Asymptotic enumeration of labelled planar graphs

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Definitions (I)

$G = (V, E)$ is planar if we can embed it in the plane.

Labelled graphs:
- $V = \{1, \ldots, n\}$
- $G, G'$ isomorphic iff $E = E'$
Definitions (II)

All the graphs in this talk will be *planar* and *labelled*.

A graph is *connected* if every two vertices are joined by a path.

A graph is *2-connected* if it is connected and by removing any vertex the graph is still connected.

\[ a_n \sim r(n) \text{ means that } \lim_{n \to \infty} \frac{a_n}{r(n)} = 1 \]
Labelled planar graphs

Let $g_n$ be the number of LPG on $n$ vertices.

<table>
<thead>
<tr>
<th>$n$</th>
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</table>
Labelled connected planar graphs

Let $c_n$ be the number of LCPG on $n$ vertices.

<table>
<thead>
<tr>
<th>$n$</th>
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Let $b_n$ be the number of L2CPG on $n$ vertices.

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<td>78702536</td>
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</table>
Asymptotic enumeration

It is known that

\[ c_n \sim \rho(n) \, \gamma^n \, n! \]

where

- \( \rho(n) \) is a subexponential function,
- \( \gamma \) is the growth constant:

\[
\lim_{n \to \infty} \frac{c_n \, (n - 1)!}{n! \, c_{n-1}} = \gamma \quad \lim_{n \to \infty} \left( \frac{c_n}{n!} \right)^{\frac{1}{n}} = \gamma
\]

Questions:
- What is the growth constant \( \gamma \)?
- What is the function \( \rho(n) \)?

We will later show that

\[ g_n \sim k \, c_n, \quad \text{where} \, 0 < k < 1 \]

The asymptotics of LCPG and LPG are almost the same.
Previous results

Bounds for $\gamma$

\begin{align*}
6 & \leq \gamma \leq 75.8 & \text{(Demise, Vasconcellos, Welsh; 1996)} \\
\gamma & \leq 37.3 & \text{(Osthus, Prömel, Taraz; 2002)} \\
\gamma & \leq 32.2 & \text{(Bonichon, Gavoille, Hanusse; 2002)} \\
26.1 & \leq \gamma & \text{(Bender, Gao, Wormald; 2002)} \\
\gamma & \leq 30.1 & \text{(B., G., H., Poulalhon, Schaeffer; 2004)} \\
27.2 & \leq \gamma & \text{(Prömel; 2003) [conference]}
\end{align*}

Asymptotic enumeration of labelled 2-connected planar graphs
(Bender, Gao, Wormald; 2002)

\[ b_n \sim \rho_b \, n^{-7/2} \, \gamma_b^n \, n! \]

with explicit expressions for $\rho_b \approx 0.37 \cdot 10^{-5}$ and $\gamma_b \approx 26.184$. 
Our results

We take the Work of [BGW] as a starting point.

We show that

\[ c_n \sim \rho_c \, n^{-7/2} \, \gamma^n \, n! \]

\[ g_n \sim k \rho_c \, n^{-7/2} \, \gamma^n \, n! \]

with explicit expressions for \( \rho_c, \ k \simeq 0.96 \) and \( \gamma \simeq 27.22687 \).

Other consequences of our work

We can give a *precise* asymptotic answer to:

- How many edges a random planar graph has?
- How many isolated vertices?
- How many connected components?
Structure of this talk
Generating Functions

- Bivariate generating functions.
- The variable $x$ counts the vertices, $y$ counts the edges.
- The GFs are exponential on $x$ and ordinary on $y$.

$$G(x, y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m \quad \text{(LPG)}$$

$$C(x, y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m \quad \text{(LCPG)}$$

$$B(x, y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m \quad \text{(L2CPG)}$$

- Univariate GFs are the corresponding bivariate GFs at $y = 1$.
- If not specified, derivatives are taken on the variable $x$.

$$C(x) := C(x, 1) \quad C'(x, y) := \frac{\partial C'}{\partial x}(x, y)$$
Three Steps

(Flajolet, Sedgewick; *Analytic Combinatorics*)

**Combinatorial Description**

Constructive, unambiguous description of the objects we are counting.

