# Asymptotic enumeration of labelled planar graphs

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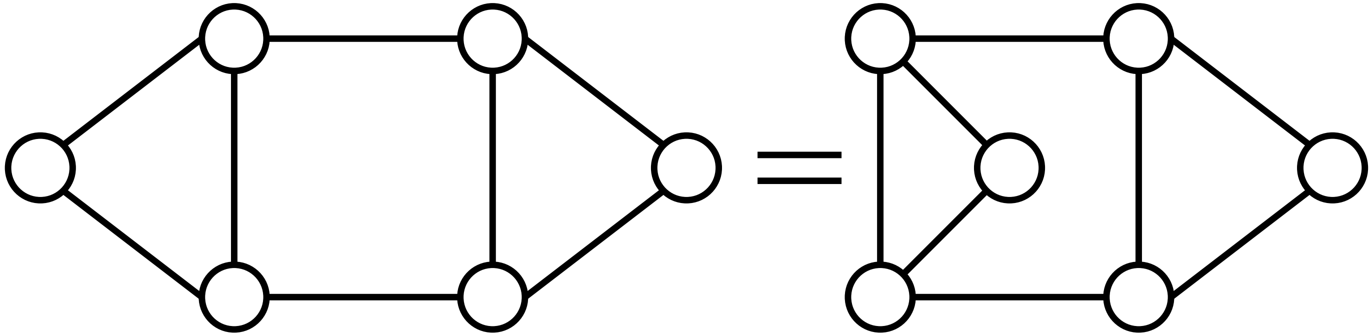
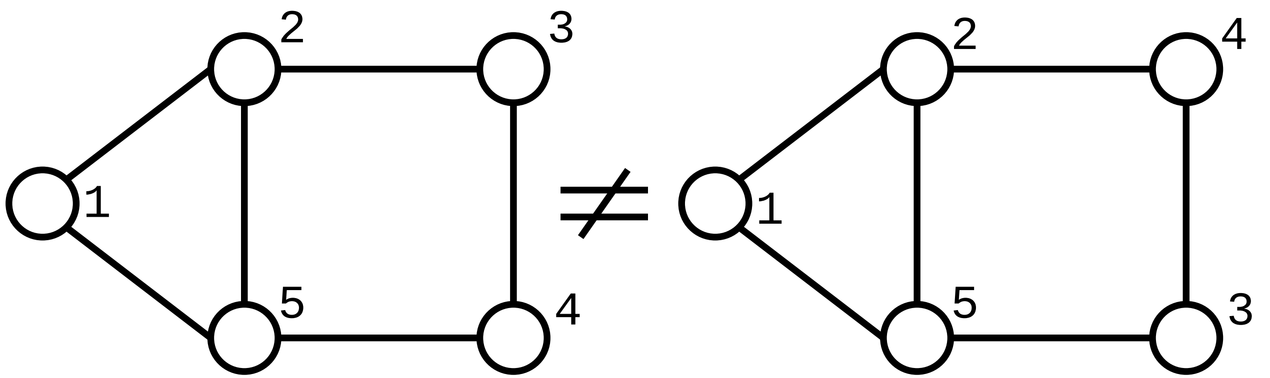
# Definitions (I)

$G = (V, E)$ is *planar* if we can embed it in the plane.



*Labelled* graphs:

- $V = \{1, \ldots, n\}$
- $G, G'$ isomorphic iff $E = E'$



# Definitions (II)

All the graphs in this talk will be *planar* and *labelled*.

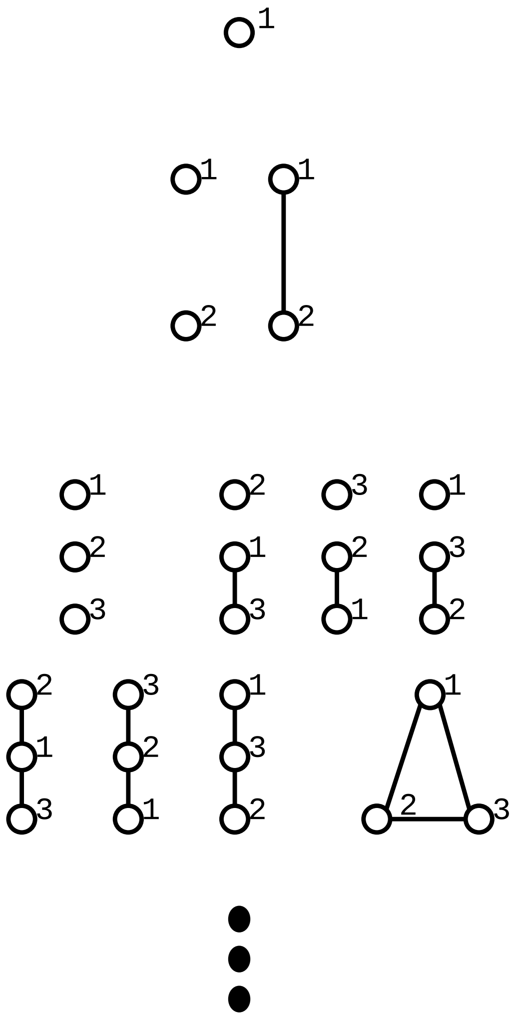
A graph is *connected* if every two vertices are joined by a path.

A graph is *2-connected* if it is connected and by removing any vertex the graph is still connected.

$a_n \sim r(n)$ means that $\lim_{n\to\infty} \frac{a_n}{r(n)} = 1$

# Labelled planar graphs

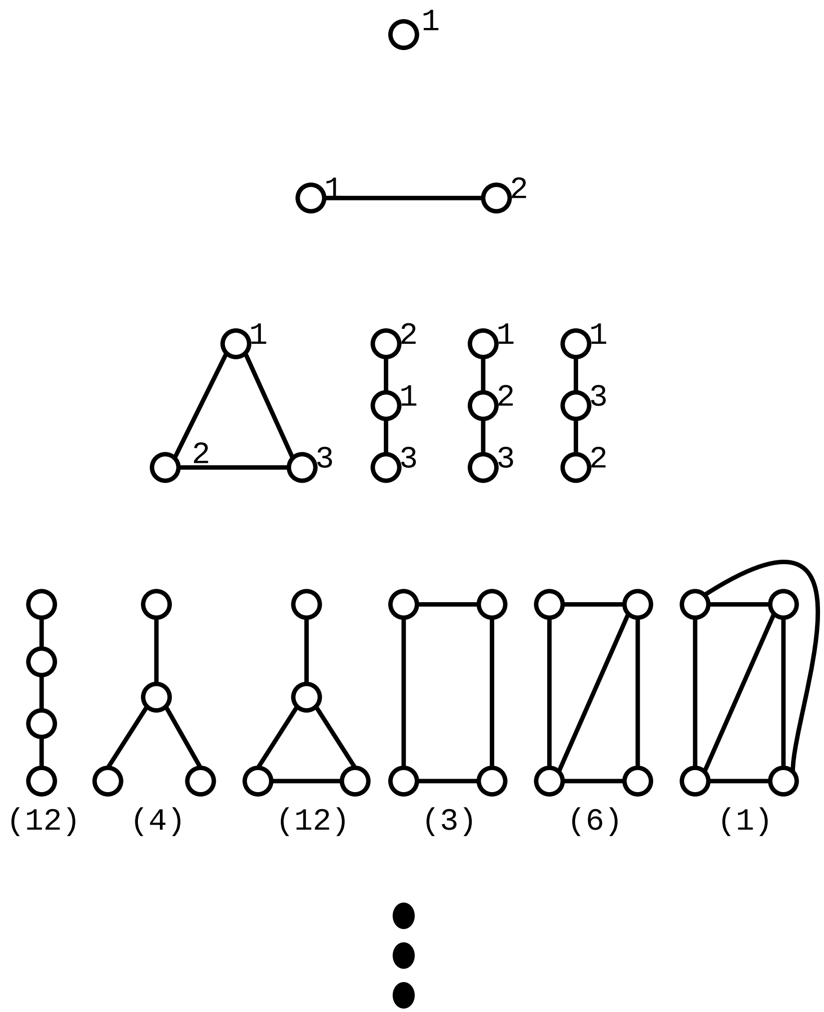
Let $g_n$ be the number of LPG on $n$ vertices.



| $n$ | $g_n$ |
|---|---:|
| 1 | 1 |
| 2 | 2 |
| 3 | 8 |
| 4 | 64 |
| 5 | 1023 |
| 6 | 32071 |
| 7 | 1823707 |
| 8 | 163947848 |
| $\vdots$ | $\vdots$ |

# Labelled connected planar graphs

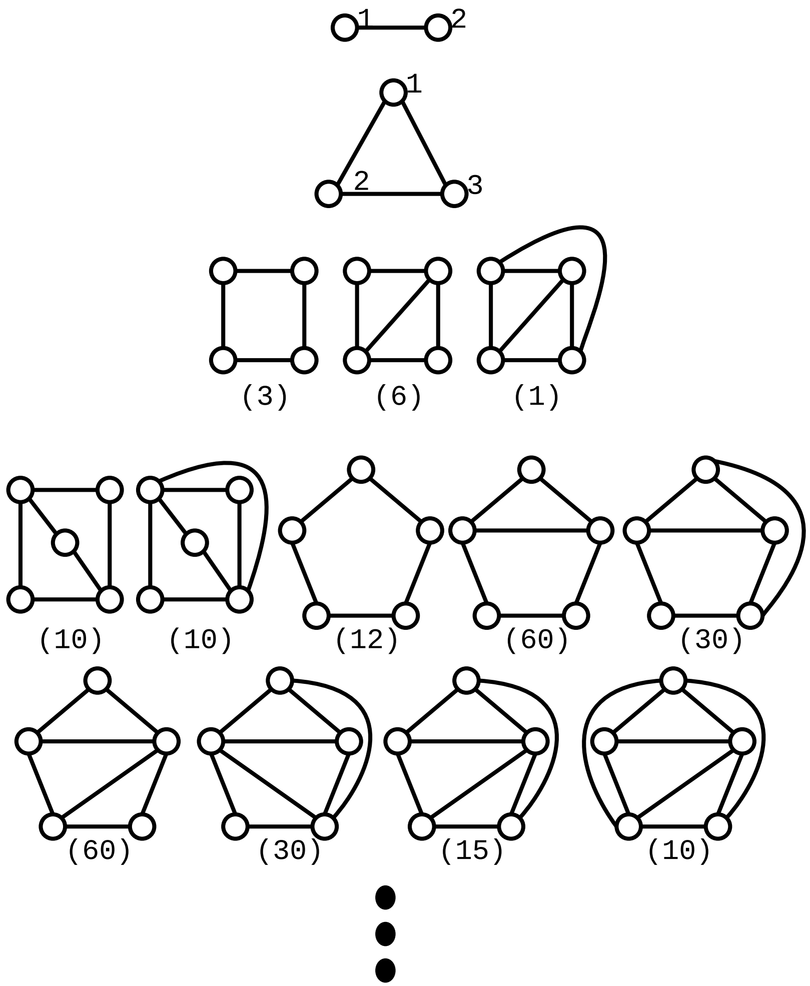
Let $c_n$ be the number of LCPG on $n$ vertices.




