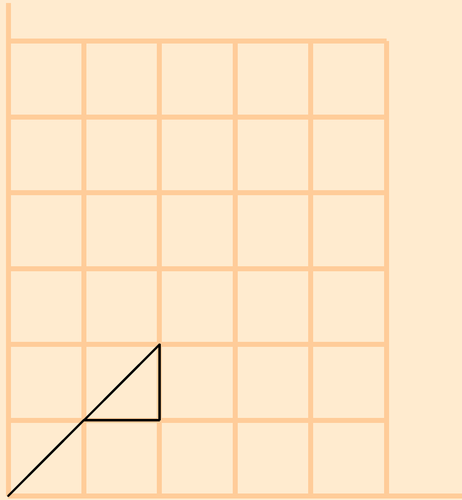

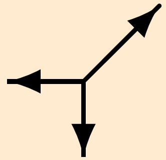

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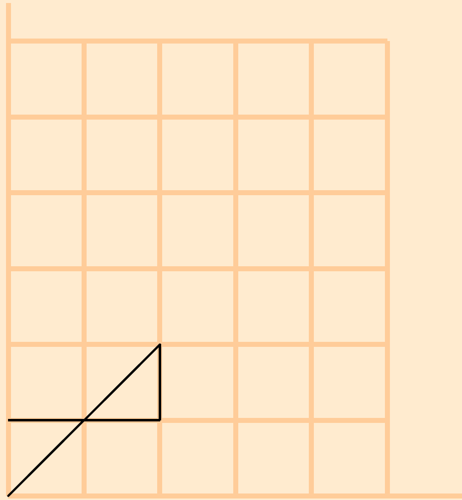

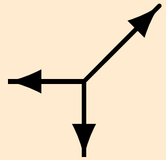

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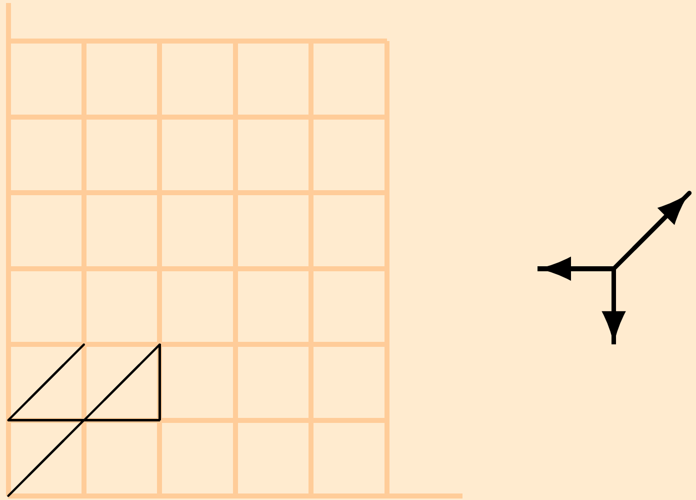

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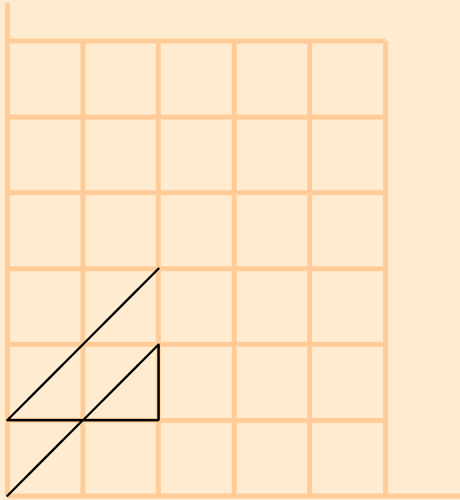

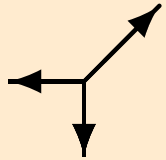

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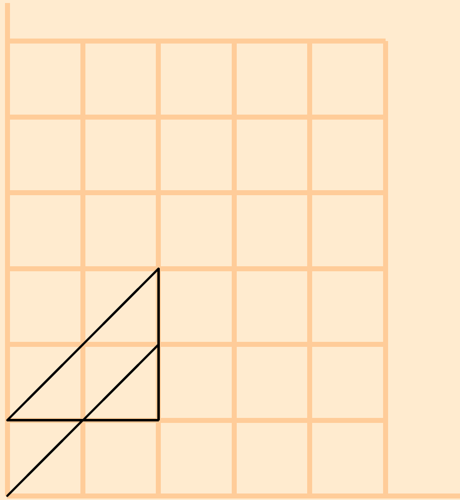
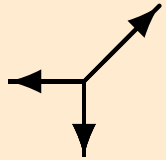




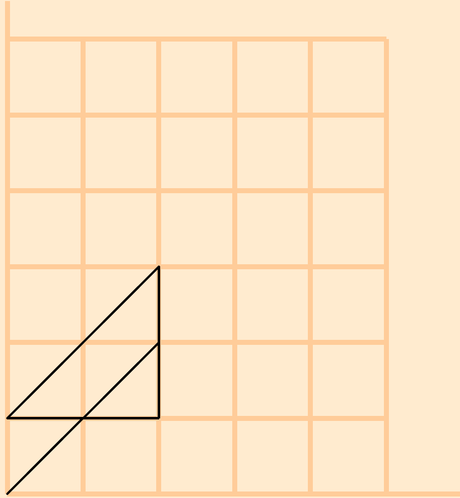
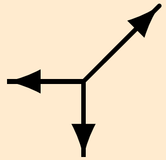
How many walks are there of length n, ending at (i, j), with steps from $\mathcal{Y} \subset \{\text{N}, \text{NW}, \text{W}, \dots, \text{NE}\}$?

Define: complete generating series $Q_{\mathcal{Y}}(x, y; t)$:

$$Q_{\mathcal{Y}}(x, y; t) = \sum_{i,j,n} c_{ij}(n) x^i y^j t^n$$

where $c_{ij}(n)$= # walks ending at (i, j), of length n

Walks in the quarter plane

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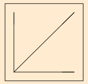
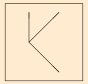
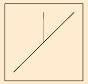
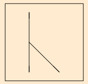
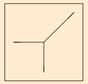
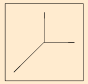
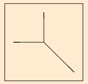
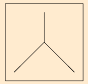
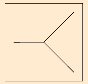
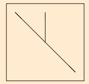
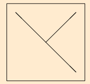
Goal: Classify $\mathcal{Y}$ according to nature of $Q_{\mathcal{Y}}(x, y; t)$

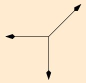
Results

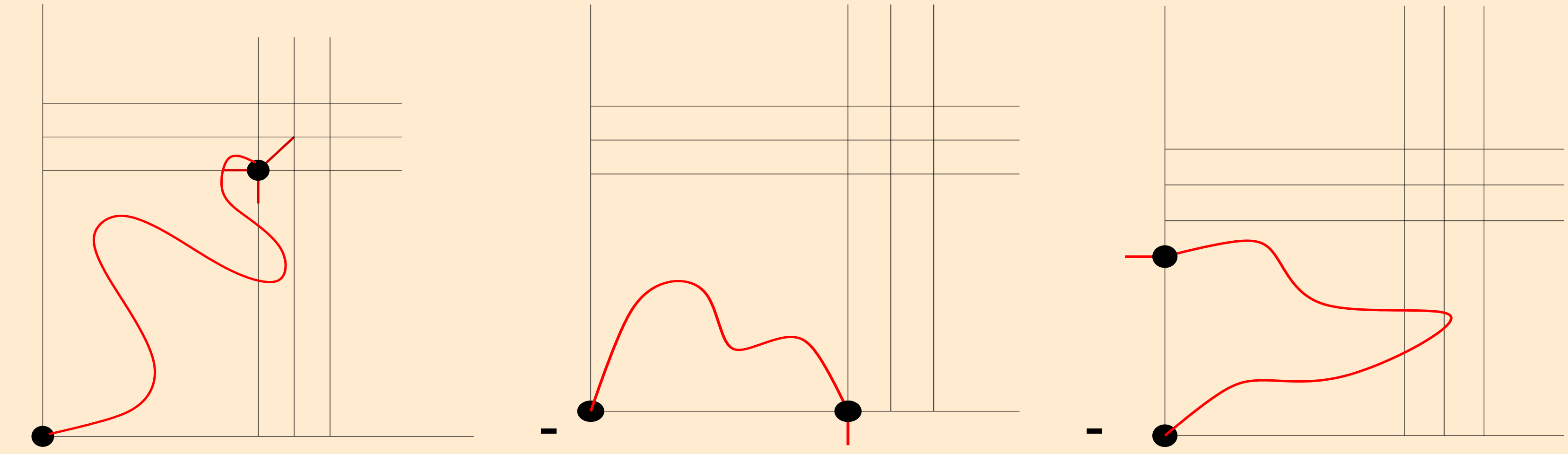
Classification when $|\mathcal{Y}| = 3$

											
Rational	x										
Algebraic	x	x	x	x	x	x	x$^+$				
D-finite	x	x	x	x	x	x	x	x	x		
Not D-finite										x*	x*

The functional equation approach

Functional equation: A walk is a walk plus a step.

$\mathcal{Y} = \{\mathsf{S}, \mathsf{W}, \mathsf{NE}\} =$ 



$$Q_{\mathcal{Y}}(x,y;t) = 1 + t\left(\tfrac{1}{x} + \tfrac{1}{y} + xy\right)Q_{\mathcal{Y}}(x,y;t) - \tfrac{t}{y}Q_{\mathcal{Y}}(x,0;t) - \tfrac{t}{x}Q_{\mathcal{Y}}(0,y;t)$$

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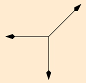
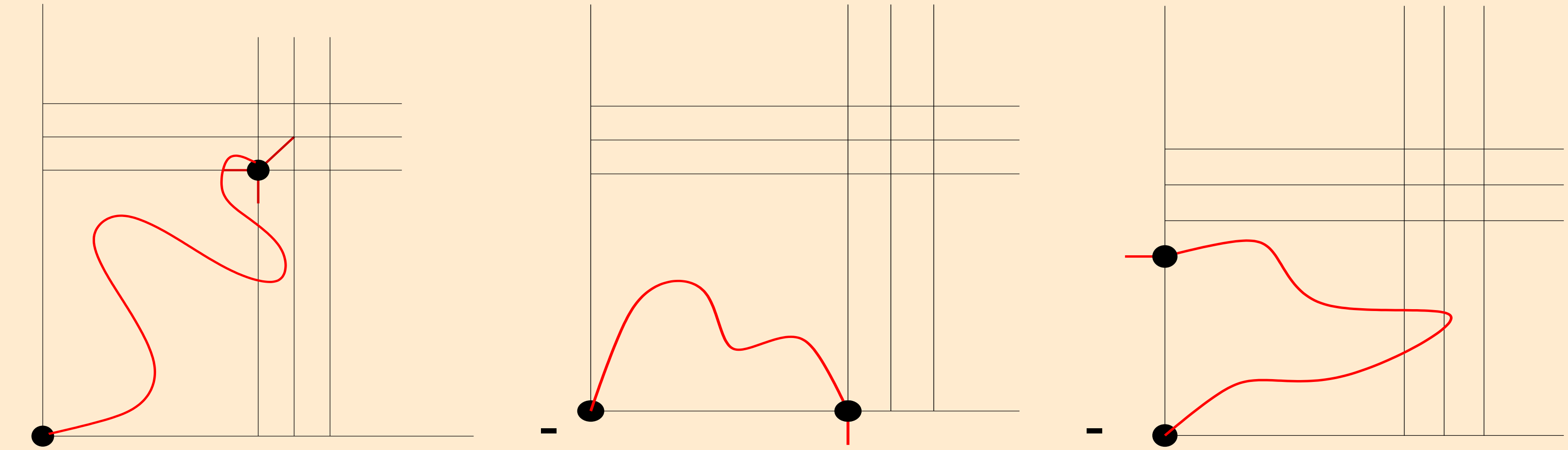
$\mathcal{Y} = \{\mathsf{S}, \mathsf{W}, \mathsf{NE}\} =$ 