**Generating Functions**

**Singularity Analysis**
Three Steps

(Flajolet, Sedgewick; *Analytic Combinatorics*)

- Combinatorial Description
- Generating Functions
  
  Translate the previous description into equations of generating functions.
- Singularity Analysis
Three Steps

(Flajolet, Sedgewick; Analytic Combinatorics)

- Combinatorial Description
- Generating Functions
- Singularity Analysis

Study the singularities of the generating functions.
LPG, LCPG and L2CPG are related by several graph decomposition theorems.

Combinatorial operations on graphs translate into operations on the GFs.

- \( A \cup B \) \( a_n + b_n \) \( A(z) + B(z) \)
- \( A \ast B \) \( \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \) \( A(z)B(z) \)
- \( A^* \) \( n a_n \) \( zA'(z) \)
- \( A^o \) \( (n + 1) a_{n+1} \) \( A'(z) \)
- \( \text{powerset}(A) \) \( \sum_{i=0}^{\infty} A(z)^i / i! = \exp(A(z)) \)
- \( A(B) \) \( \sum_{i=0}^{\infty} a_i B(z)^i = A(B(z)) \)

We proceed to show the decompositions.
Equations relating the GFs

Labelled Planar Graphs

A LPG is a powerset of LCPG (conveniently relabelled).

That is,

\[ G = \text{powerset}(C), \]

\[ G(x, y) = \exp(C(x, y)) \]
Equations relating the GFs

Planar graphs

\[ C(x, y) \quad \text{connected} \]

\[ G(x, y) \quad \text{all} \]

The arrow means that the generating functions are related by an easy (explicit) equation.
Equations relating the GFs

Labelled Connected Planar Graphs.

Non-obvious example. How can we describe a LCPG in terms of L2CPGs?
Equations relating the GFs

- Point one vertex.
- Look at the 2-connected components it belongs.
- Replace vertexes of these components by connected graphs.
Equations relating the GFs

Another example.

A pointed LCPG is a powerset of pointed L2CPG where each vertex is replaced by a pointed LCPG (everything conveniently relabelled).
Equations relating the GFs

\[ C^o = \text{multiset}(\mathcal{B}^o(C^*)) \]

\[ C''(x, y) = \exp\left( B'(xC''(x), y) \right) \]
Equations relating the GFs

Vertex rooted  Non-rooted

\[ \frac{\partial B}{\partial x}(x, y) \]

2-connected

\[ \frac{\partial C}{\partial x}(x, y) \quad C(x, y) \]

connected

\[ G(x, y) \]

all

The arrow \[ \leftrightarrow \] means that the generating functions are related by a hard (implicit) equation.
Equations relating the GFs

Labelled 2-Connected Planar Graphs.

We need a characterization of 2-connected graphs in terms of 3-connected graphs.

A network is a graph with two distinguished vertices (poles) such that the graph obtained by joining the poles if they were not joined is 2-connected.

[Trakhtenbrot, 1958] Every network is either

- A series composition of networks.
- A parallel composition of networks.
- A 3-connected graph (with two poles) where every edge has been replaced by a network.
Equations relating the GFs

Let \( D(x, y) \) be the GF of labelled planar networks with unlabelled poles.

To obtain such a network we need to
- Choose a 2-connected planar graph
- Select an edge
- Unlabel the two vertices of the edge (the poles)
- Choose which pole is the first one
- Remove or do not remove the selected edge

Hence planar networks are related to L2CPG by:

\[
D(x, y) = (1 + y) \frac{2}{x^2} \frac{\partial B}{\partial y}(x, y)
\]
Equations relating the GFs

<table>
<thead>
<tr>
<th>Vertex rooted</th>
<th>Non-rooted</th>
<th>Edge rooted</th>
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<tbody>
<tr>
<td>$\frac{\partial B}{\partial x}(x, y)$</td>
<td>$\frac{\partial B}{\partial y}(x, y) \leftrightarrow D(x, y)$</td>
<td>2-connected</td>
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<tr>
<td>$\frac{\partial C}{\partial x}(x, y)$</td>
<td>$C(x, y)$</td>
<td>connected</td>
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<td>$G(x, y)$</td>
<td>all</td>
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Planar networks counted by $D(x, y)$ are closely related to edge-rooted 2-connected graphs.
Equations relating the GFs

Let $S(x, y)$ be the GF of labelled planar networks that are series compositions.