| $n$ | $c_n$ |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 4 |
| 4 | 38 |
| 5 | 727 |
| 6 | 26013 |
| 7 | 1597690 |
| 8 | 149248656 |
| $\vdots$ | $\vdots$ |

# Labelled 2-connected planar graphs

Let $b_n$ be the number of L2CPG on $n$ vertices.




| $n$ | $b_n$ |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | 1 |
| 4 | 10 |
| 5 | 237 |
| 6 | 10707 |
| 7 | 774924 |
| 8 | 78702536 |
| $\vdots$ | $\vdots$ |

# Asymptotic enumeration

It is known that

$$c_n \sim \rho(n)\ \gamma^n\ n!$$

where

- $\rho(n)$ is a subexponential function,
- $\gamma$ is the *growth constant*:

$$\lim_{n\to\infty} \frac{c_n}{n!}\frac{(n-1)!}{c_{n-1}} = \gamma \qquad \lim_{n\to\infty} \left(\frac{c_n}{n!}\right)^{\frac{1}{n}} = \gamma$$

Questions: $\boxed{\begin{array}{l}\text{What is the growth constant } \gamma\text{?} \\ \text{What is the function } \rho(n)\text{?}\end{array}}$

We will later show that

$$g_n \sim k\ c_n, \qquad \text{where} 0 < k < 1$$

The asymptotics of LCPG and LPG are almost the same.

# Previous results

Bounds for $\gamma$

| | |
|---|---|
| $6 \leq \gamma \leq 75.8$ | (Demise, Vasconcellos, Welsh; 1996) |
| $\gamma \leq 37.3$ | (Osthus, Prömel, Taraz; 2002) |
| $\gamma \leq 32.2$ | (Bonichon, Gavoille, Hanusse; 2002) |
| $26.1 \leq \gamma$ | (Bender, Gao, Wormald; 2002) |
| $\gamma \leq 30.1$ | (B., G., H., Poulalhon, Schaeffer; 2004) |
| $27.2 \leq \gamma$ | (Prömel; 2003) [conference] |

Asymptotic enumeration of labelled *2-connected* planar graphs (Bender, Gao, Wormald; 2002)

$$b_n \sim \rho_b \, n^{-7/2} \, {\gamma_b}^n \, n!$$

with explicit expressions for $\rho_b \simeq 0.37 \cdot 10^{-5}$ and $\gamma_b \simeq 26.184$ .

# Our results

We take the Work of [BGW] as a starting point.

We show that

$$c_n \sim \rho_c \, n^{-7/2} \, \gamma^n \, n!$$

$$g_n \sim k\rho_c \, n^{-7/2} \, \gamma^n \, n!$$

with explicit expressions for $\rho_c$, $k \simeq 0.96$ and $\gamma \simeq 27.22687$ .

**Other consequences of our work**

We can give a *precise* asymptotic answer to:

- How many edges a random planar graph has?
- How many isolated vertices?
- How many connected components?

# Structure of this talk

- **Generating functions and equations of LPG**
  We introduce the generating functions for our families of graphs and the equations relating them.

- **Brief description of the work of BGW**
  Solving the asymptotic enumeration of 2-connected labelled planar graphs.

- **How to obtain *a good estimation* for $\gamma$**
  Easy way to improve the best known bounds for $\gamma$.

- **How to obtain *an exact expression* for $\gamma$**
  Solving the problem.

- **Applications**
  Distributions for the number of edges, components and isolated vertices.

# Generating Functions

- Bivariate generating functions.
- The variable $x$ counts the vertices, $y$ counts the edges.
- The GFs are exponential on $x$ and ordinary on $y$.

$$G(x,y) = \sum_{n,m} g_{n,m} \frac{x^n}{n!} y^m \quad \text{(LPG)}$$

$$C(x,y) = \sum_{n,m} c_{n,m} \frac{x^n}{n!} y^m \quad \text{(LCPG)}$$

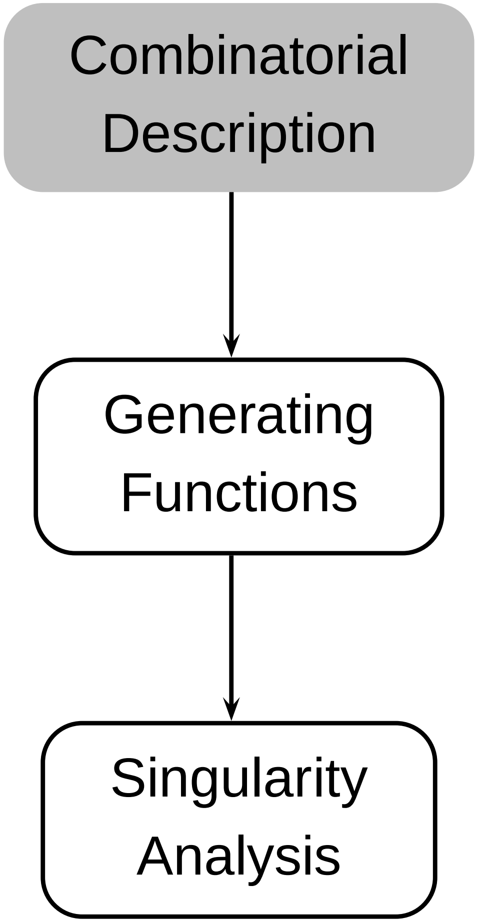
$$B(x,y) = \sum_{n,m} b_{n,m} \frac{x^n}{n!} y^m \quad \text{(L2CPG)}$$

- Univariate GFs are the corresponding bivariate GFs at $y = 1$.
- If not specified, derivatives are taken on the variable $x$.

$$C(x) := C(x,1) \qquad C'(x,y) := \frac{\partial C}{\partial x}(x,y)$$

# Three Steps

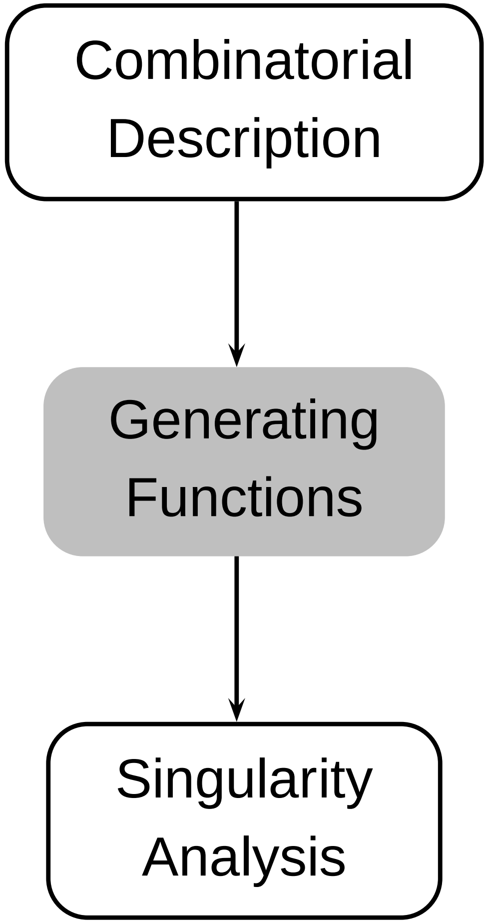
(Flajolet, Sedgewick; *Analytic Combinatorics*)




Constructive, unambiguous description of the objects we are counting.

# Three Steps

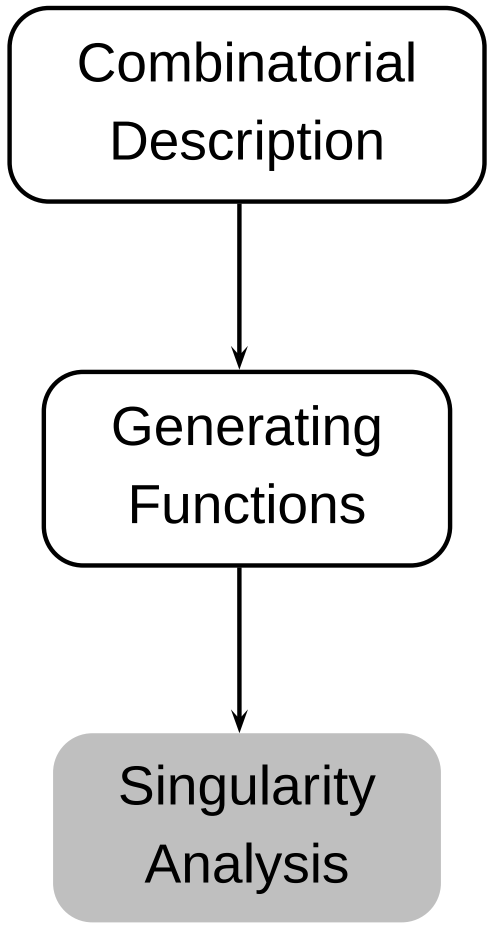
(Flajolet, Sedgewick; *Analytic Combinatorics*)




Translate the previous description into equations of generating functions.

# Three Steps

(Flajolet, Sedgewick; *Analytic Combinatorics*)




Study the singularities of the generating functions.