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Rewrite:

$$(1 - t(\bar{y} + \bar{x} + xy))Q(x,y;t) = 1 - \tfrac{t}{y}Q_0(x;t) - \tfrac{t}{x}Q_0(y;t)$$

$$(xy - t(x + y + x^2y^2))Q(x,y;t) = xy - txQ_0(x;t) - tQ_0(y;t)$$

Classification of power series

Rational

Algebraic

D-finite

Differentiably algebraic

Algebraic functions

$F(x, y, z)$ satisfi es $P(F(x, y, z), x, y, z) = 0$ for some polynomial P.
Useful properties:

- Singularities and coeff asymptotics:
  singularity: $(x - \zeta)^{-\alpha}$ coeff: $n^{\alpha-1}\zeta^{-n}$
- The Hadamard product of two rational series is algebraic.
  $\sum_n f_n x^n \times \sum_n g_n x^n = \sum_n f_n g_n x^n$

D-finite functions

$f(x_1, x_2, \ldots, x_n)$ is Differentiably finite (D-finite) with respect to $X = x_1, \ldots, x_n$ if

- For $1 \leq j \leq n$, f satisfies n linear differential equation s with polynomial coefficients:

$$\phi_0(X)f(X) + \phi_1(X)\frac{\partial f(X)}{\partial x_j} + \ldots + \phi_k(X)\frac{\partial^k f(X)}{\partial x_j^k} = 0$$

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Useful properties:

- The Hadamard product wrt any subset of variables is D-finite; ($\sum f_n x^n \times \sum g_n x^n = \sum f_n g_n x^n$)
- algebraic substitution $\implies$ still D-finite

Context of our study

- 1-Dimensional Lattice Paths

  - Flajolet, Banderier (2002)

  - Always algebraic

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- Walks in the quarter plane

  - Enumeration/ GF approach: Kreweras, Bousquet-Mélou, Gessel, Gouyou-Beauchamps, Petkovsěk

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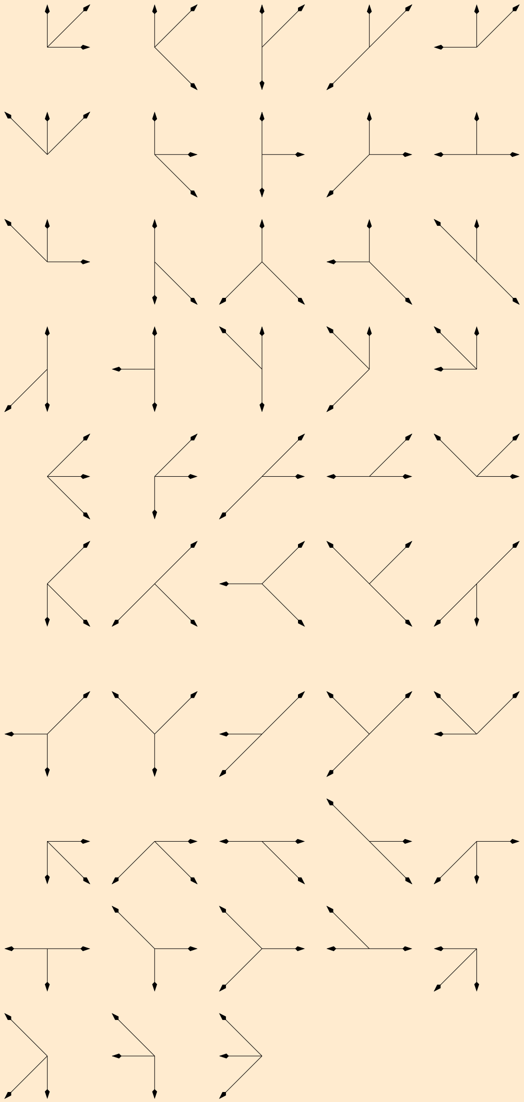
- Walks in the Slit-Plane

  -GF approach: Bousquet-Mélou, Schaeffer (2002)

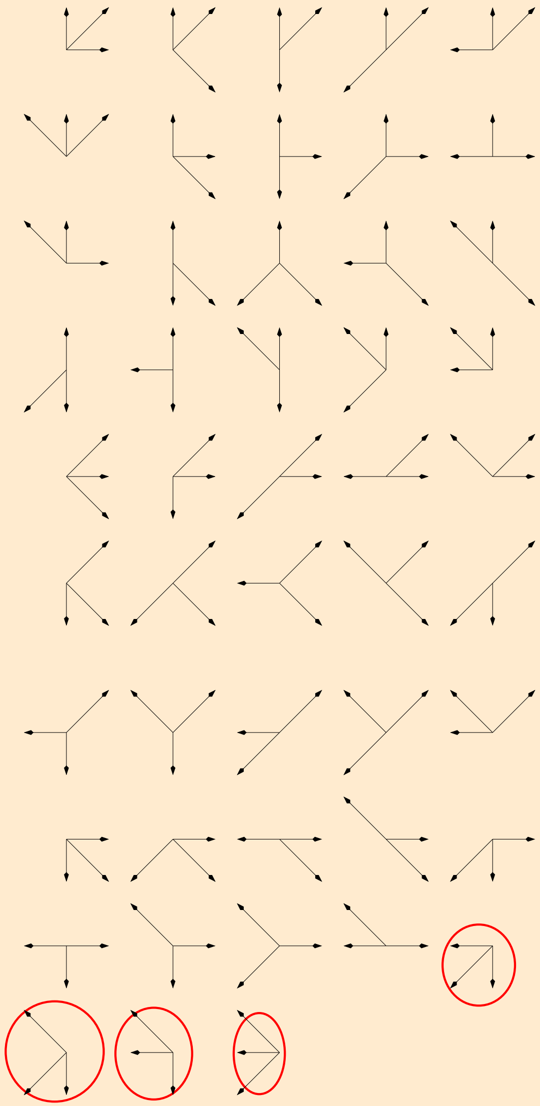
  -small height variations= Algebraic

When $|\mathcal{Y}| = 3$



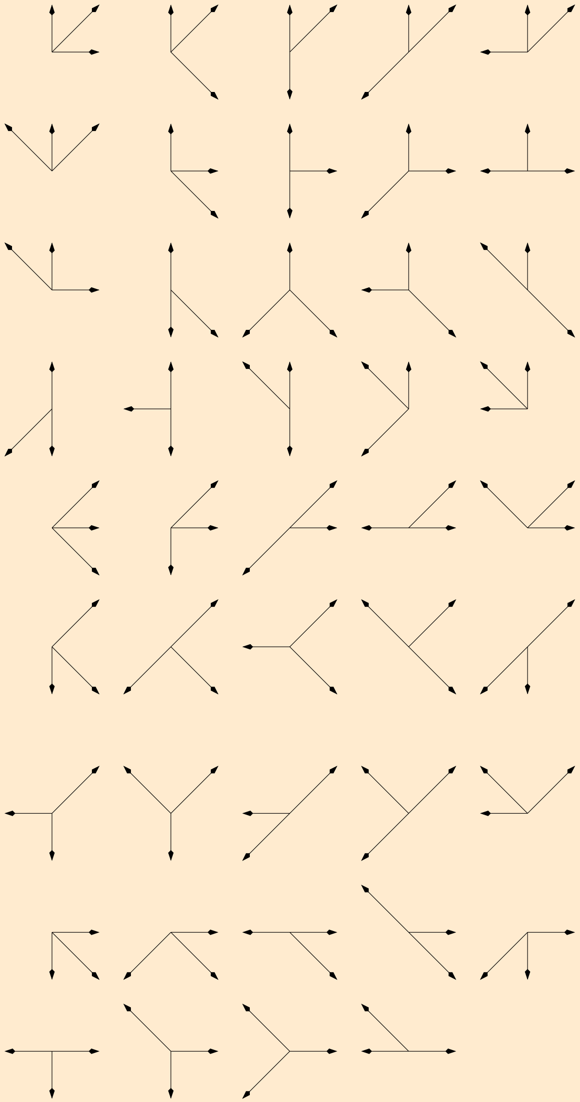
all of the possibilities

When $|\mathcal{Y}| = 3$



no walks in quarter plane

When $|\mathcal{Y}| = 3$



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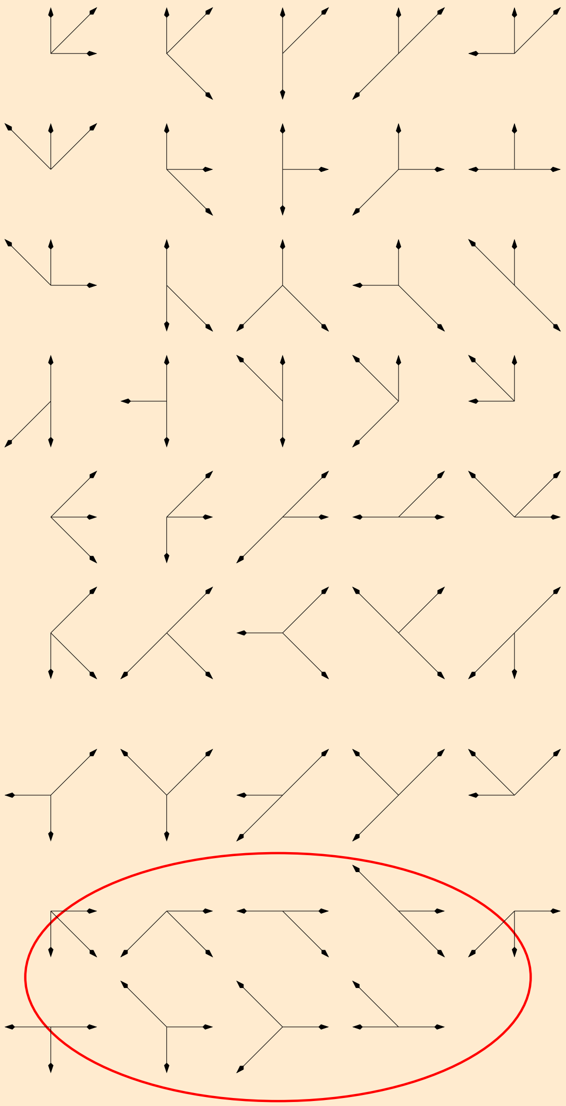
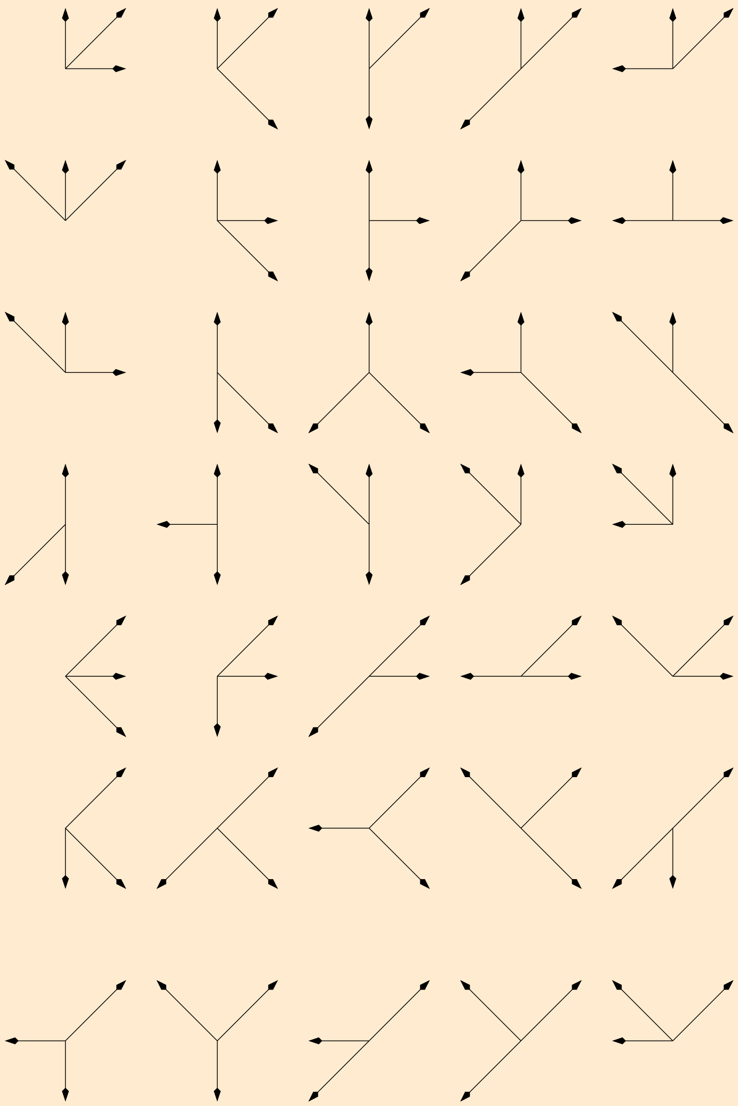