To obtain such a series network we need to

- Choose a network $N$
- Choose a non-series network $M$
- Join the second pole of $N$ with the first pole of $M$.
- Add a label to the joined poles.

Hence the GFs of all networks and series networks satisfy

$$S(x, y) = xD(x, y) \left( D(x, y) - S(x, y) \right)$$

$$S(x, y) = \frac{xD(x, y)^2}{1 + xD(x, y)}$$
Equations relating the GFs

Let $K(x, y)$ be the GF of networks with non-adjacent poles, and $U(x, y)$ of those that are non-parallel.

A network with non-adjacent poles is either
- non-parallel
- parallel, that is, a set of at least two non-parallel networks.

\[
K(x, y) = 1 + \underbrace{U(x, y)}_{\text{non-parallel}} + \left(\frac{U^2}{2!} + \frac{U^3}{3!} + \cdots\right) = \exp(U(x, y))
\]

On the other hand $D(x, y)$ and $K(x, y)$ are easy to relate, so

\[
(1 + y)K(x, y) - 1 = D(x, y) \quad K(x, y) = \frac{1 + D(x, y)}{1 + y}
\]

\[
U(x, y) = \log\left(\frac{1 + D(x, y)}{1 + y}\right)
\]
Equations relating the GFs

Let $M_3(x, y)$ be the GF of the rooted 3-connected planar maps.

To obtain a network with non-adjacent poles that is neither parallel nor series:
- Choose a rooted 3-connected planar map
- Forget which face is the root face
- Remove the root edge
- Unlabel the two distinguished vertices (poles)
- Substitute every edge by a network

So we have the equation

\[
\frac{M_3(x, D(x, y))}{2x^2 D(x, y)} = \log \left( \frac{1 + D(x, y)}{1 + y} \right) - \frac{x^2 D(x, y)}{1 + x D(x, y)}
\]

non-parallel, non-series
non-parallel
series
Equations relating the GFs

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Non-rooted</th>
<th>Edge</th>
<th>Rooted</th>
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\[
\frac{\partial B}{\partial x}(x, y) \quad \frac{\partial B}{\partial y}(x, y) \quad \frac{\partial C}{\partial x}(x, y) \quad C(x, y) \quad G(x, y)
\]

\[
M_3(x, y) \quad 3\text{-connected} \quad D(x, y) \quad 2\text{-connected} \quad \text{connected} \quad \text{all}
\]

\[M_3(x, y) \text{ and } D(x, y) \text{ are related by an implicit equation.}\]
Equations relating the GFs

Mullin, Schellenberg (1968) gave the GF for rooted 3-connected planar maps.

\[ u(x, y) = xy(1 + v(x, y))^2 \quad v(x, y) = y(1 + u(x, y))^2 \]

\[ M_3(x, y) = x^2y^2 \left( \frac{1}{1 + xy} + \frac{1}{1 + y} - 1 - \frac{(1 + u)^2(1 + v)^2}{(1 + u + v)^3} \right) \]

Fusy, Schaeffer (2003) provided a combinatorial explanation for this formula.
Equations relating the GFs

So these are all the equations:

\[
\begin{cases}
u(x, z) = xz(1 + v(x, z))^2 \\
v(x, z) = z(1 + u(x, z))^2 \\
M_3(x, z) = x^2z^2 \left( \frac{1}{1+xz} + \frac{1}{1+z} - 1 - \frac{(1+u)^2(1+v)^2}{(1+u+v)^3} \right)
\end{cases}
\]

\[
\begin{cases}
\frac{M_3(x, D)}{2x^2D} - \log \left( \frac{1+D}{1+y} \right) + \frac{xD^2}{1+xD} = 0 \\
\frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \left( \frac{1+D(x, y)}{1+y} \right)
\end{cases}
\]

\[
\begin{cases}
C'(x, y) = \exp \left( B' \left( xC'(x, y), y \right) \right) \\
G(x, y) = \exp(C'(x, y))
\end{cases}
\]
Equations relating the GFs

Vertex rooted  Non-rooted  Edge rooted

$B(x, y) \leftarrow \frac{\partial B}{\partial x}(x, y)$  $\frac{\partial B}{\partial y}(x, y) \leftrightarrow D(x, y)$

$M_3(x, y)$  3-connected

$C(x, y) \leftarrow \frac{\partial C}{\partial x}(x, y)$

$G(x, y)$  all

The arrow means that the generating functions are related by a partial derivation.
**Equations relating the GFs**

<table>
<thead>
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<th>Vertex rooted</th>
<th>Non-rooted</th>
<th>Edge rooted</th>
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We follow the path to obtain $G(x, y)$. 