# Generating Functions

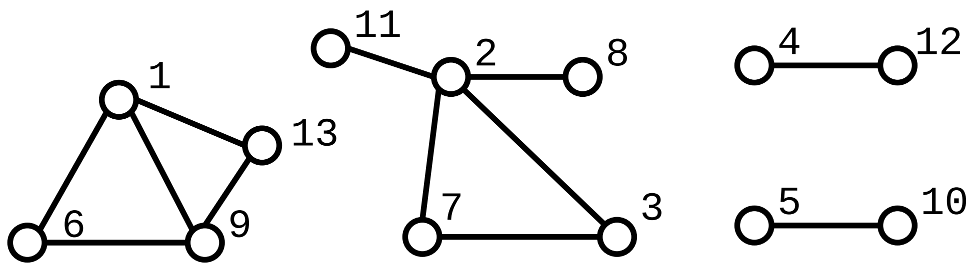
- LPG, LCPG and L2CPG are related by several graph decomposition theorems.
- Combinatorial operations on graphs translate into operations on the GFs.

| | | |
|---|---|---|
| $\mathcal{A} \cup \mathcal{B}$ | $a_n + b_n$ | $A(z) + B(z)$ |
| $\mathcal{A} * \mathcal{B}$ | $\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$ | $A(z)B(z)$ |
| $\mathcal{A}^*$ | $n\ a_n$ | $zA'(z)$ |
| $\mathcal{A}^o$ | $(n+1)\ a_{n+1}$ | $A'(z)$ |
| powerset($\mathcal{A}$) | | $\sum_{i=0}^{\infty} A(z)^i/i! = \exp(A(z))$ |
| $\mathcal{A}(\mathcal{B})$ | | $\sum_{i=0}^{\infty} a_i B(z)^i = A(B(z))$ |

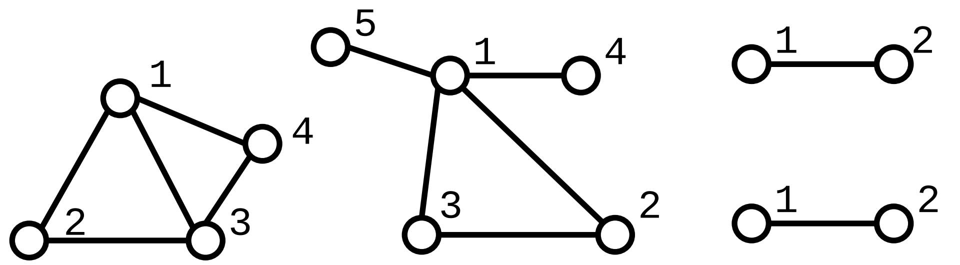
- We proceed to show the decompositions.

# Equations relating the GFs

Labelled Planar Graphs




A LPG is a *powerset* of LCPG (conveniently relabelled).




That is,

$$\mathcal{G} = \text{powerset}(\mathcal{C}),$$

$$\boxed{G(x, y) = \exp(C(x, y))}$$

# Equations relating the GFs

Planar graphs

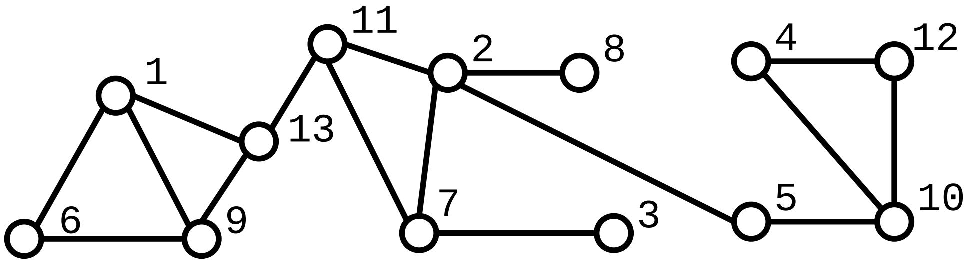
$$\begin{array}{ll} C(x,y) & \text{connected} \\ \big\updownarrow & \\ G(x,y) & \text{all} \end{array}$$

The arrow $\longleftrightarrow$ means that the generating functions are related by an easy (explicit) equation.

# Equations relating the GFs

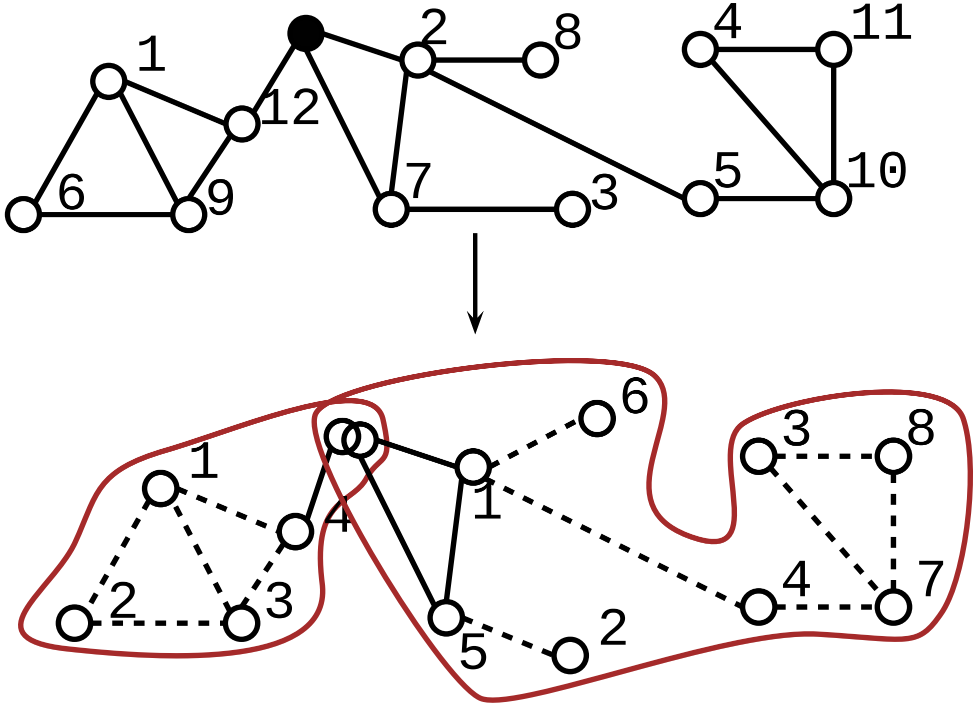
Labelled Connected Planar Graphs.

Non-obvious example. How can we describe a LCPG in terms of L2CPGs?



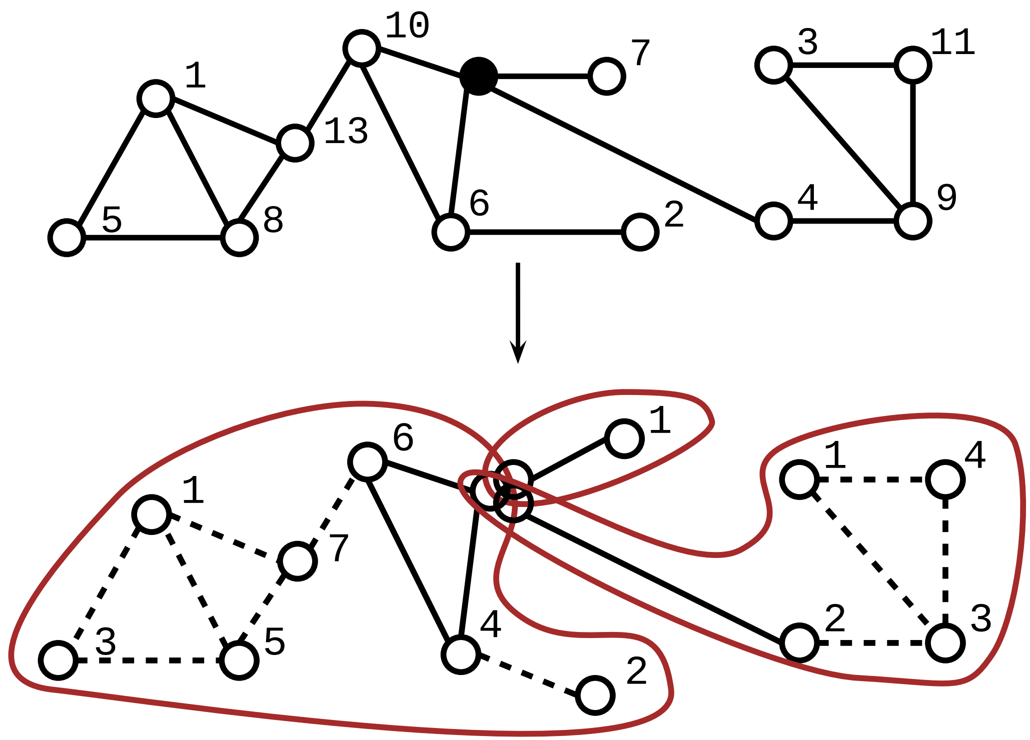
# Equations relating the GFs

- Point one vertex.
- Look at the 2-connected components it belongs.
- Replace vertexs of these components by connected graphs.



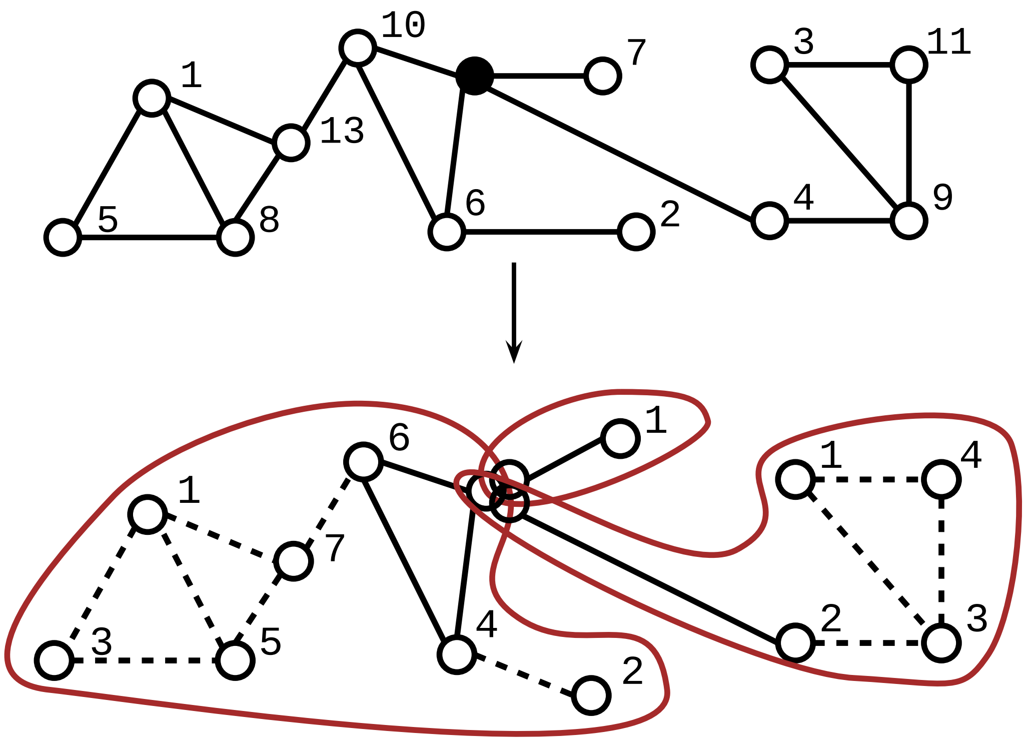
# Equations relating the GFs

Another example.