identical to
others reflected
in $x = y$

When $|\mathcal{Y}| = 3$



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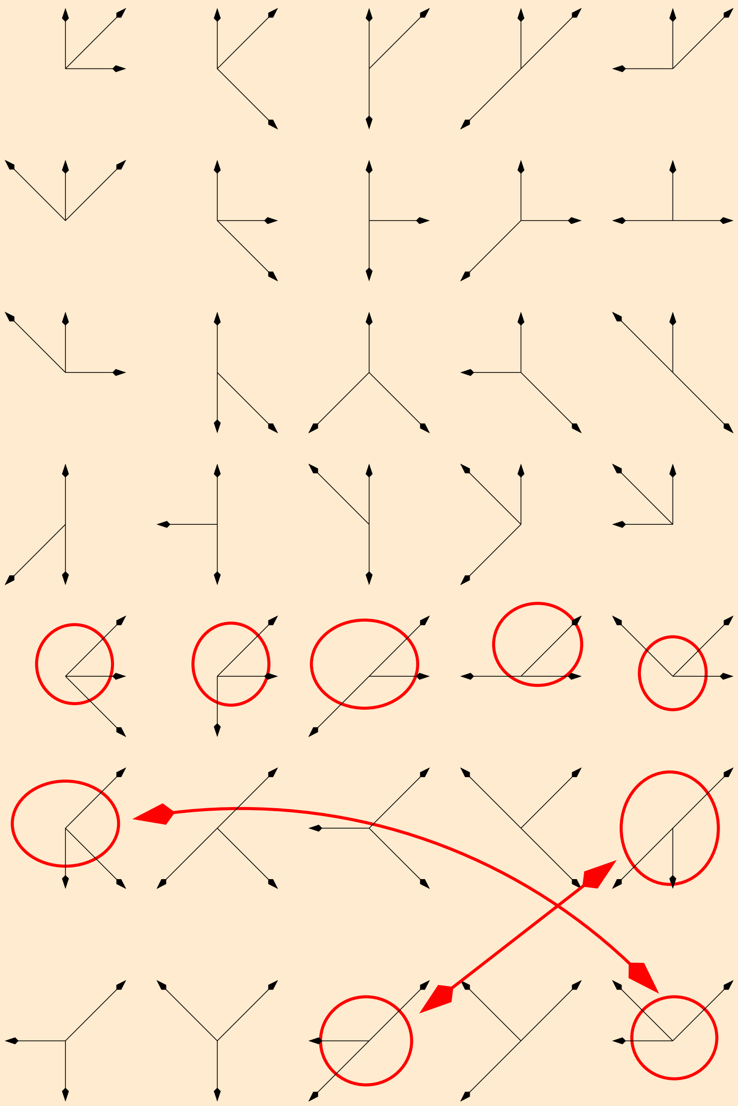
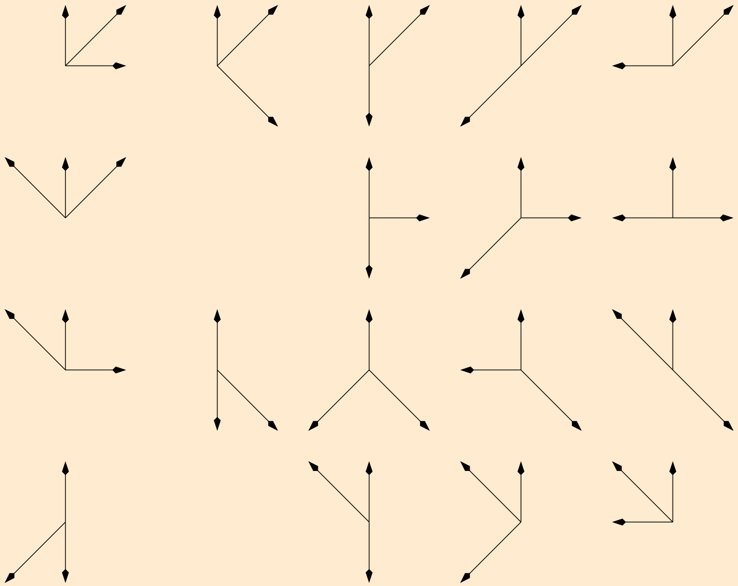


more identical pairs...

When $|\mathcal{Y}| = 3$



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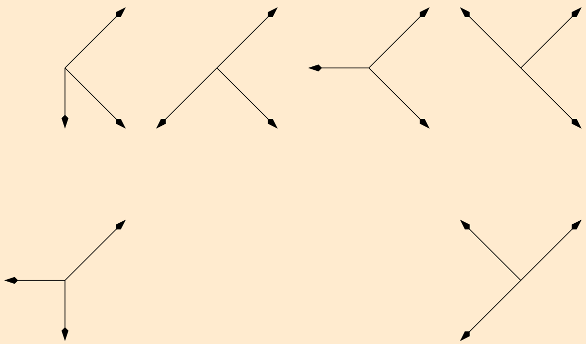

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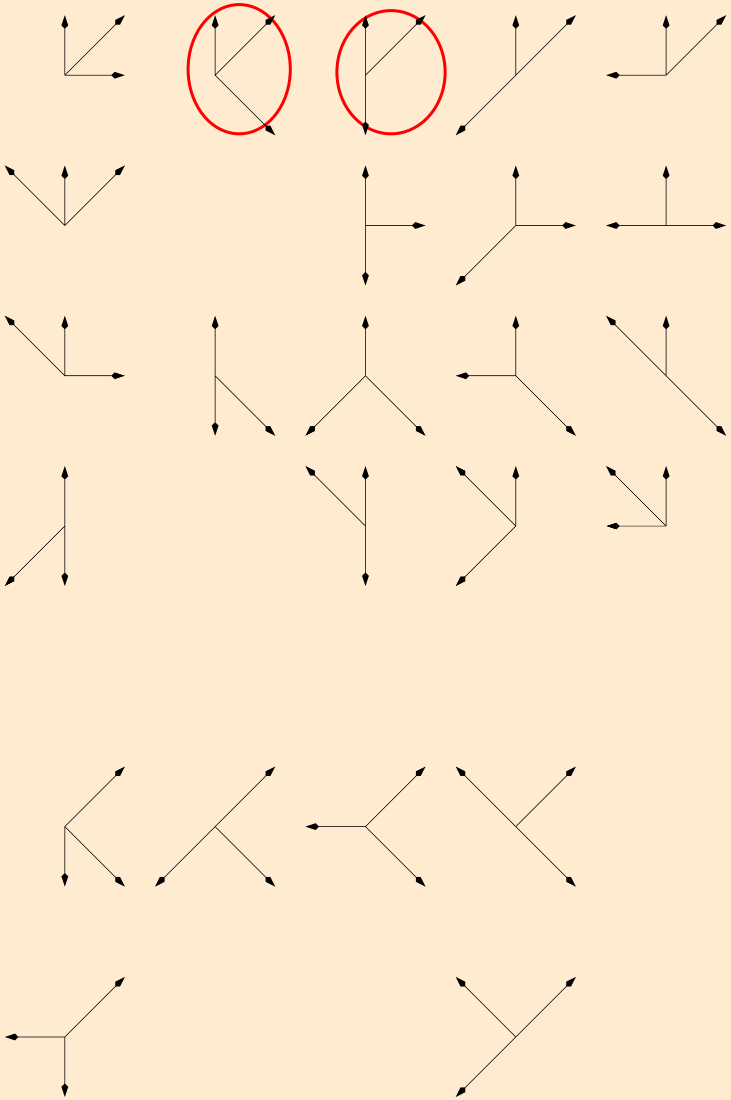
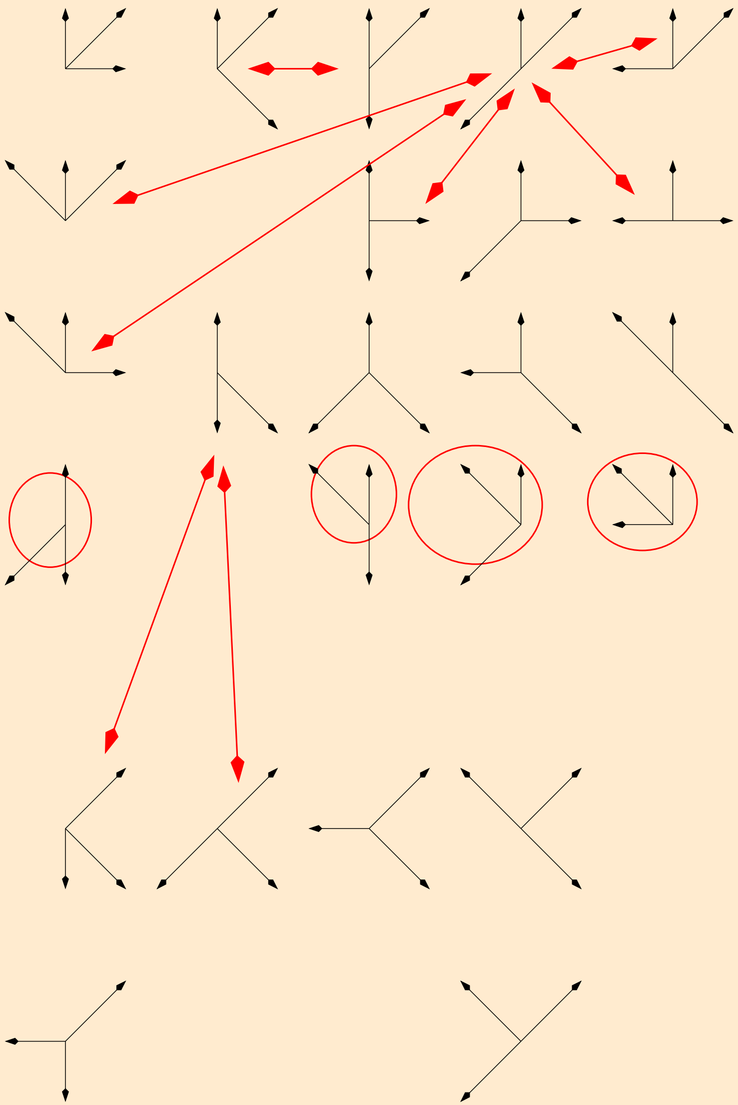