- $\frac{\partial B}{\partial x}(x, y) \xrightarrow{B(x, y)} \frac{\partial B}{\partial y}(x, y) \xrightarrow{D(x, y)} M_3(x, y)$ (3-connected)
- $\frac{\partial C}{\partial x}(x, y) \xrightarrow{C(x, y)} \frac{\partial C}{\partial x}(x, y)$ (2-connected)
- Connected
- All
2-connected graphs [BGW]

[BGW] proved that:
The singularities of $D(x, y)$ are given by those of $M_3(x, z)$.

$$\frac{M_3(x, D)}{2x^2 D} - \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} = 0$$

A parametrization $x_0(t), y_0(t)$ for the singularities of $D(x, y)$.

The expansion of $D(x, y_0)$ at a singular point $(x_0(t), y_0(t))$:

$$D(x, y_0) \sim D_0(t) + D_2(t)(1 - \frac{x}{x_0}) + D_3(t)(1 - \frac{x}{x_0})^{3/2}$$

If $t$ is such that $y_0(t) = 1$, then

$$[x^n] D(x, 1) = \frac{D_3(t)}{\Gamma(-3/2)} n^{-5/2} x_0(t)^{-n}$$
Because of
\[
\frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} \right)
\]
they can obtain the asymptotics for
\[
[x^n] \frac{\partial B}{\partial y}(x, 1) = \sum_m m b_{n,m}
\]

They also prove that

- The expected number of edges is \( \mu n \), with \( \mu \approx 2.26 \).
- The variance is \( \sigma^2 n \), with \( \sigma^2 \approx 0.4 \).

So almost all \( n \)-vertex 2-connected planar graphs (for large \( n \)) have \( \mu n \) edges,
\[
b_n \sim \frac{\sum_m m b_{n,m}}{\mu n} \sim \frac{x_0^2 D_3}{4\mu \Gamma(-3/2)} n^{-7/2} x_0^{-n}
\]
2-connected graphs [BGW]

Summary:
- We know about \( \frac{\partial B}{\partial y}(x, y) \).
- We know about the coefficients of \( B \) and \( \frac{\partial B}{\partial y} \).
- But \( B(x, y) \) is hidden behind an integral.

\[
B(x, y) = \frac{x^2}{2} \int_0^y \frac{1 + D(x, s)}{1 + s} ds
\]

Just understanding \( D(x, y) \) at the singularity point is not enough for evaluating \( B(x, y) \).
Estimating $\gamma$

The growth constant $\gamma$ is the inverse of the radius of converge of $C(x)$.

Question:
Where does $C(x)$ stop being defined?
Lower bound: \( \gamma > 27.22685 \)

We use the equation

\[
C'(x) = \exp \left( \frac{\partial B}{\partial x}(xC'(x), 1) \right).
\]

Define

\[
\tilde{C}(x) := xC'(x), \quad \Psi(u) := u \exp \left( -\frac{\partial B}{\partial x}(u, 1) \right);
\]

then \( \tilde{c}_n = n \ c_n \), and

\[
\tilde{C}(x) = x \exp \left( \frac{\partial B}{\partial x}(\tilde{C}(x), 1) \right)
\]

\[
\tilde{C}(x) \exp \left( -\frac{\partial B}{\partial x}(\tilde{C}(x), 1) \right) = x
\]

\[
\boxed{\Psi(\tilde{C}(x)) = x}
\]

That is, \( \Psi(u) \) is the inverse of \( \tilde{C}(x) \).
Lower bound: $\gamma > 27.22685$

$$\Psi(u) = u \exp \left(-\frac{\partial B}{\partial x}(u, 1)\right)$$

$\Psi(u)$ stops being defined at the radius of convergence of $B(u, 1): \gamma_b^{-1} \approx 0.0383$. Hence $\tilde{C}(x)$ is bounded by $\gamma_b^{-1}$. 