A *pointed* LCPG is a *powerset* of *pointed* L2CPG where each vertex is *replaced* by a *pointed* LCPG (everything conveniently relabelled).

# Equations relating the GFs




$$\mathcal{C}^o = \text{multiset}(\mathcal{B}^o(\mathcal{C}^*))$$

$$C'(x,y) = \exp\left(B'(xC'(x),y)\right)$$

# Equations relating the GFs

| Vertex rooted | Non-rooted | |
|---|---|---|
| $\frac{\partial B}{\partial x}(x,y)$ | | 2-connected |
| $\frac{\partial C}{\partial x}(x,y)$ | $C(x,y)$ | connected |
| | $G(x,y)$ | all |

The arrow $\longleftrightarrow$ means that the generating functions are related by a hard (implicit) equation.

# Equations relating the GFs

Labelled 2-Connected Planar Graphs.

We need a characterization of 2-connected graphs in terms of 3-connected graphs.

A *network* is a graph with two distinguished vertices (poles) such that the graph obtained by joining the poles if they were not joined is 2-connected.

[Trakhtenbrot, 1958] Every network is either

- A *series* composition of networks.
- A *parallel* composition of networks.
- A *3-connected graph* (with two poles) where every edge has been replaced by a network.

# Equations relating the GFs

Let $D(x, y)$ be the GF of labelled planar networks with unlabelled poles.

To obtain such a network we need to

- Choose a 2-connected planar graph
- Select an edge
- Unlabel the two vertices of the edge (the poles)
- Choose which pole is the first one
- Remove or do not remove the selected edge

Hence planar networks are related to L2CPG by:

$$D(x, y) = (1 + y)\frac{2}{x^2}\frac{\partial B}{\partial y}(x, y)$$

# Equations relating the GFs

| Vertex rooted | Non-rooted | Edge rooted | |
|---|---|---|---|
| $\dfrac{\partial B}{\partial x}(x,y)$ | | $\dfrac{\partial B}{\partial y}(x,y) \leftrightarrow D(x,y)$ | 2-connected |
| $\updownarrow$ | | | |
| $\dfrac{\partial C}{\partial x}(x,y)$ | $C(x,y)$ | | connected |
| | $\updownarrow$ | | |
| | $G(x,y)$ | | all |

Planar networks counted by $D(x,y)$ are closely related to edge-rooted 2-connected graphs.

# Equations relating the GFs

Let $S(x,y)$ be the GF of labelled planar networks that are series compositions.

To obtain such a *series* network we need to

- Choose a network $N$
- Choose a non-series network $M$
- Join the second pole of $N$ with the first pole of $M$.
- Add a label to the joined poles.

Hence the GFs of all networks and series networks satisfy

$$S(x,y) = xD(x,y)\left(D(x,y) - S(x,y)\right)$$

$$S(x,y) = \frac{xD(x,y)^2}{1 + xD(x,y)}$$

# Equations relating the GFs

Let $K(x,y)$ be the GF of networks with non-adjacent poles, and $U(x,y)$ of those that are non-parallel.

A network with non-adjacent poles is either

- non-parallel
- parallel, that is, a set of at least two non-parallel networks.

$$K(x,y) = 1 + \underbrace{U(x,y)}_{\text{non-parallel}} + \underbrace{\left(\frac{U^2}{2!} + \frac{U^3}{3!} + \cdots\right)}_{\text{parallel}} = \text{exp}(U(x,y))$$

On the other hand $D(x,y)$ and $K(x,y)$ are easy to relate, so

$$(1+y)K(x,y) - 1 = D(x,y) \qquad K(x,y) = \frac{1+D(x,y)}{1+y}$$

$$U(x,y) = \log\left(\frac{1+D(x,y)}{1+y}\right)$$

# Equations relating the GFs

Let $M_3(x, y)$ be the GF of the rooted 3-connected planar maps.

To obtain a network with non-adjacent poles that is neither parallel nor series:

- Choose a rooted 3-connected planar map
- Forget which face is the root face
- Remove the root edge
- Unlabel the two distinguished vertices (poles)
- Susbtitute every edge by a network

So we have the equation

$$\underbrace{\frac{M_3(x, D(x, y))}{2x^2 D(x, y)}}_{\text{non-parallel,non-series}} = \underbrace{\log\left(\frac{1 + D(x, y)}{1 + y}\right)}_{\text{non-parallel}} - \underbrace{\frac{x^2 D(x, y)}{1 + xD(x, y)}}_{\text{series}}$$

# Equations relating the GFs

$$
\begin{tikzcd}
\text{Vertex rooted} & \text{Non-rooted} & \text{Edge rooted} & & \\
 & & & M_3(x,y) \arrow[d, leftrightarrow] & \text{3-connected} \\
\dfrac{\partial B}{\partial x}(x,y) \arrow[d, leftrightarrow] & & \dfrac{\partial B}{\partial y}(x,y) \arrow[r, leftrightarrow] & D(x,y) & \text{2-connected} \\
\dfrac{\partial C}{\partial x}(x,y) & C(x,y) \arrow[d, leftrightarrow] & & & \text{connected} \\
 & G(x,y) & & & \text{all}
\end{tikzcd}
$$

$M_3(x,y)$ and $D(x,y)$ are related by an implicit equation.

# Equations relating the GFs

Mullin, Schellenberg (1968) gave the GF for rooted 3-connected planar maps.

$$u(x,y) = xy(1+v(x,y))^2 \quad v(x,y) = y(1+u(x,y))^2$$

$$M_3(x,y) = x^2y^2\left(\frac{1}{1+xy}+\frac{1}{1+y}-1-\frac{(1+u)^2(1+v)^2}{(1+u+v)^3}\right)$$

Fusy, Schaeffer (2003) provided a combinatorial explanation for this formula.

# Equations relating the GFs

So these are all the equations:

$$\begin{cases} u(x,z) = xz(1+v(x,z))^2 \\ v(x,z) = z(1+u(x,z))^2 \\ M_3(x,z) = x^2z^2\left(\dfrac{1}{1+xz} + \dfrac{1}{1+z} - 1 - \dfrac{(1+u)^2(1+v)^2}{(1+u+v)^3}\right) \end{cases}$$

$$\begin{cases} \dfrac{M_3(x,D)}{2x^2D} - \log\left(\dfrac{1+D}{1+y}\right) + \dfrac{xD^2}{1+xD} = 0 \\ \dfrac{\partial B}{\partial y}(x,y) = \dfrac{x^2}{2}\left(\dfrac{1+D(x,y)}{1+y}\right) \end{cases}$$

$$\begin{cases} C'(x,y) = \exp\left(B'(xC'(x,y),y)\right) \\ \\ G(x,y) = \exp(C(x,y)) \end{cases}$$

# Equations relating the GFs

Vertex rooted | Non-rooted | Edge rooted

$$
\begin{tikzcd}
 & & & M_3(x,y) \arrow[d, leftrightarrow] & \text{3-connected} \\
\frac{\partial B}{\partial x}(x,y) \arrow[d, leftrightarrow] & B(x,y) \arrow[l] \arrow[r] & \frac{\partial B}{\partial y}(x,y) \arrow[r, leftrightarrow] & D(x,y) & \text{2-connected} \\
\frac{\partial C}{\partial x}(x,y) & C(x,y) \arrow[l] \arrow[d, leftrightarrow] & & & \text{connected} \\
 & G(x,y) & & & \text{all}
\end{tikzcd}
$$

The arrow $\longrightarrow$ means that the generating functions are related by a partial derivation.

# Equations relating the GFs

$$
\begin{tikzcd}
\text{Vertex rooted} & \text{Non-rooted} & \text{Edge rooted} & & \\
 & & M_3(x,y) \arrow[dr, shift left] & & \text{3-connected} \\
\frac{\partial B}{\partial x}(x,y) \arrow[d, shift left] \arrow[r, shift left] & B(x,y) \arrow[l, shift left] \arrow[r, shift left] & \frac{\partial B}{\partial y}(x,y) \arrow[l, shift left] \arrow[r, shift left] & D(x,y) \arrow[l, shift left] \arrow[ul, shift left] & \text{2-connected} \\
\frac{\partial C}{\partial x}(x,y) \arrow[u, shift left] \arrow[r, shift left] & C(x,y) \arrow[l, shift left] \arrow[d, shift left] & & & \text{connected} \\
 & G(x,y) \arrow[u, shift left] & & & \text{all}
\end{tikzcd}
$$

We follow the path $\longrightarrow$ to obtain $G(x,y)$.

# 2-connected graphs [BGW]

[BGW] proved that:
The singularities of $D(x,y)$ are given by those of$M_3(x,z)$.

$$\frac{M_3(x,D)}{2x^2D} - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0$$

A parametrization $x_0(t), y_0(t)$ for the singularities of $D(x,y)$.