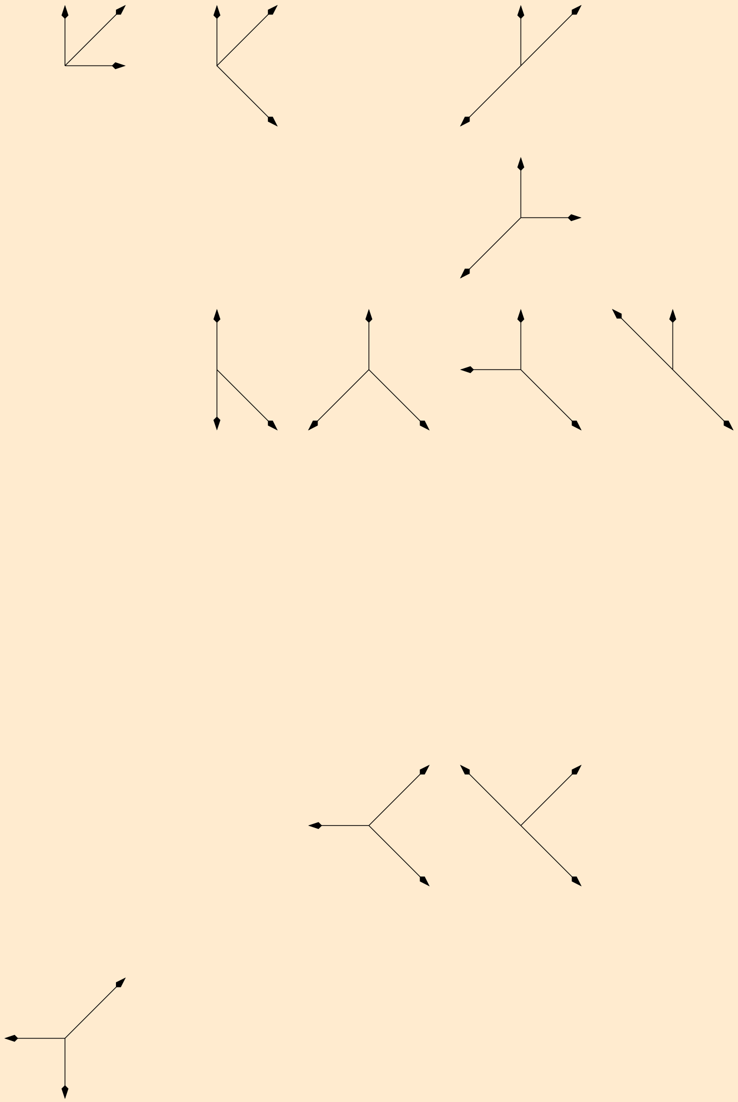
These two are governed by the same inequality:
$\#a + \#b \geq \#c$

When $|\mathcal{Y}| = 3$



More cases of this + 1 dimensional cases

When $|\mathcal{Y}| = 3$



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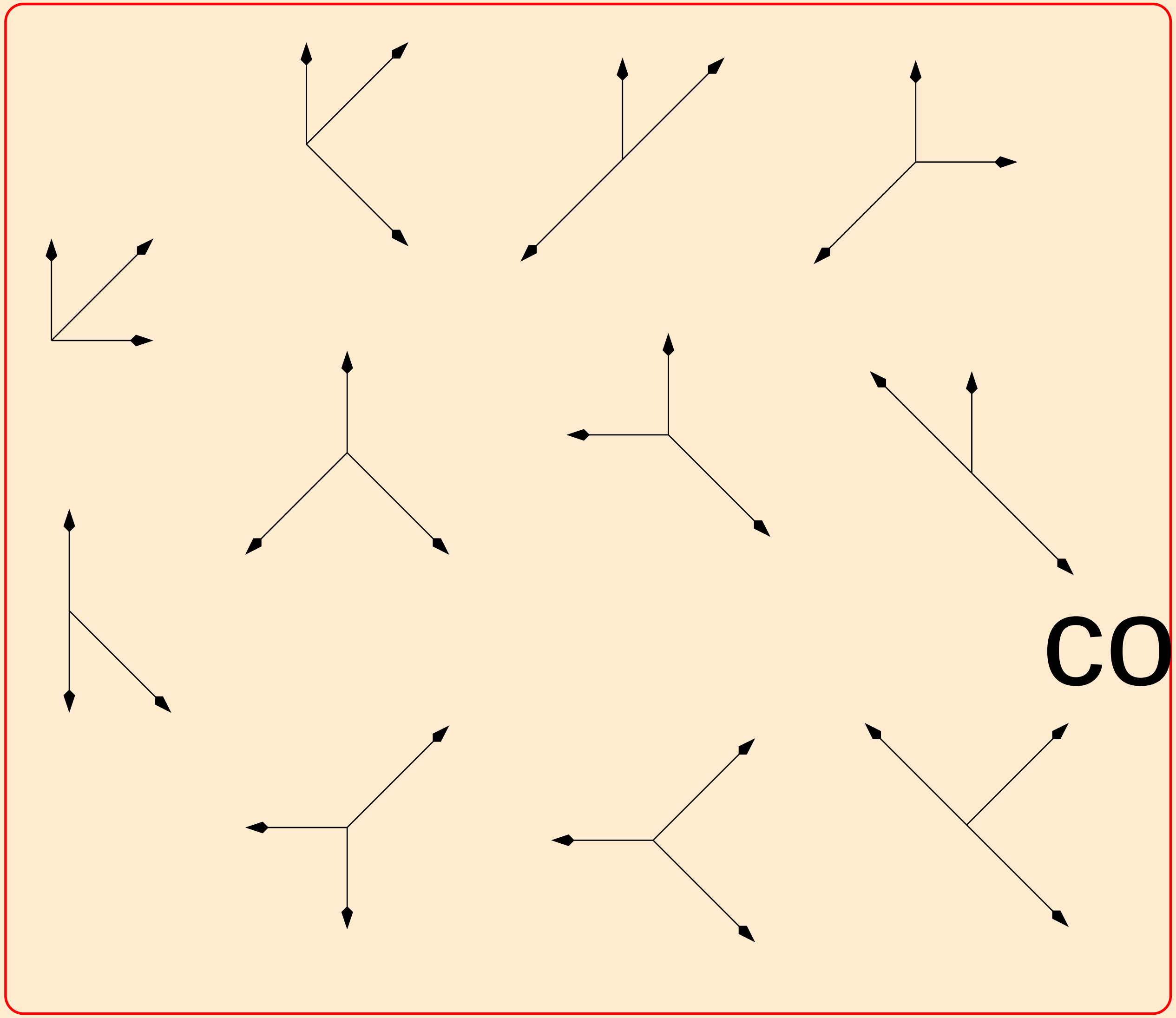
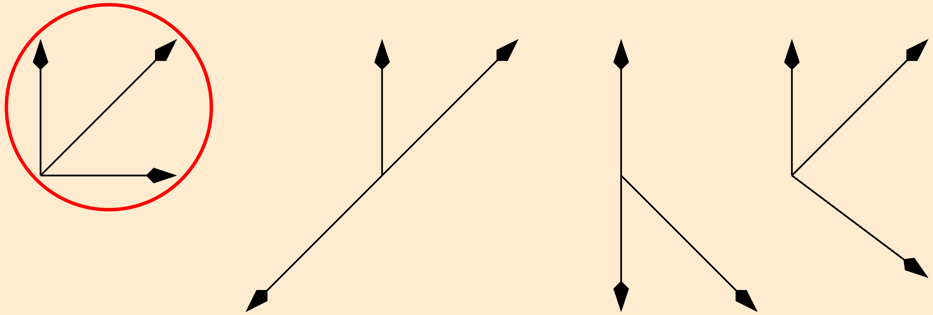


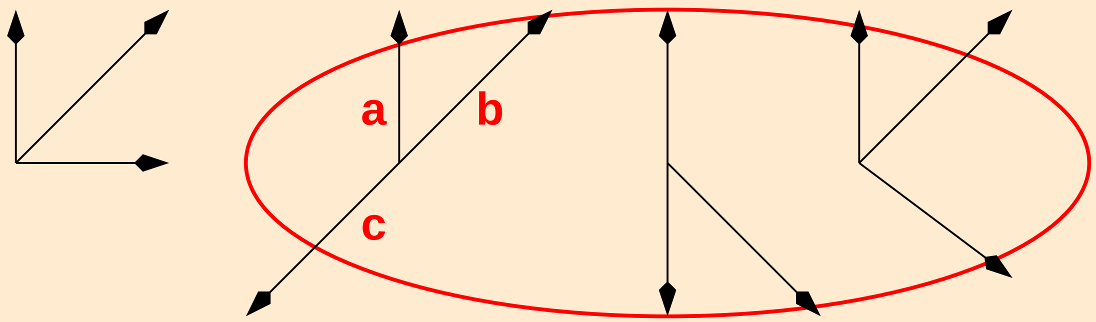
Final contenders!

Easy cases



- Rational: Only one, unrestricted case

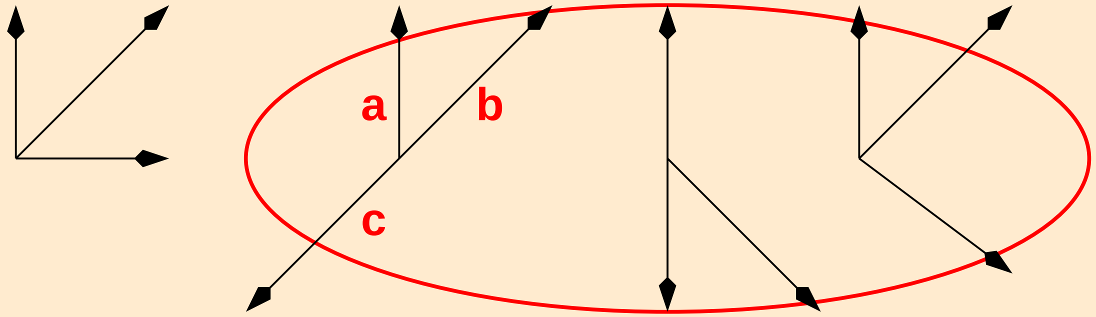
Easy cases



- Rational: Only one, unrestricted case
- "Easy algebraic": Only one true condition.
  - Reduces to half plane condition $\implies$ algebraic
  - One condition $\implies$ can be recognized by a pushdown automaton (CFL)