![Graph showing the lower bound and the function $\Psi(u)$]
Lower bound: $\gamma > 27.22685$

Consider the truncations of $\tilde{C}(x)$ and $B(u, 1)$. That is, if $\tilde{C}(x) = \sum_{i=0}^{\infty} \tilde{c}_i x^i$, then let

$$\tilde{C}_n(x) := \sum_{i=0}^{n} \tilde{c}_i x^i.$$  

$\tilde{C}_n(x)$ is a lower bound of $\tilde{C}(x)$.

Similarly, if $B(u, 1) = \sum_{i=0}^{\infty} b_i u^i$, then let

$$B_n(u) := \sum_{i=0}^{n} b_i u^i; \quad \Psi_n(u) := u \exp \left( - \frac{\partial B_n}{\partial x}(u) \right)$$

Since $B_n(u)$ is a lower bound of $B(u, 1)$,

$\Psi_n(u)$ is an upper bound of $\Psi(u)$. 

- p. 40/63
We can use $\tilde{C}_n(x)$ or $\Psi_n(u)$ to bound the region where $\tilde{C}(x)$ can be.

\[ \tilde{C}_3(x) = x + x^2 + 2x^3 \]

\[ \gamma_b^{-1} = 26.1^{-1} \]

$\tilde{C}(x)$ can not be here
Lower bound: $\gamma > 27.22685$

This gives us
- a lower bound on the region where $\tilde{C}(x)$ is defined,
- an upper bound on the radius of convergence $\gamma^{-1}$,
- a lower bound on the growth constant $\gamma$.

By considering $\tilde{C}_n(x)$ and $\Psi_n(u)$ for greater $n$ we can improve the previous bounds.

| $n$ | $\max\{x|\tilde{C}_n(x) < \gamma^{-1}_b\}^{-1}$ (bound from $\tilde{C}_n(x)$) | $\max\{\Psi_n(u)|u \in (0, \gamma^{-1}_b)\}^{-1}$ (bound from $\Psi_n(u)$) |
|-----|------------------------------------------------------------------|--------------------------------------------------|
| 5   | 27.226147                                                        | 27.226399                                        |
| 10  | 27.226779                                                        | 27.226790                                        |
| 15  | 27.226833                                                        | 27.226837                                        |
| 20  | 27.226851                                                        | 27.226853                                        |
Upper bound: \( \gamma < 27.22688 \)

To prove \( \gamma < 27.22688 \) we need the opposite:
- An upper bound on the region where \( \bar{C}(x) \) is defined,
- a lower bound on the radius of convergence \( \gamma^{-1} \),
- an upper bound on the growth constant \( \gamma \).

\[ c < \Psi(u_0) \quad \Rightarrow \quad \gamma^{-1} > c \quad \Rightarrow \quad \gamma < c^{-1} \]
Upper bound: $\gamma < 27.22688$

We want $u_0$ to be as close to $\gamma_b^{-1}$ as possible.

\[ c < \Psi(u_0) \quad \Rightarrow \quad \gamma^{-1} > c \quad \Rightarrow \quad \gamma < c^{-1} \]
Upper bound: \( \gamma < 27.22688 \)

Truncations \( \bar{C}_n \) and \( \Psi_n \) bound in the wrong direction.

We try to evaluate \( \Psi(u_0) \) using the equations:

\[
\Psi(u) = u \exp \left( -\frac{\partial B}{\partial x}(u, 1) \right)
\]

\[
\frac{\partial B}{\partial x}(u, 1) = \frac{\partial}{\partial x} \int_0^1 \frac{\partial B}{\partial y}(u, t) dt
\]

\[
\frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} - 1 \right)
\]

\[
\frac{M_3(x, D)}{2x^2D} - \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} = 0
\]

\[
M_3(x, y) = x^2y^2 \left( \frac{1}{1 + xy} + \frac{1}{1 + y} - 1 - \frac{(1 + u)^2(1 + v)^2}{(1 + u + v)^3} \right)
\]

\[
u = xy(1 + v)^2 \quad v = y(1 + u)^2
\]
Upper bound: $\gamma < 27.22688$

Things to take care of:

\[ \frac{\partial B}{\partial x}(u, 1) = \frac{\partial}{\partial x} \int_0^1 \frac{\partial B}{\partial y}(u, t)dt \]