The expansion of $D(x,y_0)$ at a singular point $(x_0(t), y_0(t))$:

$$D(x,y_0) \sim D_0(t) + D_2(t)(1-\frac{x}{x_0}) + D_3(t)(1-\frac{x}{x_0})^{3/2}$$

If $t$ is such that $y_0(t)=1$, then

$$[x^n]D(x,1) = \frac{D_3(t)}{\Gamma(-3/2)} n^{-5/2} x_0(t)^{-n}$$

# 2-connected graphs [BGW]

Because of

$$\frac{\partial B}{\partial y}(x,y) = \frac{x^2}{2}\left(\frac{1+D(x,y)}{1+y}\right)$$

they can obtain the asymptotics for

$$[x^n]\frac{\partial B}{\partial y}(x,1) = \sum_m m b_{n,m}$$

They also prove that

- The expected number of edged is $\mu n$, with $\mu \simeq 2.26$.
- The variance is $\sigma^2 n$, with $\sigma^2 \simeq 0.4$.

So almost all $n$-vertex 2-connected planar graphs (for large $n$) have $\mu n$ edges,

$$b_n \sim \frac{\sum_m m b_{n,m}}{\mu n} \sim \frac{x_0^2 D_3}{4\mu\Gamma(-3/2)} n^{-7/2} x_0^{-n}$$

# 2-connected graphs [BGW]

Summary:

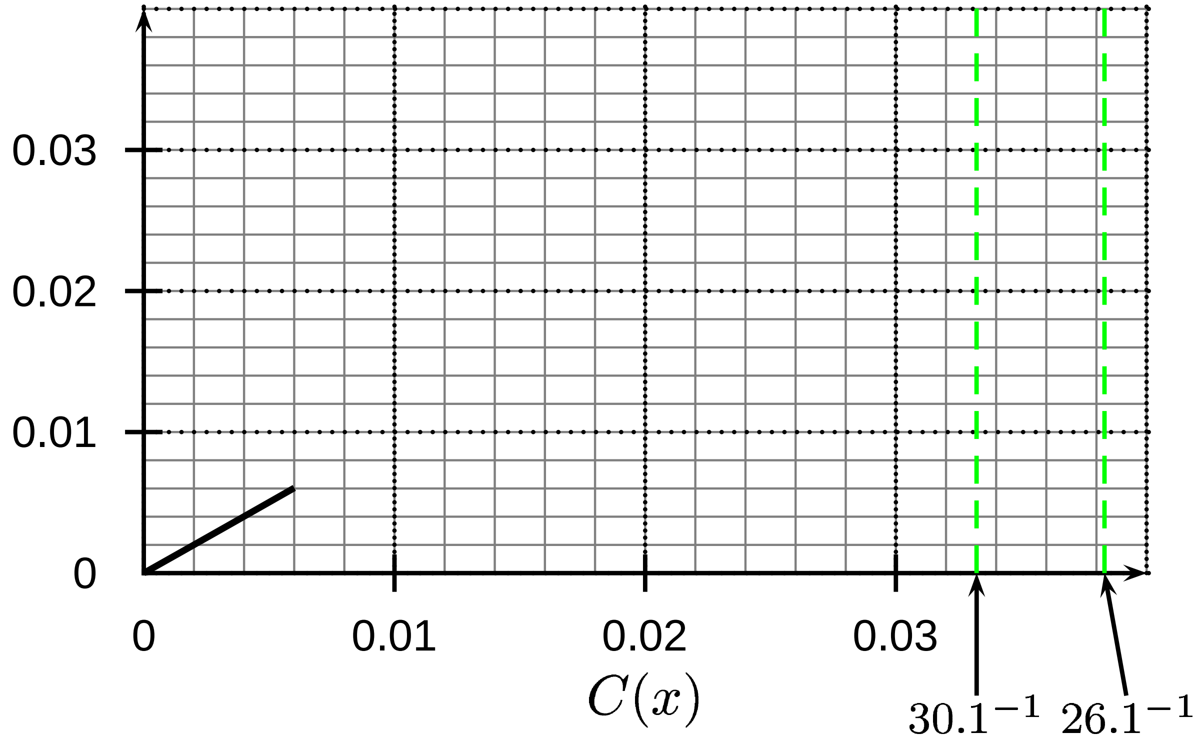
- We know about $\dfrac{\partial B}{\partial y}(x, y)$.
- We know about the *coefficients* of $B$ and $\dfrac{\partial B}{\partial y}$.
- But $B(x, y)$ is hidden behing an integral.

$$B(x, y) = \frac{x^2}{2} \int_0^y \frac{1 + D(x, s)}{1 + s} \mathrm{d}s$$

Just understanding $D(x, y)$ at the singularity point is not enough for evaluating $B(x, y)$.

# Estimating $\gamma$

The growth constant $\gamma$ is the inverse of the radius of converge of $C(x)$.




Question:

Where does $C(x)$ stop being defined?

# Lower bound: $\gamma > 27.22685$

We use the equation

$$C'(x) = \exp\left(\frac{\partial B}{\partial x}(xC'(x), 1)\right).$$

Define

$$\bar{C}(x) := xC'(x), \qquad \Psi(u) := u \exp\left(-\frac{\partial B}{\partial x}(u, 1)\right);$$

then $\bar{c}_n = n\, c_n$, and

$$\bar{C}(x) = x \exp\left(\frac{\partial B}{\partial x}(\bar{C}(x), 1)\right)$$

$$\bar{C}(x) \exp\left(-\frac{\partial B}{\partial x}(\bar{C}(x), 1)\right) = x$$

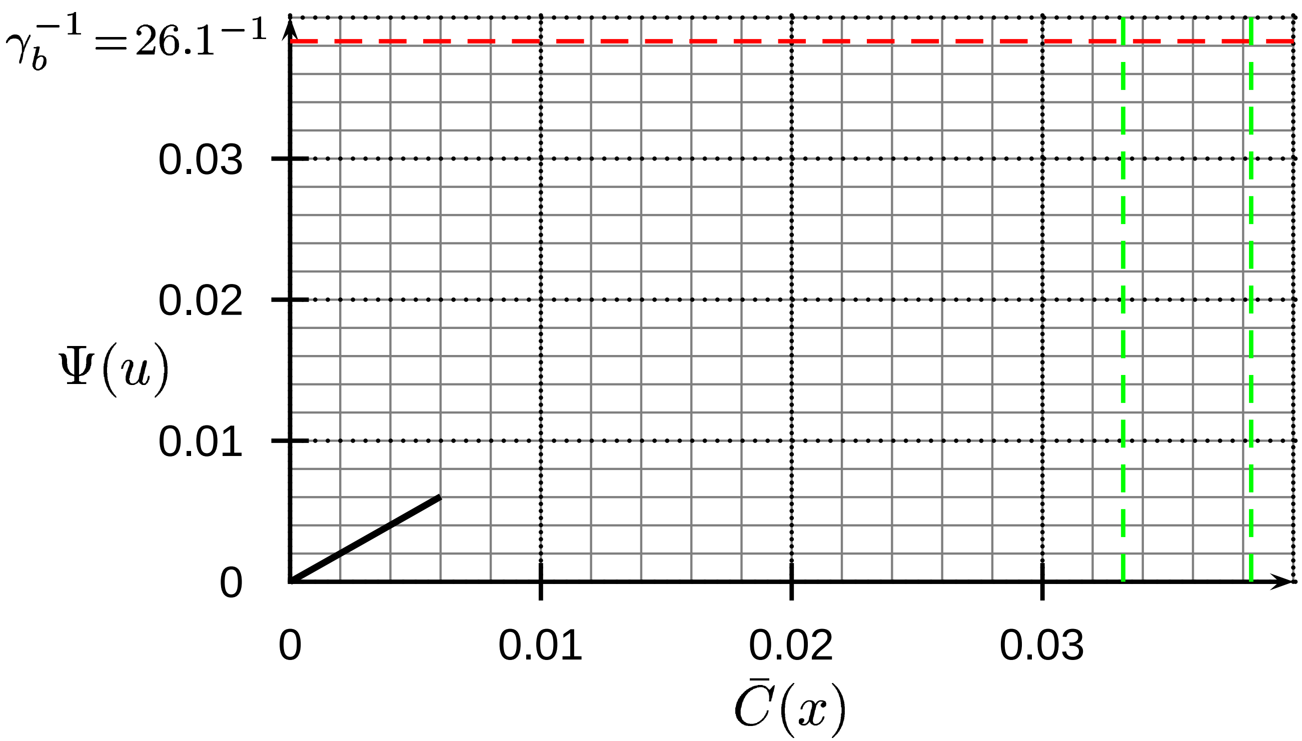
$$\boxed{\Psi(\bar{C}(x)) = x}$$

That is, $\Psi(u)$ is the inverse of $\bar{C}(x)$.

# Lower bound: $\gamma > 27.22685$

$$\Psi(u) = u \exp\left(-\frac{\partial B}{\partial x}(u,1)\right)$$

$\Psi(u)$ stops being defined at the radius of convergence of $B(u,1)$ : $\gamma_b^{-1} \simeq 0.0383$. Hence $\bar{C}(x)$ is bounded by $\gamma_b^{-1}$.