$$V : a + b \geq c \quad H : b \geq c \quad H \implies V$$

Easy cases



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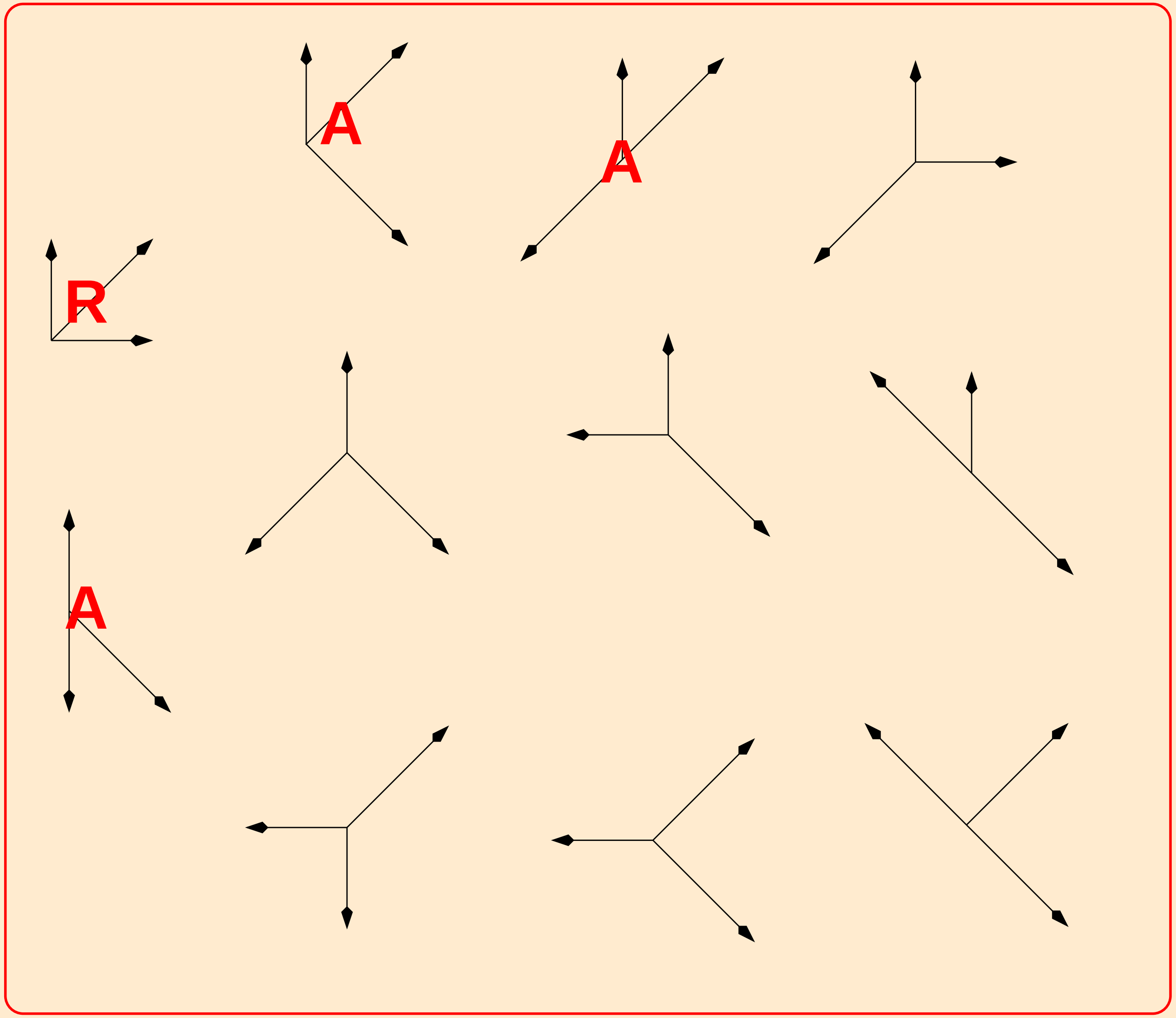
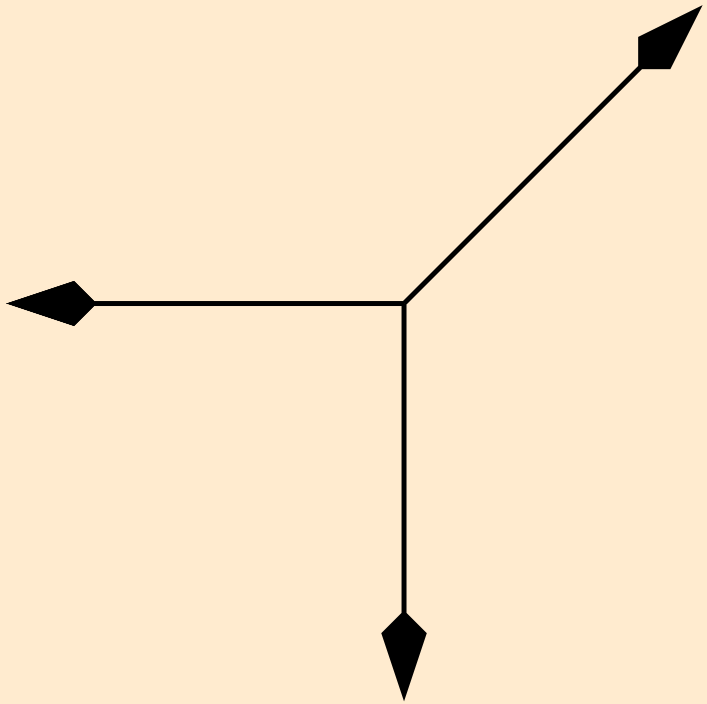
Later, we will call these cases singular

Update



The novelty of Kreweras' model



- Well studied: Kreweras (60s) – MBM(04)
- NOT "easy algebraic" (Not context-free!)
- Algebraic, with explicit description of GF

Algebraic Kernel Method

Recall a general form of the equation satisfied by $Q(x, y; t)$

Rational Kernel: K_r

The coefficient of $Q(x, y; t)$: $K_r = 1 - t \sum_{(i,j)\in\mathcal{Y}} x^i y^j$

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The group of the walk: G

Write K as a function of y : $K = a(x, t)y^2 + b(x, t)y + c(x, t) \implies$ roots: Y_0, Y_1. Define $\tau_y : (x, y) \mapsto (x, Y_0 Y_1 / y)$ and similarly τ_x.

The group of the walk $G(\mathcal{Y})$:= The group generated by τ_x, τ_y.

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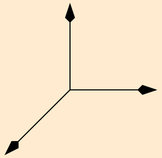
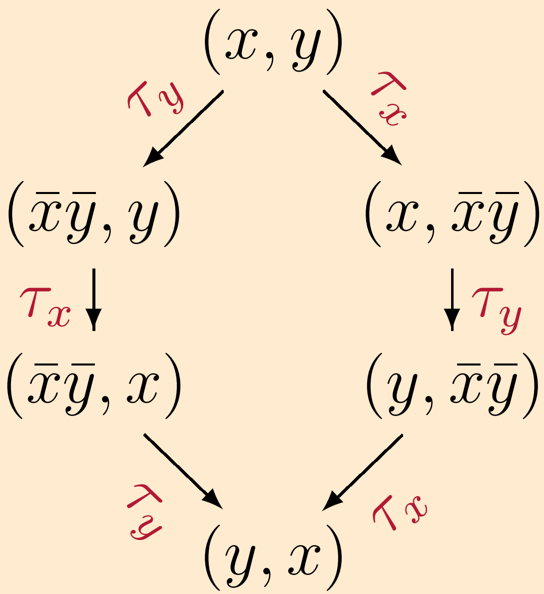
Lemma: $K_r(x,y) = K_r(\rho(x,y))$ for $\rho \in G(\mathcal{Y})$

Reverse Kreweras walks



$$K_r = 1 - t(x + y + \bar{x}\bar{y}) \quad Y_0 Y_1 = 1/x = \bar{x}$$



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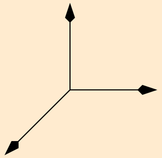

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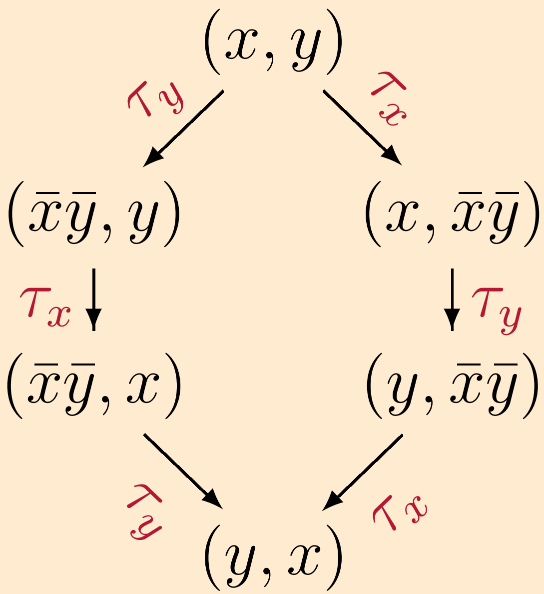

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Apply to fundamental equation

$$\begin{aligned}
(1) \quad \bar{x}\bar{y}\bar{K}_r Q(\bar{x}, \bar{y}; t) &= \bar{x}\bar{y} - tQ_0(\bar{x}) - tQ_0(\bar{y}) + tQ_{00}(t) \\
(2) \quad y\bar{K}_r Q(\bar{x}, xy; t) &= y - tQ_0(\bar{x}) - tQ_0(xy) + tQ_{00}(t) \\
(3) \quad x\bar{K}_r Q(xy, \bar{y}; t) &= x - tQ_0(xy) - tQ_0(\bar{y}) + tQ_{00}(t)
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New equation: (1)+(2)-(3)

$$\begin{aligned}
& \bar{x}\bar{y}Q(\bar{x},\bar{y};t) + yQ(\bar{x},xy;t) - xQ(xy,\bar{y};t) \\
&= \frac{1}{\bar{K}_r}\left(\bar{x}\bar{y} + y - x - 2tQ_0(\bar{x}) + tQ_{00}(t)\right)
\end{aligned}$$

Reverse Kreweras walks

New equation: (1)+(2)-(3)

$$\bar{x}\bar{y}Q(\bar{x},\bar{y};t) + yQ(\bar{x},xy;t) - xQ(xy,\bar{y};t)$$
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Extract constant term wrt y:

$$xQ_d(x) = \frac{1}{\sqrt{\Delta(x)}}\left(2Y_0 - x - 2tQ_0(\bar{x}) + tQ_{00}(t)\right)$$

$Y_0 = \frac{1-t\bar{x}-\sqrt{\Delta}}{2tx}$

Reverse Kreweras walks

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Canonical factorization of $f \in C[x,\bar{x}][[t]]$ into $f^+(x,t)f(t)f(\bar{x},t)$.

Extract non-positive powers of x to get $Q_0(\bar{x})$.

Reverse Kreweras walks

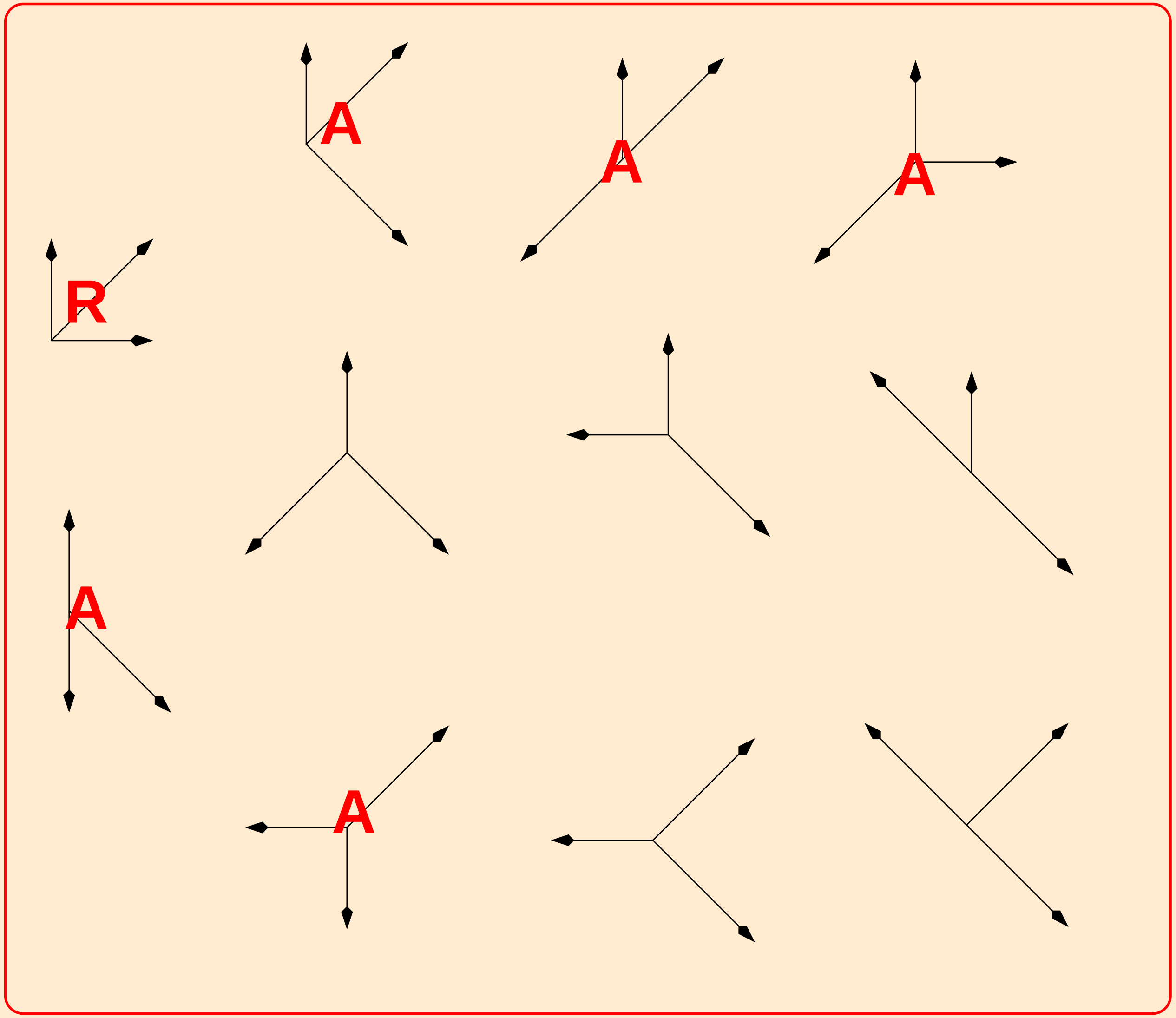
Theorem:

Let $\mathcal{Y} = \{\text{N}, \text{E}, \text{SW}\}$. Then:

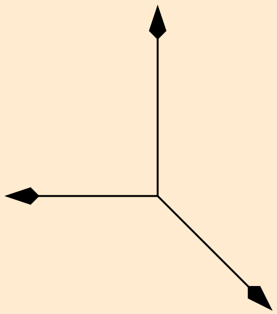
$$2Q_{\mathcal{Y}}(x,0;t) = \frac{4T - T^2}{8t} + \left(\frac{-2x}{Tt}\left(1 - \frac{T^2}{2x}\right) + \frac{1}{tx}\right)\sqrt{U} + \left(1 - tx - \frac{t}{x^2}\right)xt^{-2}.$$

$T = t(2 + T^3)$, $U = 1 - xT(1 + T^3/4) + x^2T^2/4$

Update

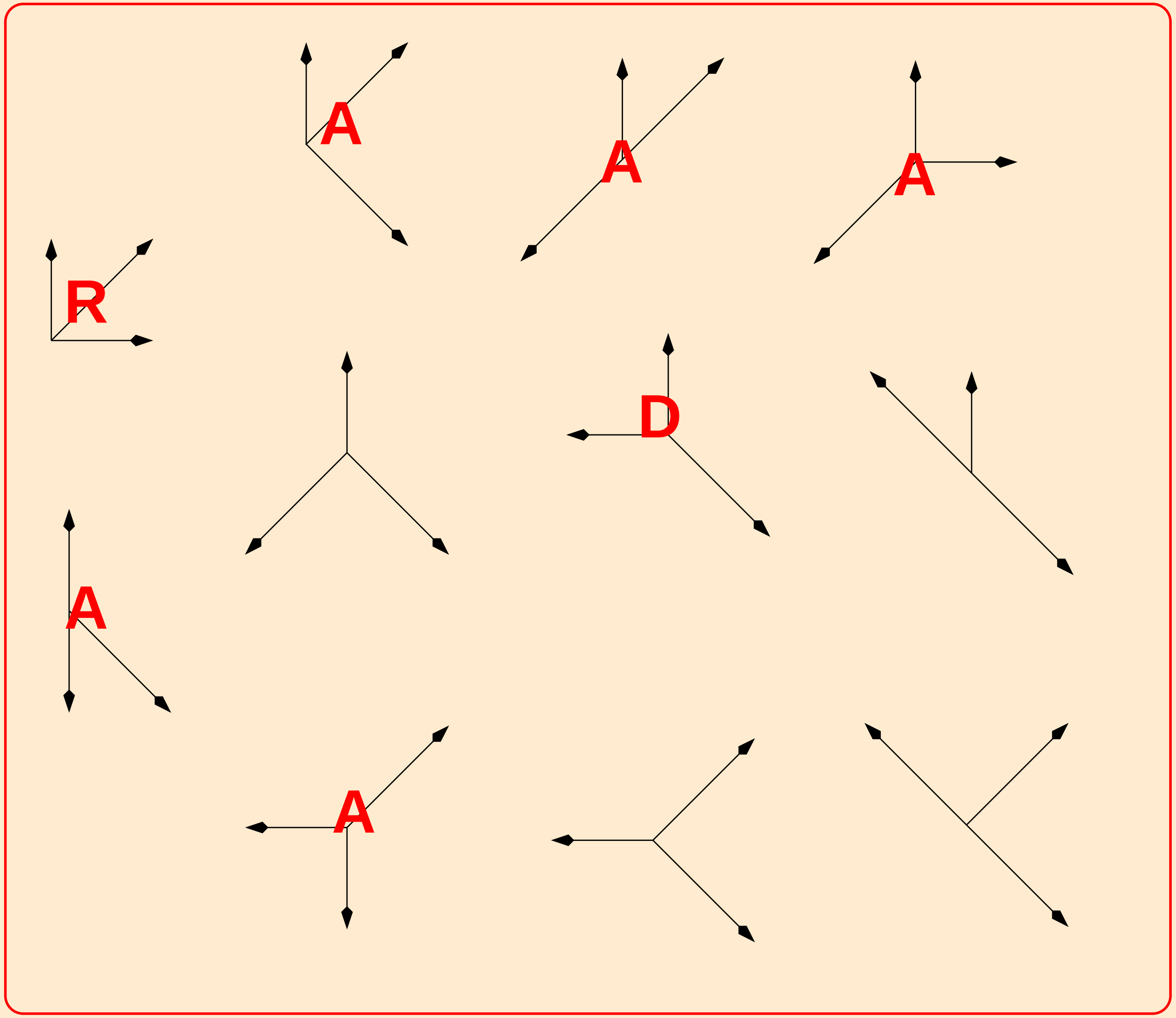


An interesting case



- In bijection with Young tableaux of height 3 $(\#a \geq \#b \geq \#c) \implies$ explicit formula for complete gf. (D-finite; non-algebraic)
- Regev (1981) gives explicit form for counting gf: algebraic!
- Motzkin!

Update



A D-finiteness criterion

Theorem. [Bousquet-Mélou (Petkovšěk)]
Let $\mathcal{Y}$ be a finite subset of $\{\pm 1, 0\} \times \mathbb{Z} \setminus \{(0,0)\}$ that is symmetric with respect to the y-axis. Then the complete generating function $Q(x, y; t)$ for walks that start from $(0,0)$, take their steps in $\mathcal{Y}$ and stay in the first quadrant is D-finite.

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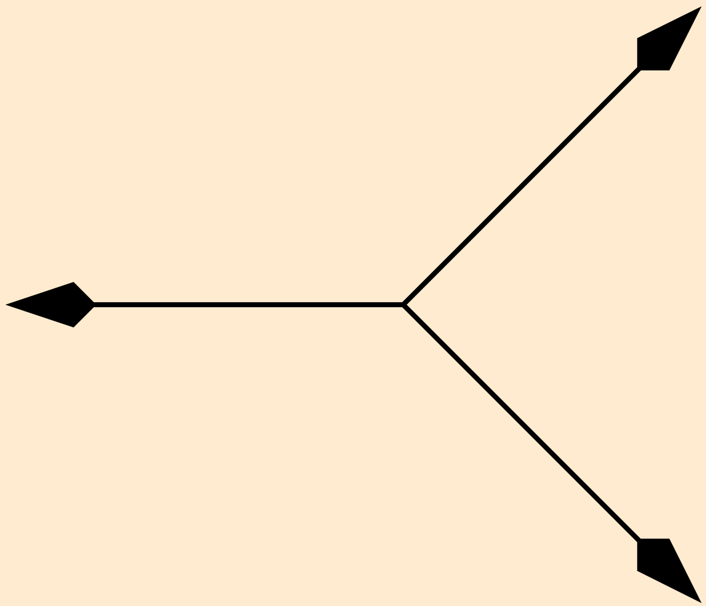
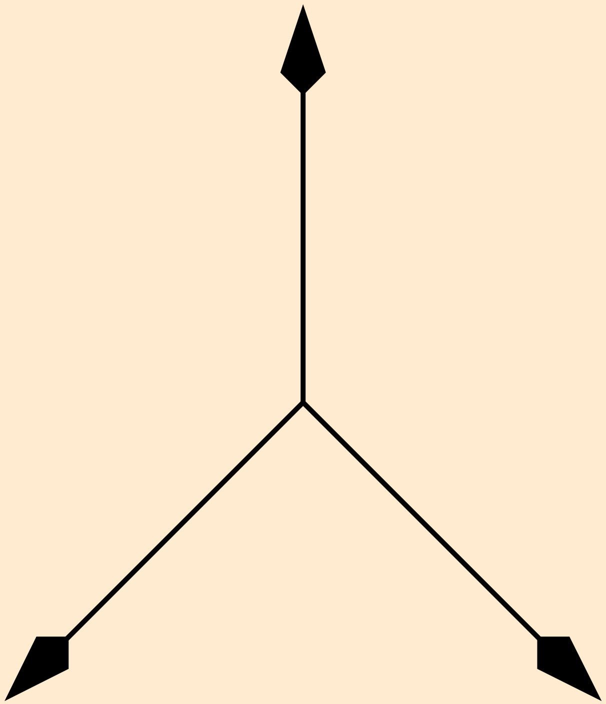
Example:





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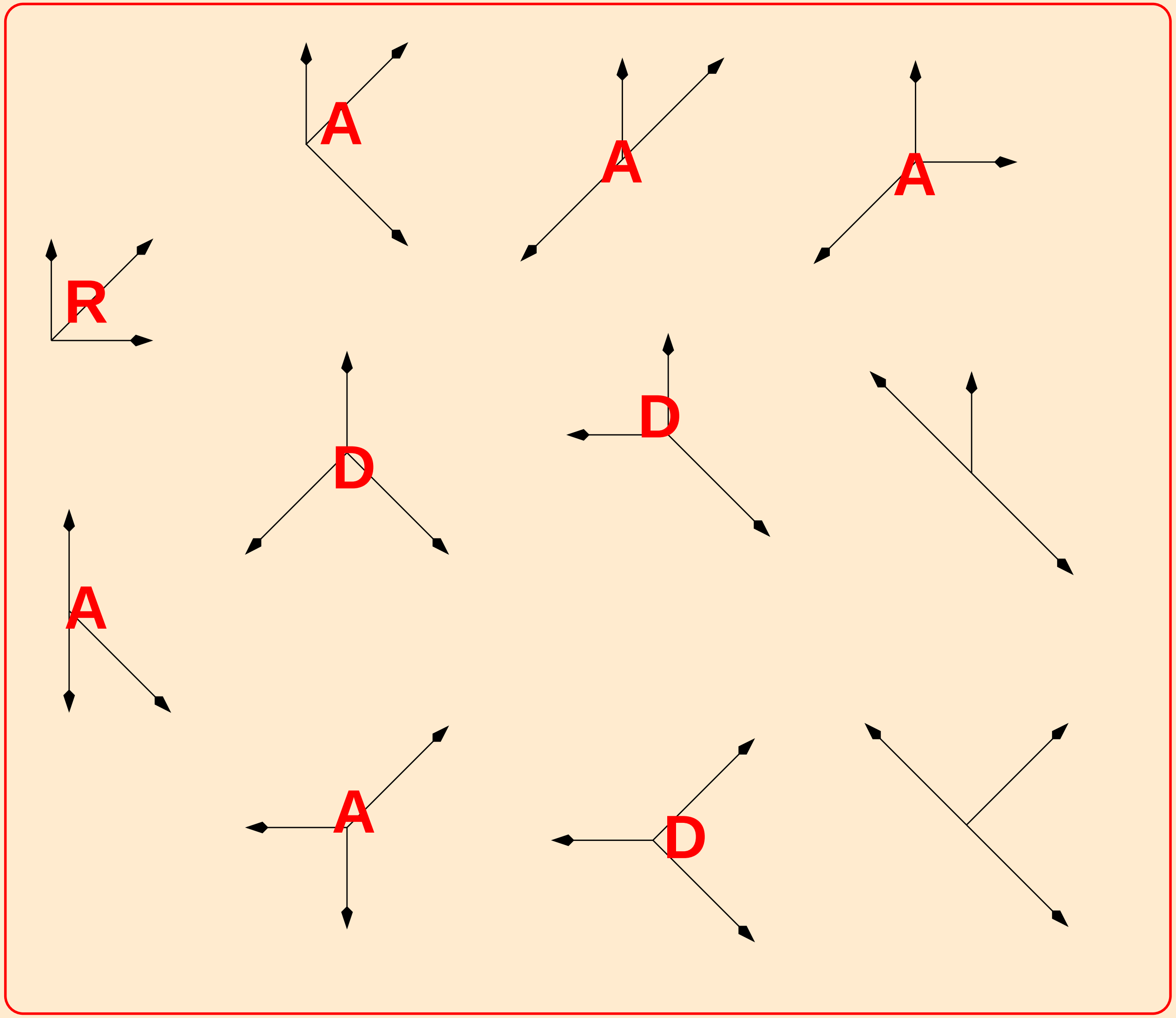
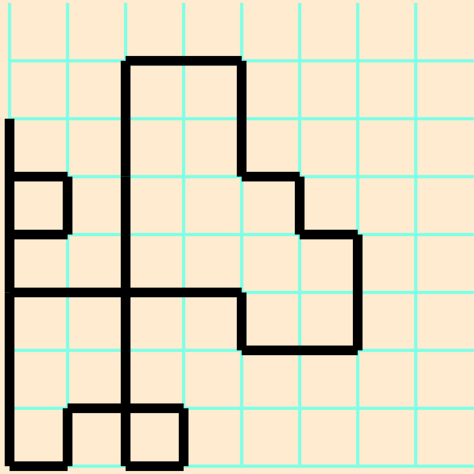


Another D-finite example

Ex: $\mathcal{Y} = \{\mathsf{N}, \mathsf{E}, \mathsf{S}, \mathsf{W}\}$



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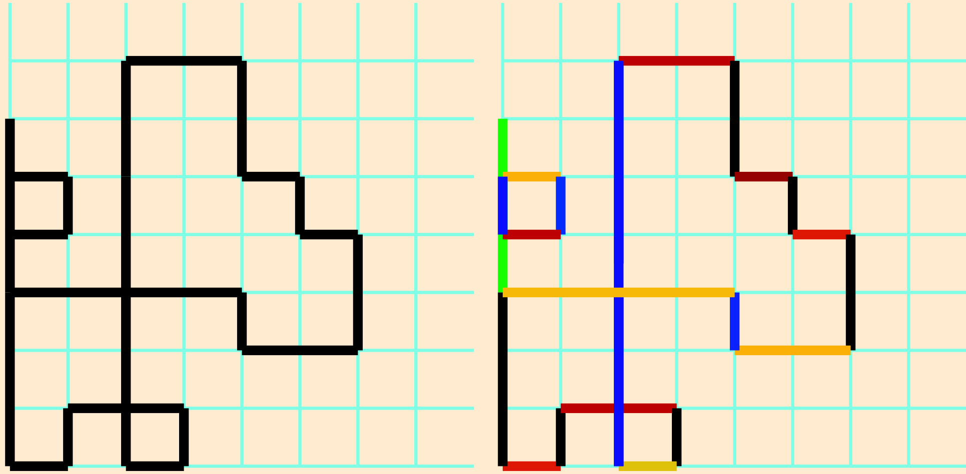

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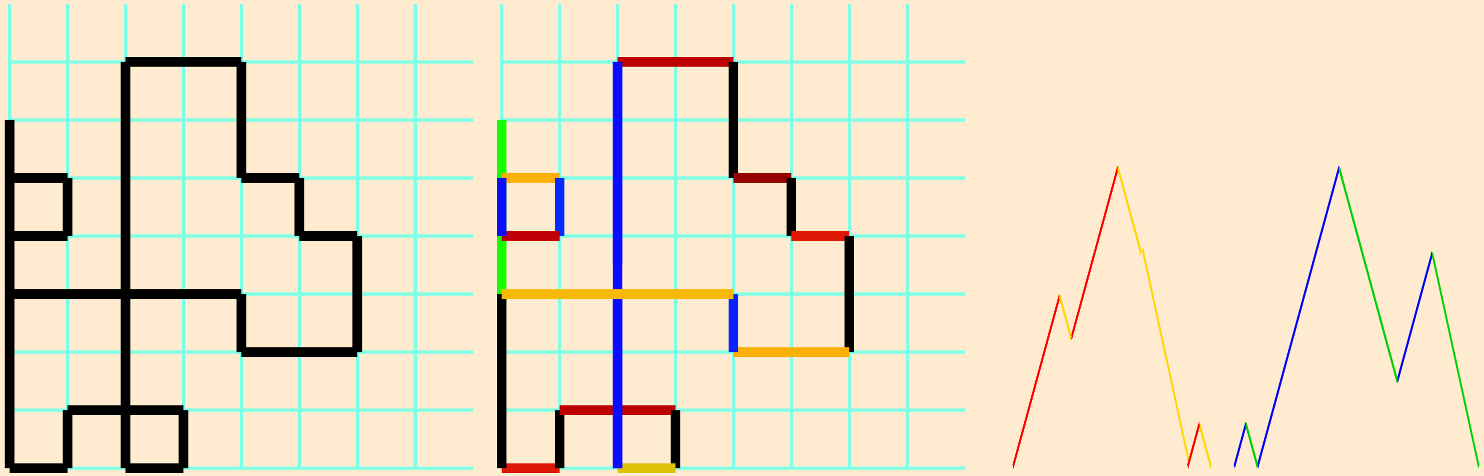

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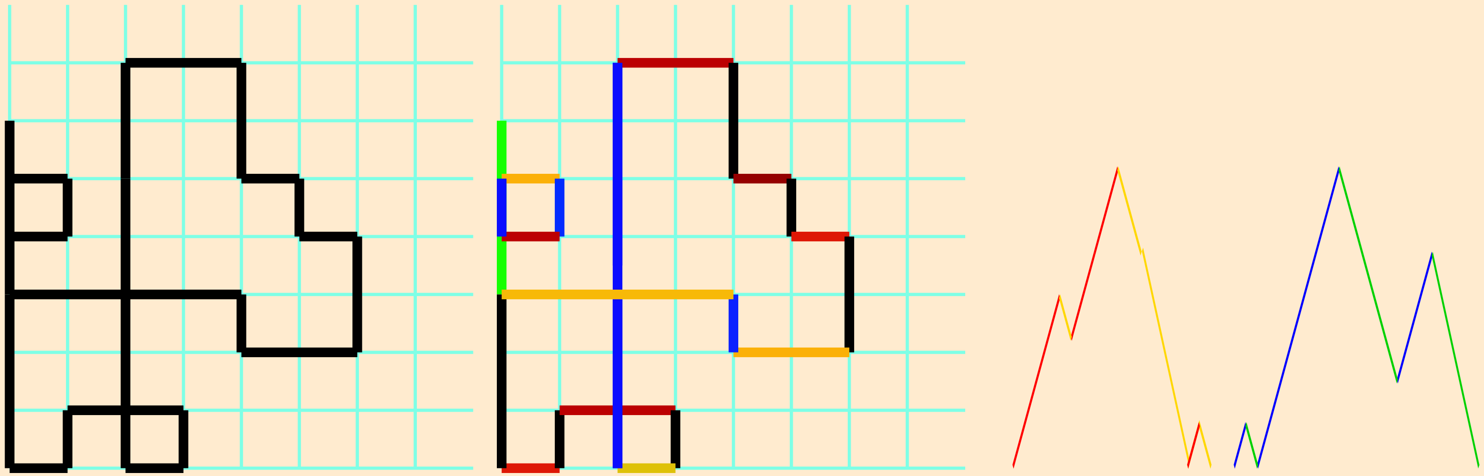
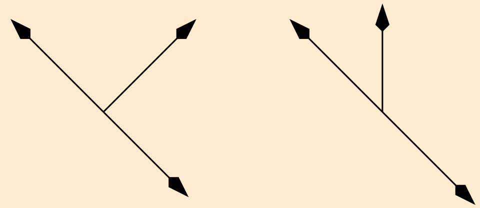


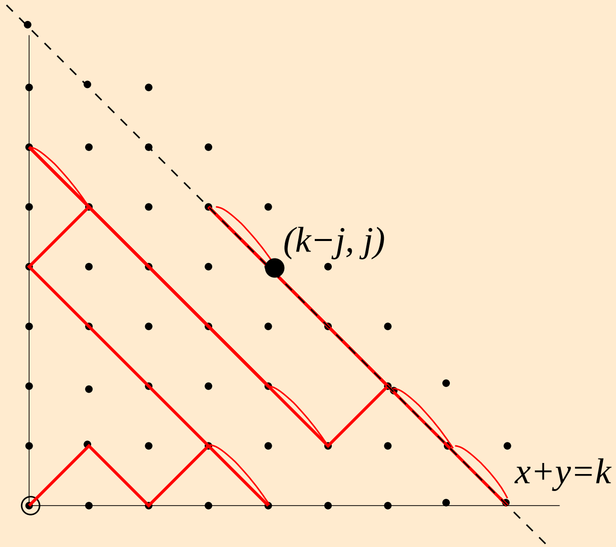
A walk = A pair of Dyck path prefixes implies

$$Q_{\{\mathsf{N,E,S,W}\}}(t) = \sum_{m,n\geq 0} \binom{m+n}{m}\binom{m}{\lfloor m/2 \rfloor}\binom{n}{\lfloor n/2 \rfloor} t^{m+n}$$

which is *not* algebraic, but *is* D-finite.

A non-D-finite example...