- We have to bound $\frac{\partial B}{\partial x}(u_0, 1)$.
- We use that $\frac{\partial^2 B}{\partial x^2}(u, y)$ is positive.

\[ B'(u_0) \leq \frac{B(u_0 + \epsilon) - B(u_0)}{\epsilon} \]

- We need to evaluate $\int \frac{\partial B}{\partial y}$ at $u_0$ and at $u_0 + \epsilon$. 
Upper bound: \( \gamma < 27.22688 \)

Things to take care of:

\[
\frac{\partial B}{\partial x}(u, 1) = \frac{\partial}{\partial x} \int_{0}^{1} \frac{\partial B}{\partial y}(u, t) \, dt
\]

- Numerical methods to estimate the integral.
- Errors in Newton-Cotes integration methods (Simpson rule, midpoint rule, etc.) are evaluations of a derivative of the integrand.
- Integrand \( \frac{\partial B}{\partial y} \) has positive derivatives.
- We know the sign of the derivative, so we know the sign of the error.
- We are obtaining bounds, not just approximations.
Upper bound: $\gamma < 27.22688$

Things to take care of:

\[
\frac{M_3(x, D)}{2x^2 D} - \log \left( \frac{1 + D}{1 + y} \right) + \frac{x D^2}{1 + x D} = 0
\]

- $D(x, y)$ is singular in $(\gamma_b^{-1}, 1)$.
- We need to be carefull when solving for $D(x, y)$ if $(x, y)$ is close to $(\gamma_b^{-1}, 1)$.
- $D(x, y)$ does not go to infinity at the singularity.
Upper bound: $\gamma < 27.22688$

- Maple 9
- 25 precision digits.
- $u_0 = 0.038191096$ ($\gamma_b^{-1} - u_0 \simeq 10^{-8}$).
- $\epsilon = 1.6 \cdot 10^{-9}$.
- Integration methods: repeated midpoint rule, trapezoid rule.
- 30000 equispaced evaluations.

Computational results:

$$\gamma^{-1} > \Psi(u_0) > 0.036728410432$$

$$\gamma < 27.2268797$$
Singularity type

**Transfer theorems** (Flajotet, Odlyzko): The singularity type of $C(x)$ determines the subexponential behaviour of its coefficients $c_n$.

Question: *How does $C(x)$ stop being defined?*

Since its inverse $\Psi(u)$ is defined in $(0, \gamma_b^{-1})$, there are two different possibilities:

1. $\exists u_0 \in (0, \gamma_b^{-1}) \quad \Psi'(u_0) = 0$
2. $\forall u \in (0, \gamma_b^{-1}) \quad \Psi'(u) > 0$

![Graphs showing two cases](image)

*(case A)*  
*(case B)*
(case A): $\tilde{C}$ has a square root-type singularity ($Z^{1/2}$):

\[ \Psi(u) = \Psi(u_0) + A(u - u_0)^2 + \mathcal{O} \left( (u - u_0)^3 \right) \]

\[ \tilde{C}(x) = u_0 - B \sqrt{1 - \frac{x}{\Psi(u_0)}} + \mathcal{O} \left( (x - \Psi(u_0))^2 \right) \]

By the Transfer theorems,

\[ \tilde{c}_n \sim \rho \, n^{-3/2} \, \gamma^n n! \quad \text{and} \quad c_n \sim \rho \, n^{-5/2} \, \gamma^n n! \]
(case B): \( \tilde{C} \) has the same singularity type than \( \Psi \).

The singularity type of \( B \) is \( Z^{5/2} \) (BGW; 2002). Thus the singularity type of \( B' \) is \( Z^{3/2} \).

\( \Psi(u) = u \exp(-B'(u, 1)) \) has the same singularity type that \( B' \).

\[
\bar{c}_n \sim \rho \ n^{-5/2} \ \gamma^n n! \\
c_n \sim \rho \ n^{-7/2} \ \gamma^n n!
\]
How to obtain exact results

\[ \exists u_0 \in (0, \gamma_b^{-1}) \quad \Psi'(u_0) = 0? \]

\[ \Psi'(u) = \exp(-B'(u))(1 - uB''(u)) \]

If \( 1 - R_bB''(R_b) > 0 \) it follows that \( \Psi'(u) > 0 \) for all \( u \). So:

- Evaluate \( B''(R_b, y) \) to obtain the singularity type of \( C \).
- If \( 1 - R_bB''(R_b) > 0 \), then
  - the singularity type would be \( Z^{5/2} \),
  - and \( \gamma^{-1} \) would be \( R_b \exp(-B'(R_b, y)) \).