# Lower bound: $\gamma > 27.22685$

Consider the truncations of $\bar{C}(x)$ and $B(u,1)$. That is, if $\bar{C}(x) = \sum_{i=0}^{\infty} \bar{c}_i x^i$, then let

$$\bar{C}_n(x) := \sum_{i=0}^{n} \bar{c}_i x^i.$$

$\boxed{\bar{C}_n(x) \text{ is a } \textit{lower bound} \text{ of } \bar{C}(x).}$

Similarly, if $B(u,1) = \sum_{i=0}^{\infty} b_i u^i$, then let

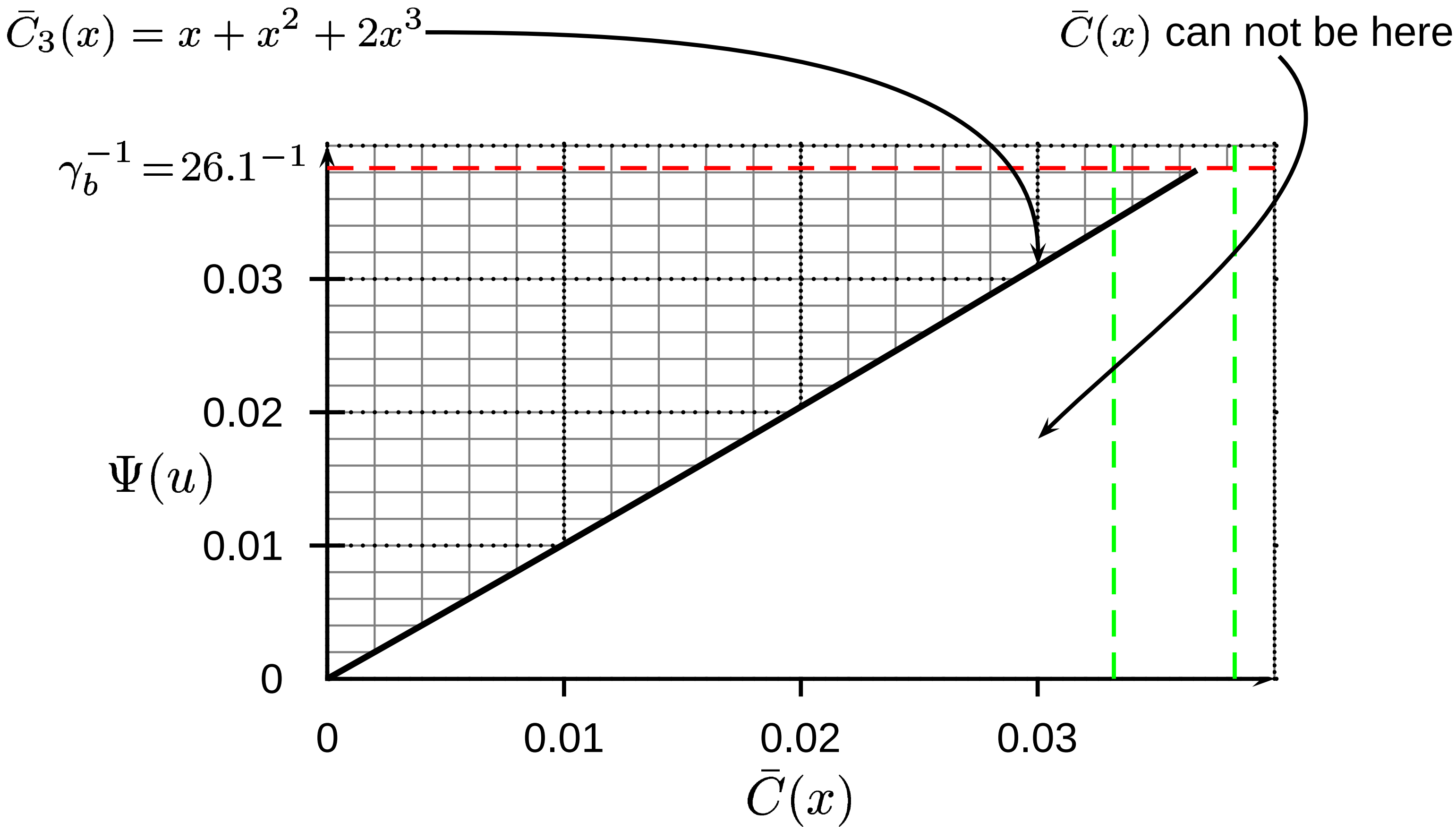
$$B_n(u) := \sum_{i=0}^{n} b_i u^i; \qquad \Psi_n(u) := u \exp\left(-\frac{\partial B_n}{\partial x}(u)\right)$$

Since $B_n(u)$ is a *lower bound* of $B(u,1)$,

$\boxed{\Psi_n(u) \text{ is an } \textit{upper bound} \text{ of } \Psi(u).}$

# Lower bound: $\gamma > 27.22685$

We can use $\bar{C}_n(x)$ or $\Psi_n(u)$ to bound the region where $\bar{C}(x)$ can be.



# Lower bound: $\gamma > 27.22685$

This gives us

- a *lower bound* on the region where $\bar{C}(x)$ is defined,
- an *upper bound* on the radius of convergence $\gamma^{-1}$,
- a *lower bound* on the growth constant $\gamma$.

By considering $\bar{C}_n(x)$ and $\Psi_n(u)$ for greater $n$ we can improve the previous bounds.

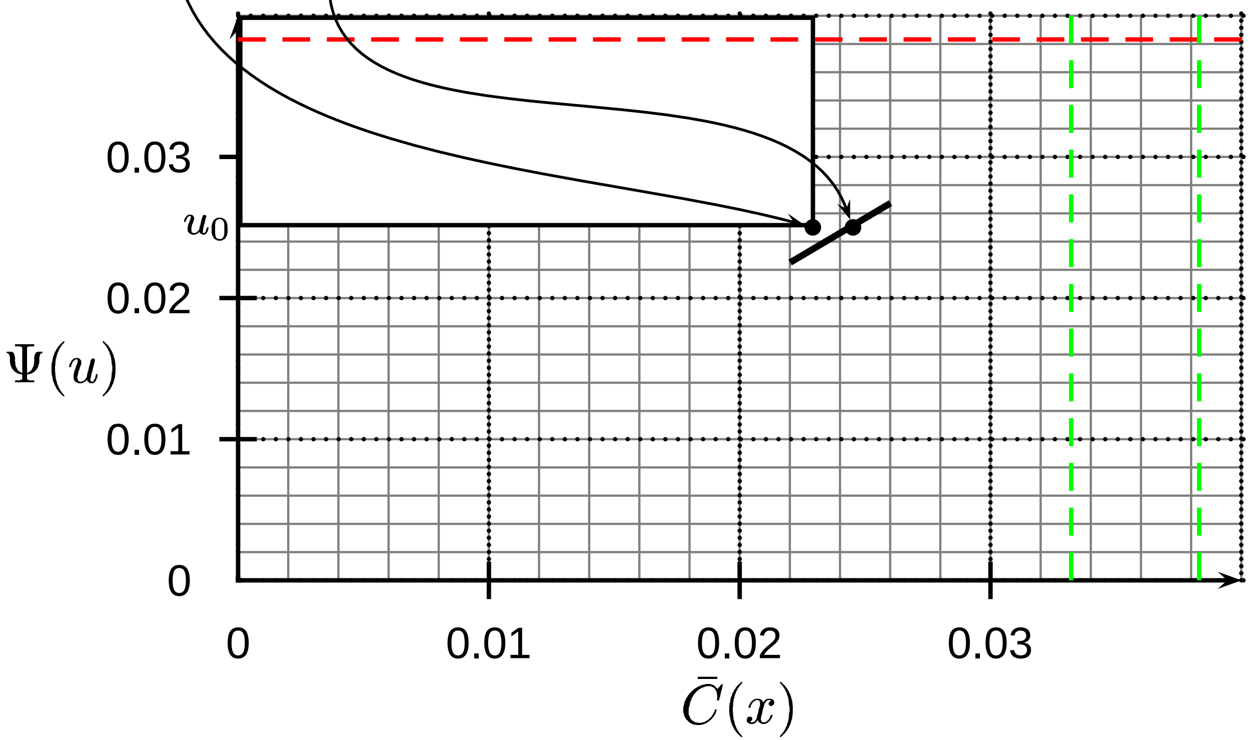
| $n$ | $\max\{x \mid \bar{C}_n(x) < \gamma_b^{-1}\}^{-1}$ (bound from $\bar{C}_n(x)$) | $\max\{\Psi_n(u) \mid u \in (0, \gamma_b^{-1})\}^{-1}$ (bound from $\Psi_n(u)$) |
|---|---|---|
| 5 | 27.226147 | 27.226399 |
| 10 | 27.226779 | 27.226790 |
| 15 | 27.226833 | 27.226837 |
| 20 | 27.226851 | 27.226853 |

# Upper bound: $\gamma < 27.22688$

To prove $\gamma < 27.22688$ we need the opposite:

- An *upper bound* on the region where $\bar{C}(x)$ is defined,
- a *lower bound* on the radius of convergence $\gamma^{-1}$,
- an *upper bound* on the growth constant $\gamma$.

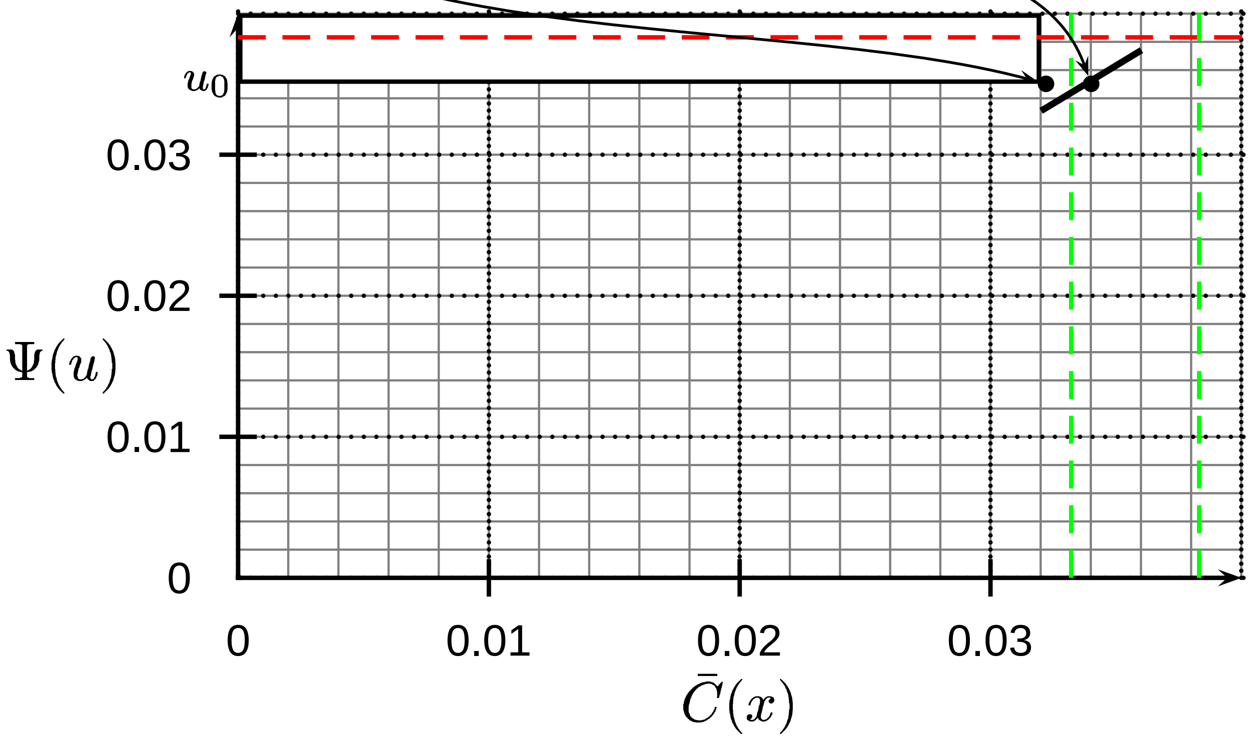
$$\underbrace{c < \Psi(u_0)}_{} \quad \Rightarrow \quad \gamma^{-1} > c \quad \Rightarrow \quad \gamma < c^{-1}$$



# Upper bound: $\gamma < 27.22688$

We want $u_0$ to be as close to $\gamma_b^{-1}$ as possible.

$$c < \underbrace{\Psi(u_0)} \quad \Rightarrow \quad \gamma^{-1} > c \quad \Rightarrow \quad \gamma < c^{-1}$$



# Upper bound: $\gamma < 27.22688$

Truncations $\bar{C}_n$ and $\Psi_n$ bound in the wrong direction.