A walk to $(k-j,j)$
= walk to $x+y=k-2$ + step
+ walk along $x+y=k$

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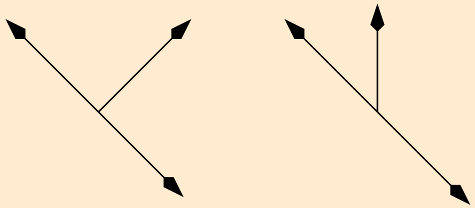

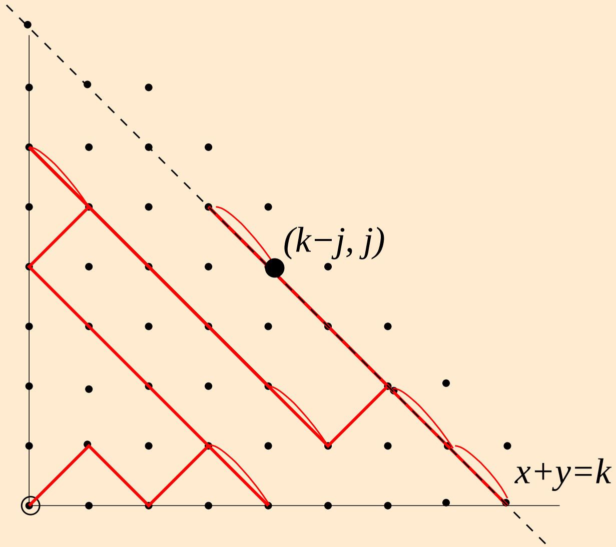

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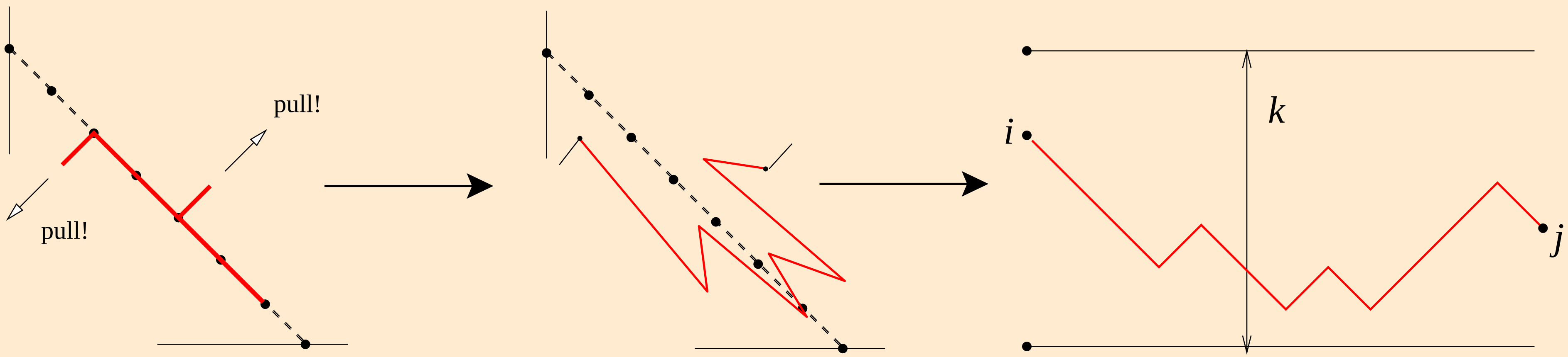
Bounded Dyck paths



Generating function:

$$H^k_{ij}(t) = t^{i-j}\frac{f_{j+1}f_{k-i+1}}{f_{k+2}} \qquad (i \geq j)$$

Fibonacci polynomials:

$$f_{n+1} = f_n - t^2 f_{n-1}, \qquad f_0 = 0, \qquad f_1 = 1$$

OGF for walks ending on $x + y = k$

$S_{k,j} :=$ ogf walks that end at $(k-j, j)$

$$B_k(y) := \sum_{j=0}^{k} S_{k,j} y^j$$

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Use:

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to derive recurrence for $\hat{B}_k := B_k(y)\big|_{t \mapsto \frac{q}{1+q^2}}$

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$$\hat{B}_k(y) = \frac{q^3 \hat{B}_{k-2}(q)(y^{k+2}+1) - qy^2 \hat{B}_{k-2}(y)(q^{k+2}+1)}{(q^{k+2}+1)(yq-1)(y-q)}.$$

These walks are not D-finite...

$$R(s, y, q) := \sum \hat{B}_k(y, q) s^k$$

Lemma: $\hat{B}_k(y)$ is rational, with poles at $q^{k+2} = -1$ ($q^2 \neq 1$)

Proof: Can show $\hat{B}_k(q) \neq 0$ when $q^{k+2} = -1$ and $q^2 \neq -1$.

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Corollary: $R(s,y,q)$ is not D-finite.

Proof: $F(x,y) = \sum_n x^n c_n(y)$ D-finite in x, with $c_n(y)$ rational. For $n \geq 0$ let S_n = poles of $c_n(y)$, and let $S = \bigcup S_n$. Then

S has only a finite number of accumulation points.

In this case, set of poles $\Longrightarrow$ unit circle.

These walks are not D-finite...

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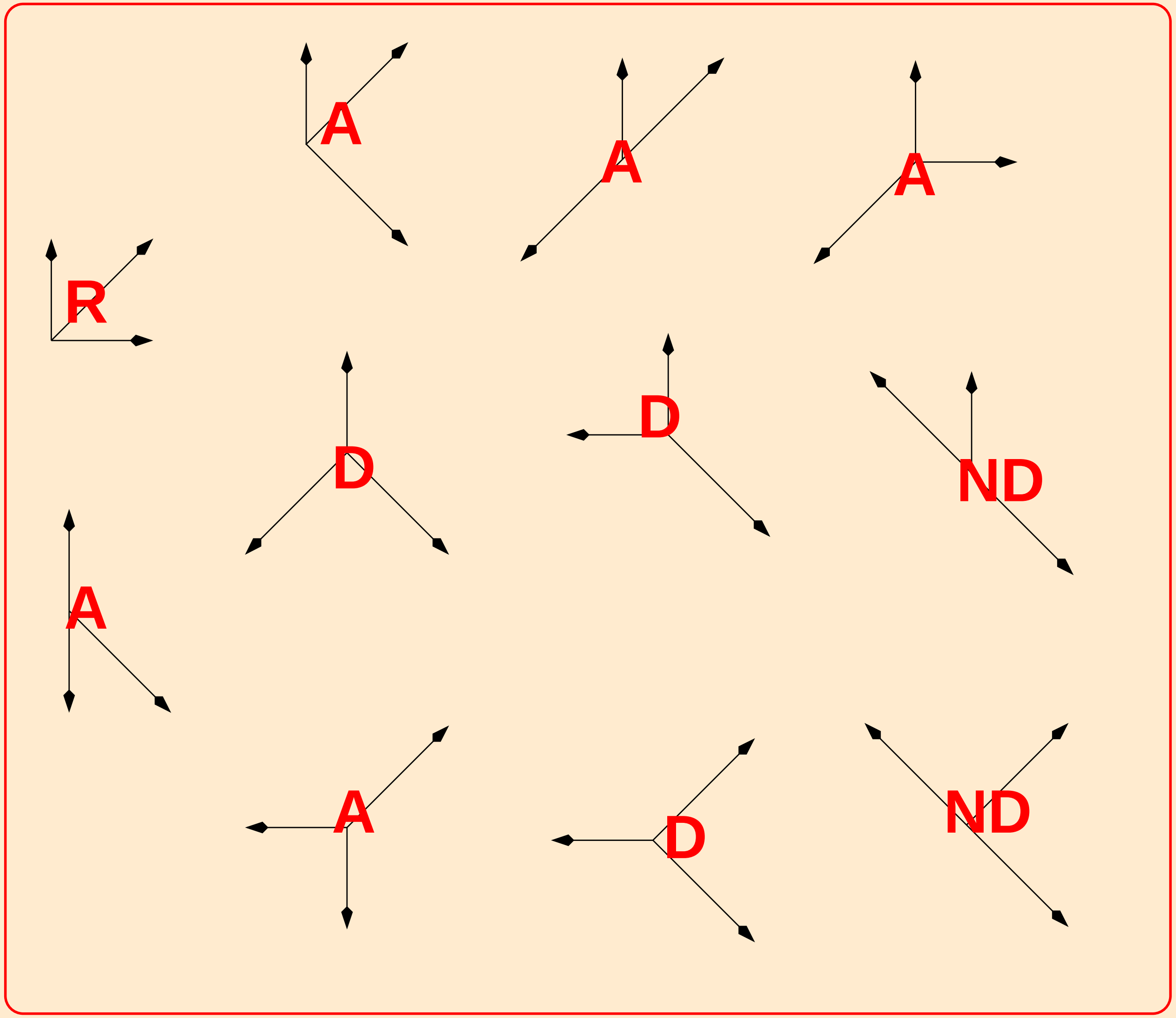
Corollary: $R(s,y,q)$ is not D-finite.

Theorem: $Q(x,y;t)$ is not D-finite.

Proof: $Q(x,y;t) = R(x, y/x, \frac{1-\sqrt{1-4t^2}}{2t})$ and

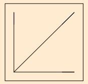
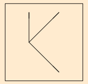
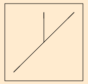
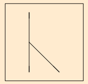
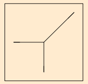
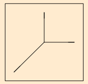
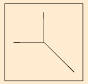
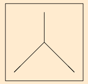
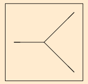
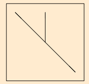
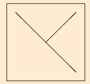
$R(s,y,q) = Q(s, ys; \frac{q}{1+q^2})$.

Update



Results

Classification when $|\mathcal{Y}| = 3$

											
Rational	x										
Algebraic	x	x	x	x	x	x	x^+				
D-finite	x	x	x	x	x	x	x	x	x		
Not D-finite										x^*	x^*

A conjectured criteria for D-finiteness

Conjecture 1.
If $Q_{\mathcal{Y}}$ is transcendental, then $Q_{\mathcal{Y}}$ is D-finite if and only if at least one of the following holds

- $\mathcal{Y}$ is x– or y– axis symmetric;

$\mathsf{rev}(\mathcal{Y})$: each step of $\mathcal{Y}$ in reverse. e.g. $\mathsf{rev}(\mathsf{N}) = \mathsf{S}, \mathsf{rev}(\mathsf{SE}) = \mathsf{NW}$.

$\mathsf{reflect}(\mathcal{Y})$: reflect the stepset in $x = y$.

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Is the answer in the group?

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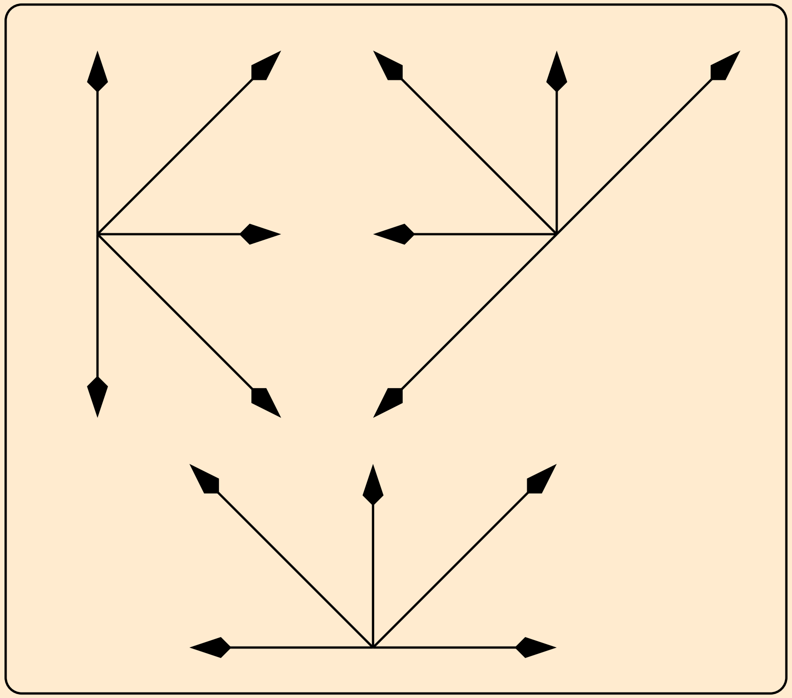
Suppose the step set group $\mathcal{Y}$ is not singular. Then $Q_{\mathcal{Y}}$ is D-finite if and only if the group of its step set is finite.

Singular Walks



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Possible route?? FIM: *The group is finite if there exists an abelian integral of the third kind, having logarithmic singularities at two points, and represented as the logarithm of an algebraic function* ϕ*, which belongs to ... .*

Reducing one conj. to the other

Lemma:

In the case of restricted amplitude walks in the quarter plane, suppose $\mathcal{Y}$ is not singular. Then $G(\mathcal{Y})$ is finite if one of the following

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Challenge: Generalize to non-restricted amplitude case!

Other related questions

- Combinatorial actions that preserve either algebraicity or D-finiteness. e.g. reversal?
- Walks of greater amplitude. How generalizable are these arguments?
- General (automatic) asymptotics.

fin.