Problem: We know about \( \partial B/\partial y \), but not about \( B \).

\[ \frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y} \]

\[ B(x, y) = \frac{x^2}{2} \int_0^y \frac{1 + D(x, s)}{1 + s} ds \]
How to obtain exact results

Three lemmas about integrating functions through its inverses.

Let $f(z)$ be an invertible function such that $f(0) = 0$, and let $f^{-1}(u)$ be its inverse. Then

\[ \int_{0}^{Z} f(z) dz = Z f(Z) - \int_{0}^{f(Z)} f^{-1}(u) du \]

\[ \int_{0}^{Z} f(z) a'(z) dz = a(Z) f(Z) - \int_{0}^{f(Z)} a(f^{-1}(u)) du \]

\[ \int_{0}^{Z} \Phi(z, f(z)) dz = \int_{0}^{f(Z)} \Phi(f^{-1}(u), u) \frac{\partial f^{-1}(u)}{\partial u} du \]
How to obtain exact results

\[
\frac{M_3(x, D)}{2x^2D} - \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} = 0
\]

The inverse of \(D(x, y)\) with respect to \(y\) is explicit:

\[
y = \exp \left( -\frac{M_3(x, D)}{2x^2D} + \log (1 + D) - \frac{xD^2}{1 + xD} \right) - 1
\]

So we can integrate \(D(x, y)\) through its inverse:

\[
\int_0^y \frac{1 + D(x, s)}{1 + s} ds = D(x, y) \log(1+y) - \int_0^{D(x,y)} \log(1+D^{-1}(x, t)) dt =
\]

\[
D(x, y) \log(1+y) + \int_0^{D(x,y)} \frac{M_3(x, t)}{2x^2t} - \log(1 + t) + \frac{xt^2}{1 + xt} dt =
\]

\[
\int_0^{D(x,y)} \frac{M_3(x, t)}{2x^2t} - \log(1 + t) + \log(1+y) + \frac{xt^2}{1 + xt} dt
\]
How to obtain exact results

\[
\int_0^{D(x,y)} \frac{M_3(x, t)}{2x^2t} - \log(1 + t) + \log(1 + y) + \frac{xt^2}{1 + xt} dt
\]

But \( \int \frac{M_3(x, t)}{t} dt \) is not easy, because \( M_3 \) depends on algebraic functions \( u \) and \( v \):

\[
\begin{align*}
    u(x, z) &= xz(1 + v(x, z))^2 \\
    v(x, z) &= z(1 + u(x, z))^2 \\
    u(x, z) &= xz \left( 1 + z(1 + u(x, z))^2 \right)^2
\end{align*}
\]

To obtain the inverses of \( u(x, z) \) with respect to \( z \) we need to solve a cubic equation.
How to obtain exact results

But if we consider $w(x, z) = z(1 + u(x, z))$ it will be easier, because

$$w(x, z) = z(1 + u(x, z)) = z(1 + xz(1 + z(1 + u(x, z))^2)^2) =$$

$$z + xz^2(1 + \frac{w(x, z)^2}{z})^2 = z + x(z + w(x, z)^2)^2$$

so to obtain inverse of $w(x, z)$ we only need to solve a quadratic equation.

- We express $M_3(x, z)$ in terms of $w(x, z)$.
- We integrate through the inverse of $w$.
- We replace every resulting $w(x, z)$ by $z(1 + u(x, z))$.
- So we obtain a long but explicit expression for $B(x, y)$:

$$B(x, y) = \Phi(x, y, D(x, y), u(x, D(x, y)))$$

From here we can obtain $B'(x, y)$, $B''(x, y)$, etc.
How to obtain exact results