We try to evaluate $\Psi(u_0)$ using the equations:

$$\Psi(u) = u \exp\left(-\frac{\partial B}{\partial x}(u,1)\right)$$

$$\frac{\partial B}{\partial x}(u,1) = \frac{\partial}{\partial x}\int_0^1 \frac{\partial B}{\partial y}(u,t)\mathrm{d}t$$

$$\frac{\partial B}{\partial y}(x,y) = \frac{x^2}{2}\left(\frac{1+D(x,y)}{1+y} - 1\right)$$

$$\frac{M_3(x,D)}{2x^2 D} - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0$$

$$M_3(x,y) = x^2 y^2 \left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+u)^2(1+v)^2}{(1+u+v)^3}\right)$$

$$u = xy(1+v)^2 \qquad v = y(1+u)^2$$

# Upper bound: $\gamma < 27.22688$

Things to take care of:

$$\frac{\partial B}{\partial x}(u,1) = \frac{\partial}{\partial \mathbf{x}} \int_0^1 \frac{\partial B}{\partial y}(u,t)\mathrm{d}t$$

- We have to bound $\dfrac{\partial B}{\partial x}(u_0, 1)$.
- We use that $\dfrac{\partial^2 B}{\partial x^2}(u, y)$ is positive.

$$B'(u_0) \leq \frac{B(u_0+\epsilon) - B(u_0)}{\epsilon}$$

- We need to evaluate $\int \partial B/\partial y$ at $u_0$ and at $u_0 + \epsilon$.

# Upper bound: $\gamma < 27.22688$

Things to take care of:

$$\frac{\partial B}{\partial x}(u,1) = \frac{\partial}{\partial x}\int_{\mathbf{0}}^{\mathbf{1}} \frac{\partial B}{\partial y}(u,t)\mathrm{d}t$$

- Numerical methods to estimate the integral.
- Errors in Newton-Cotes integration methods (Simpson rule, midpoint rule, etc.) are evaluations of a derivative of the integrand.
- Integrand $\partial B/\partial y$ has positive derivatives.
- We know the sign of the derivative, so we know the sign of the error.
- We are obtaining bounds, not just approximations.

# Upper bound: $\gamma < 27.22688$

Things to take care of:

$$\frac{M_3(x,D)}{2x^2D} - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0$$

- $D(x,y)$ is singular in $(\gamma_b^{-1}, 1)$.
- We need to be carefull when solving for $D(x,y)$ if $(x,y)$ is close to $(\gamma_b^{-1}, 1)$.
- $D(x,y)$ does not go to infinity at the singularity.

# Upper bound: $\gamma < 27.22688$

- Maple 9
- 25 precission digits.
- $u_0 = 0.038191096$ ($\gamma_b^{-1} - u_0 \simeq 10^{-8}$).
- $\epsilon = 1.6 \cdot 10^{-9}$.
- Integration methods: repeated midpoint rule, trapezoid rule.
- 30000 equispaced evaluations.

Computational results:

$$\gamma^{-1} > \Psi(u_0) > 0.036728410432$$

$$\gamma < 27.2268797$$

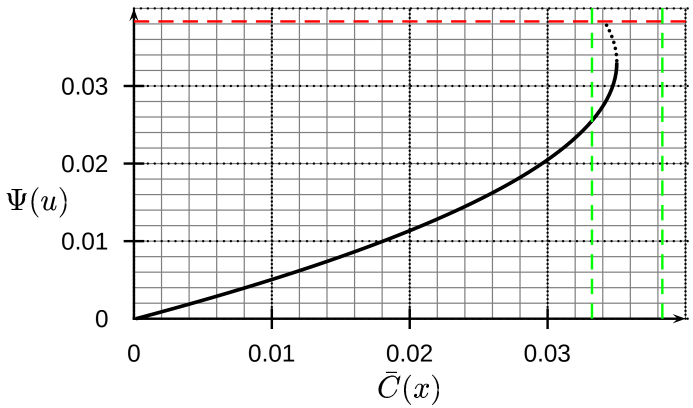
# Singularity type

**Transfer theorems** (Flajotet, Odlyzko): The singularity type of $C(x)$ determines the subexponential behaviour of its coefficients $c_n$.

Question: *How* does $C(x)$ stop being defined?

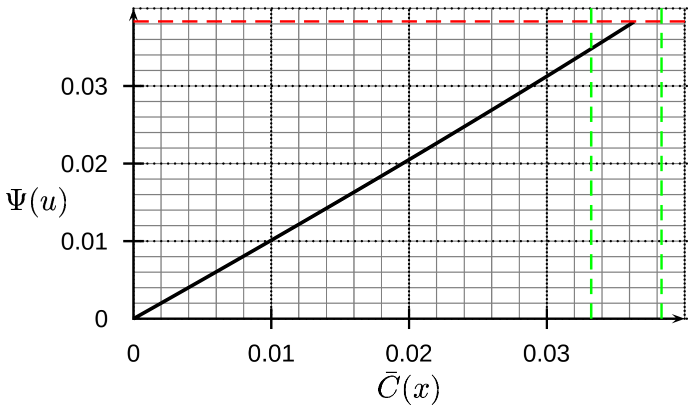
Since its inverse $\Psi(u)$ is defined in $(0, \gamma_b^{-1})$, there are two different possibilities:

$\exists u_0 \in (0, \gamma_b^{-1}) \quad \Psi'(u_0) = 0$



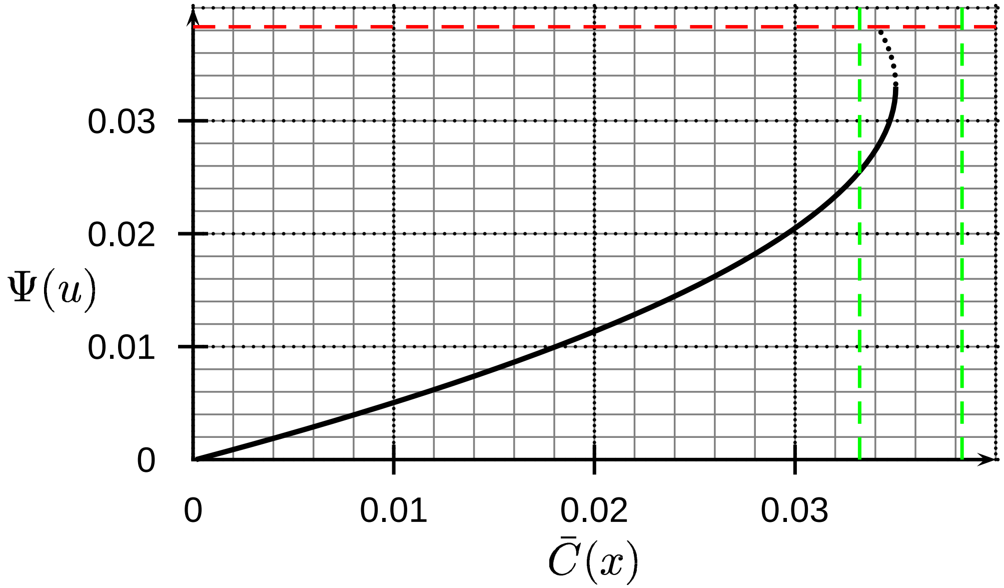

(case A)

$\forall u \in (0, \gamma_b^{-1}) \quad \Psi'(u) > 0$




(case B)

# Singularity type




**(case A)**: $\bar{C}$ has a square root-type singularity ($Z^{1/2}$):

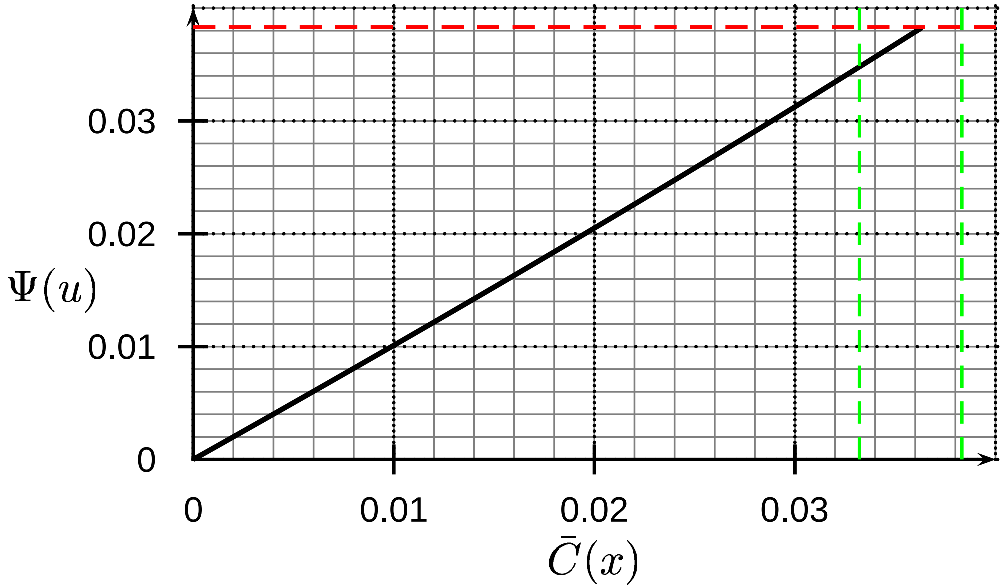
$$\Psi(u) = \Psi(u_0) + A(u - u_0)^2 + \mathcal{O}\left((u - u_0)^3\right)$$

$$\bar{C}(x) = u_0 - B\sqrt{1 - x/\Psi(u_0)} + \mathcal{O}\left((x - \Psi(u_0))^2\right)$$

By the Transfer theorems,

$$\bar{c}_n \sim \rho\, n^{\mathbf{-3/2}}\, \gamma^n n! \qquad c_n \sim \rho\, n^{\mathbf{-5/2}}\, \gamma^n n!$$

# Singularity type




**(case B)**: $\bar{C}$ has the same singularity type than $\Psi$.

The singularity type of $B$ is $Z^{5/2}$ (BGW; 2002). Thus the singularity type of $B'$ is $Z^{3/2}$.

$\Psi(u) = u \exp(-B'(u,1))$ has the same singularity type that $B'$.

$$\bar{c}_n \sim \rho\, n^{-\mathbf{5/2}}\, \gamma^n n! \qquad\qquad c_n \sim \rho\, n^{-\mathbf{7/2}}\, \gamma^n n!$$

# How to obtain exact results

$$\exists u_0 \in (0, \gamma_b^{-1}) \quad \Psi'(u_0) = 0?$$

$$\Psi'(u) = \exp(-B'(u))(1 - uB''(u))$$

If $1 - R_b B''(R_b) > 0$ it follows that $\Psi'(u) > 0$ for all $u$. So:

- Evaluate $B''(R_b, y)$ to obtain the singularity type of $C$.
- If $1 - R_b B''(R_b) > 0$, then
  - the singularity type would be $Z^{5/2}$,
  - and $\gamma^{-1}$ would be $R_b \exp(-B'(R_b, y))$.

Problem: We know about $\partial B / \partial y$, but not about $B$.

$$\frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \frac{1 + D(x, y)}{1 + y}$$

$$B(x, y) = \frac{x^2}{2} \int_0^y \frac{1 + D(x, s)}{1 + s} \mathrm{d}s$$

# How to obtain exact results

Three lemmas about integrating functions through its inverses.

Let $f(z)$ be an invertible function such that $f(0) = 0$, and let $f^{-1}(u)$ be its inverse. Then

$$\boxed{\int_0^Z f(z)\mathrm{d}z = Zf(Z) - \int_0^{f(Z)} f^{-1}(u)\mathrm{d}u}$$

$$\boxed{\int_0^Z f(z)a'(z)\mathrm{d}z = a(Z)f(Z) - \int_0^{f(Z)} a(f^{-1}(u))\mathrm{d}u}$$

$$\boxed{\int_0^Z \Phi(z, f(z))\mathrm{d}z = \int_0^{f(Z)} \Phi(f^{-1}(u), u)\frac{\partial f^{-1}(u)}{\partial u}\mathrm{d}u}$$

# How to obtain exact results

$$\frac{M_3(x,D)}{2x^2D} - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0$$

The inverse of $D(x,y)$ with respect to $y$ is explicit:

$$y = \exp\left(-\frac{M_3(x,D)}{2x^2D} + \log\left(1+D\right) - \frac{xD^2}{1+xD}\right) - 1$$

So we can integrate $D(x,y)$ through its inverse:

$$\int_0^y \frac{1+D(x,s)}{1+s}\mathrm{d}s = D(x,y)\log(1+y) - \int_0^{D(x,y)} \log(1+D^{-1}(x,t))\mathrm{d}t =$$

$$D(x,y)\log(1+y) + \int_0^{D(x,y)} \frac{M_3(x,t)}{2x^2t} - \log\left(1+t\right) + \frac{xt^2}{1+xt}\mathrm{d}t =$$

$$\int_0^{D(x,y)} \frac{M_3(x,t)}{2x^2t} - \log\left(1+t\right) + \log(1+y) + \frac{xt^2}{1+xt}\mathrm{d}t$$

# How to obtain exact results

$$\int_0^{D(x,y)} \frac{M_3(x,t)}{2x^2t} - \log(1+t) + \log(1+y) + \frac{xt^2}{1+xt}\mathrm{d}t$$

But $\displaystyle\int \frac{M_3(x,t)}{t}\mathrm{d}t$ is not easy, because $M_3$ depends on algebraic functions $u$ and $v$:

$$u(x,z) = xz(1+v(x,z))^2 \qquad v(x,z) = z(1+u(x,z))^2$$

$$u(x,z) = xz\left(1 + z(1+u(x,z))^2\right)^2$$

To obtain the inverses of $u(x,z)$ with respect to $z$ we need to solve a cubic equation.

# How to obtain exact results

But if we consider $w(x,z) = z(1+u(x,z))$ it will be easier, because

$$w(x,z) = z(1+u(x,z)) = z(1+xz\left(1+z(1+u(x,z))^2\right)^2) =$$

$$z + xz^2(1+\frac{w(x,z)^2}{z})^2 = z + x(z+w(x,z)^2)^2$$

so to obtain inverse of $w(x,z)$ we only need to solve a quadratic equation.

- We express $M_3(x,z)$ in terms of $w(x,z)$.
- We integrate through the inverse of $w$.
- We replace every resulting $w(x,z)$ by $z(1+u(x,z))$.
- So we obtain a long but explicit expression for $B(x,y)$:

$$B(x,y) = \Phi(x,y,D(x,y),u(x,D(x,y)))$$

From here we can obtain $B'(x,y)$, $B''(x,y)$, etc.

# How to obtain exact results

One last application of the integrating-through-inverse lemmas:

$$\bar{C}(x,y) = xC'(x,y) \qquad \bar{C} \text{ and } \Psi \text{ are inverses}$$

$$C(x,y) = \int_0^x C'(s,y)\mathrm{d}s = \int_0^x \frac{\bar{C}(s,x)}{s}\mathrm{d}s =$$

$$\log(x)\bar{C}(x,y) - \int_0^{\bar{C}(x,y)} \log(\Psi(t,y))\mathrm{d}t =$$

$$\log(x)\bar{C}(x,y) - \int_0^{\bar{C}(x,y)} \log(t\exp(-B'(t,y)))\mathrm{d}t =$$

$$\log(x)\bar{C}(x,y) - \int_0^{\bar{C}(x,y)} \log(t)\mathrm{d}t + \int_0^{\bar{C}(x,y)} B'(t,y)\mathrm{d}t =$$

$$\log(x)\bar{C}(x,y) - \bar{C}(x,y)\log(\bar{C}(x,y)) + \bar{C}(x,y)B(\bar{C}(x,y),y)$$

# Applications

How many isolated vertices has a random planar graph?

Consider the following GF, where $t$ counts the number of isolated vertices

$$G(x,t) = \exp\left(C(x) - x + tx\right)$$

The probability that a $n$-vertex graf has $k$ isolated vertices is

$$\frac{[x^n][t^k]G(x,t)}{[x^n]G(x)} = \frac{[x^n]\exp\left(C(x)-x\right)[t^k]\exp(tx)}{[x^n]\exp\left(C(x)\right)} \sim$$

$$\frac{R^k[x^n]\exp\left(C(x)\right)}{e^R k![x^n]\exp\left(C(x)\right)} = \frac{R^k}{e^R k!}$$

Answer: Poisson distribution of parameter $R = \gamma^{-1}$.

# Applications

We generalize the previous result.

Let $\mathcal{A}$ be a *not too large* family of planar graphs, and $A(x)$ its GF.

$\boxed{\text{How many } \mathcal{A}\text{-components has a random planar graph?}}$

$$\frac{[x^n][t^k]G(x,t)}{[x^n]G(x,1)} = \frac{[x^n]\exp\left(C(x)-A(x)\right)[t^k]\exp(tA(x))}{[x^n]\exp\left(C(x)\right)} \sim$$

$$\frac{A(R)^k[x^n]\exp\left(C(x)\right)}{e^{A(R)}k![x^n]\exp\left(C(x)\right)} = \frac{A(R)^k}{e^{A(R)}k!}$$

Answer: Poisson distribution of parameter $A(R)$.

# Applications

What is the probability of being connected?

Consider the GF $G(x,t) = \exp(tC(x))$, where $t$ counts the number of connected components.

The probability of having $k$ components is given by

$$\frac{[x^n][t^k]G(x,t)}{[x^n]G(x)} = \frac{[x^n][t^k]\exp(tC(x))}{[x^n]\exp(G(x))} \sim$$

$$\frac{[x^n]C(x)^k}{\exp(C(R))k![x^n]C(x)}$$

If we set $k = 1$ we obtain the probability of being connected:

$$\frac{[x^n]C(x)}{\exp(C(R))[x^n]C(x)} = \exp(-C(R))$$

# Applications

How many components has a random planar graph?

$$\frac{[x^n][t^k]G(x,t)}{[x^n]G(x)} \sim \frac{[x^n]C(x)^k}{\exp(C(R))k![x^n]C(x)}$$

The important terms of the expansion of $C(x)$ close to the singularity are $C(R) + b(1-\frac{x}{R})^{5/2}$, so

$$[x^n]C(x) \sim b[x^n](1-\frac{x}{R})^{5/2}$$

$$[x^n]C(x)^k \sim kC(R)^{k-1}b[x^n](1-\frac{x}{R})^{5/2}$$

$$\frac{[x^n]C(x)^k}{\exp(C(R))k![x^n]C(x)} \sim \frac{C(R)^{k-1}}{\exp(C(R))(k-1)!}$$

Answer: (1+#components) follows a Poisson distribution of parameter $C(R)$.

# Applications

How many edges has a random planar graph?

- $C(x,y) \sim (1-\frac{x}{R(y)})^{7/2}$ around the singularity in a neighbourhood of $y=1$.
- We apply the quasi powers theorem.
- It follows that
  - The distribution of the number of edges follows a Normal law.
  - The expected number of edges is $\mu n$.
  - The variation is $\sigma^2 n$.
  - Concentration: almost all planar graphs has $\mu n$ edges.
- $\mu$ is given by $R'(1)/R(1)$, and can be computed using that

$$R'(y) = \frac{\partial}{\partial y}\left(R_b(y)\exp(B'(R_b(y),y))\right)$$

.