One last application of the integrating-through-inverse lemmas:

\[ \tilde{C}(x, y) = xC'(x, y) \quad \tilde{C} \text{ and } \Psi \text{ are inverses} \]

\[ C(x, y) = \int_0^x C'(s, y)ds = \int_0^x \frac{\tilde{C}(s, x)}{s}ds = \]

\[ \log(x)\tilde{C}(x, y) - \int_0^{\tilde{C}(x, y)} \log(\Psi(t, y))dt = \]

\[ \log(x)\tilde{C}(x, y) - \int_0^{\tilde{C}(x, y)} \log(t \exp(-B'(t, y)))dt = \]

\[ \log(x)\tilde{C}(x, y) - \int_0^{\tilde{C}(x, y)} \log(t)dt + \int_0^{\tilde{C}(x, y)} B'(t, y)dt = \]

\[ \log(x)\tilde{C}(x, y) - \tilde{C}(x, y)\log(\tilde{C}(x, y)) + \tilde{C}(x, y)B(\tilde{C}(x, y), y) \]
Applications

How many isolated vertices has a random planar graph?

Consider the following GF, where $t$ counts the number of isolated vertices

$$G(x, t) = \exp (C(x) - x + tx)$$

The probability that a $n$-vertex graph has $k$ isolated vertices is

$$\frac{[x^n][t^k]G(x, t)}{[x^n]G(x)} = \frac{[x^n] \exp (C(x) - x) [t^k] \exp(tx)}{[x^n] \exp (C(x))} \sim$$

$$\frac{R^k [x^n] \exp (C(x))}{e^{Rk} [x^n] \exp (C(x))} = \frac{R^k}{e^{Rk}}$$

Answer: Poisson distribution of parameter $R = \gamma^{-1}$. 
Applications

We generalize the previous result.

Let $\mathcal{A}$ be a *not too large* family of planar graphs, and $A(x)$ its GF.

How many $\mathcal{A}$-components has a random planar graph?

\[
\frac{[x^n][t^k]G(x, t)}{[x^n]G(x, 1)} = \frac{[x^n]\exp(C(x) - A(x))[t^k]\exp(tA(x))}{[x^n]\exp(C(x))} \sim
\]

\[
\frac{A(R)^k[x^n]\exp(C(x))}{e^{A(R)k}[x^n]\exp(C(x))} = \frac{A(R)^k}{e^{A(R)k}}
\]

Answer: Poisson distribution of parameter $A(R)$. 
What is the probability of being connected?

Consider the GF \( G(x, t) = \exp(tC(x)) \), where \( t \) counts the number of connected components.

The probability of having \( k \) components is given by

\[
\frac{[x^n][t^k]G(x, t)}{[x^n]G(x)} = \frac{[x^n][t^k] \exp(tC(x))}{[x^n] \exp(G(x))} \sim \\
\frac{[x^n]C(x)^k}{\exp(C(R))k![x^n]C(x)}
\]

If we set \( k = 1 \) we obtain the probability of being connected:

\[
\frac{[x^n]C(x)}{\exp(C'(R))[x^n]C(x)} = \exp(-C(R))
\]
Applications

How many components has a random planar graph?

\[
\frac{[x^n] [t^k] G(x, t)}{[x^n] G(x)} \sim \frac{[x^n] C(x)^k}{\exp(C(R)) k! [x^n] C(x)}
\]

The important terms of the expansion of \( C(x) \) close to the singularity are \( C(R) + b(1 - \frac{x}{R})^{5/2} \), so

\[
[x^n] C(x) \sim b [x^n] (1 - \frac{x}{R})^{5/2}
\]

\[
[x^n] C(x)^k \sim k C(R)^{k-1} b [x^n] (1 - \frac{x}{R})^{5/2}
\]

\[
\frac{[x^n] C(x)^k}{\exp(C(R)) k! [x^n] C(x)} \sim \frac{C(R)^{k-1}}{\exp(C(R)) (k - 1)!}
\]

Answer: (1+\#components) follows a Poisson distribution of parameter \( C(R) \).
Applications

How many edges has a random planar graph?

- $C(x, y) \sim (1 - \frac{x}{R(y)})^{7/2}$ around the singularity in a neighbourhood of $y = 1$.
- We apply the quasi powers theorem.
- It follows that
  - The distribution of the number of edges follows a Normal law.
  - The expected number of edges is $\mu n$.
  - The variation is $\sigma^2 n$.
  - Concentration: almost all planar graphs has $\mu n$ edges.
- $\mu$ is given by $R'(1)/R(1)$, and can be computed using that

$$R'(y) = \frac{\partial}{\partial y} \left( R_b(y) \exp(B'(R_b(y), y)) \right) .$$