The Fermat cubic, elliptic functions, continued fractions, and a combinatorial tale

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Abstract

Elliptic functions considered by Dixon in the nineteenth century and related to Fermat's cubic, $x^3 + y^3 = 1$ lead to a new set of continued fraction expansions with sextic numerators and cubic denominators. The functions and the fractions are surrounded by interesting combinatorics, including a special Pólya urn, a continuous-time branching process of the Yule type, as well as permutations satisfying constraints of various types—either by level or by repeated patterns. The combinatorial models are related to but different from models of elliptic functions earlier introduced by Viennot, Flajolet, Dumont, and Françon.

1 CONTINUED FRACTIONS

In number theory: $x = \lfloor x \rfloor + \{x\}$ & $x \mapsto 1/x$

$$\frac{1+\sqrt{5}}{2} = \mathbf{1} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\ddots}}}, \quad \pi = \mathbf{3} + \cfrac{1}{\mathbf{7} + \cfrac{1}{\mathbf{15} + \cfrac{1}{\ddots}}}.$$

Theorem: $x \in \mathbb{R} \setminus \mathbb{Q}$ *iff* $\mathbf{CF}(x)$ *is infinite.*

+ Ergodic/metric properties; relation to Euclid's algorithm.

Cf Brigitte Vallée *et al.* @ ALGO Sem $\rightsquigarrow$ dynamics of $x \mapsto \{1/x\}$.

Continued fractions for power series.

Example:

$$\tan z = \cfrac{z}{1 + \cfrac{z^2}{3 + \cfrac{z^2}{5 + \cfrac{z^2}{\ddots}}}}.$$

Theorem [Lambert 1773] π is irrational.

(Proof. $\tan(p/q) \in \mathbb{R} \setminus \mathbb{Q}$.)

Continued fractions for power series (cont'd).

• Decide that "integral part" of power series is constant term. Write:

$$f = \lfloor f \rfloor + z\{f\}, \qquad \lfloor f \rfloor := f(0), \quad \{f\} := \frac{f(z) - f(0)}{z}.$$

Iterate by taking inverses. This is called an S-fraction (Stieltjes).

• Alternatively, decide that "integral part" is linear term and get a J-fraction (Jacobi):

$$f = \lfloor f \rfloor + z^2\{f\}, \qquad \lfloor f \rfloor := f(0) + zf'(0), \quad \{f\} := \frac{f(z) - f(0) - zf'(0)}{z^2}.$$

Related to orthogonal polynomials Padé approximants, moment problems, summation of divergent series, etc.

Summary: J-fractions have linear denominators; S-fractions have constant denominators. (There are known reductions.)

Explicit CFs are *very rare*, due to highly nonlinear algorithm.

From classics by Perron, Wall, etc, *there are less than 100 continued fractions known for special functions.* Main ones are due to:

— Lambert, Euler, Gauß, Jacobi, Eisenstein, Stieltjes, Rogers, Ramanujan, . . .

Theorem (Apéry 1978): $\zeta(3) = \sum 1/n^3$ is irrational.

$$\zeta(3) = \cfrac{6}{\varpi(0) - \cfrac{1^6}{\varpi(1) - \cfrac{2^6}{\varpi(2) - \cfrac{3^6}{\ddots}}}}, \tag{1}$$

where $\qquad \varpi(n) = (2n+1)(17n(n+1)+5).$

The nth stage of the fraction involves the *sextic* numerator n^6, while the corresponding numerator is a *cubic* polynomial in n.

A continued fraction due to Stieltjes later rediscovered by Ramanujan (Berndt89, Ch.12)),

$$\psi''(z+1) = \cfrac{-2}{\sigma(0) - \cfrac{1^6}{\sigma(1) - \cfrac{2^6}{\sigma(2) - \cfrac{3^6}{\ddots}}}}, \tag{2}$$

where $\sigma(n) = (2n+1)(2z^2+2z+2n+1)$,

and $\psi(z) = \frac{d}{dz}\log\Gamma(z)$.

Theorem (Conrad 2002): For a certain function sm:

$$\int_0^\infty \mathrm{sm}(u)e^{-u/x}\,du = \cfrac{x^2}{1+b_0x^3-\cfrac{1\cdot 2^2\cdot 3^2\cdot 4\,x^6}{1+b_1x^3-\cfrac{4\cdot 5^2\cdot 6^2\cdot 7\,x^6}{1+b_2x^3-\cfrac{7\cdot 8^2\cdot 9^2\cdot 10\,x^6}{\ddots}}}},$$

where $\quad b_n = 2(3n+1)((3n+1)^2+1).$

The function sm is the inverse of an Abelian integral over $\mathbf{F}_3$ and equivalently the inverse of a ${}_2F_1$:

$$\mathrm{sm}(z) = \mathrm{Inv}\int_0^z \frac{dt}{(1-t^3)^{2/3}} = \mathrm{Inv}\; z\cdot{}_2F_1\left[\frac{1}{3},\frac{2}{3},\frac{4}{3};z^3\right].$$

Plan: some cute combinatorics surrounding the functions

— The Fermat cubic $x^3 + y^3 = 1$ and Dixonian functions

— Connections with Pólya urns and Yule (branching) process

— A first model of Dixonian function by permutations

— A second model of Dixonian function by permutations

2 FERMAT CURVES: CIRCLE & CUBIC

The Fermat curve $\mathbf{F}_m$ is the complex algebraic curve

$$x^m + y^m = 1.$$

(Fermat-Wiles: no nontrivial rational point for $m \geq 3$.)

Start with $\mathbf{F}_2$, the circle. Consider two functions from $\mathbb{C}$ to $\mathbb{C}$ defined by the *linear* differential system,

$$\boxed{s' = c, \quad c' = -s} \quad \text{with} \quad s(0) = 0, \quad c(0) = 1.$$

The transcendental functions s, c do parameterize the circle,

$$s(z)^2 + c(z)^2 = 1,$$

$$\text{since} \qquad (s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0.$$

One switches to conventional notations:

$$s(z) \equiv \sin(z), \qquad c(z) \equiv \cos(z).$$

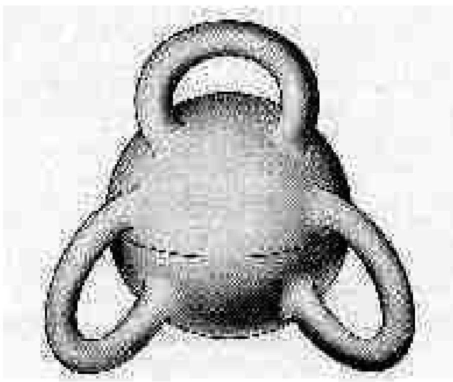
These functions are also obtained by inversion from an "*abelian integral*"[a] on $\mathbf{F}_2$:

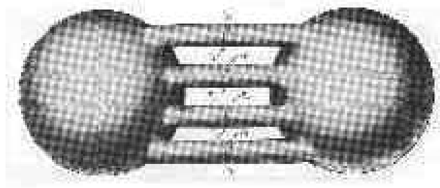
$$\int_0^{\sin z} \frac{dt}{\sqrt{1-t^2}} = z, \qquad \cos(z) = \sqrt{1-\sin(z)^2}$$

For combinatorialists: $\tan z = \dfrac{\sin z}{\cos z}, \qquad \sec z = \dfrac{1}{\cos z}$ enumerate alternating (aka up-and-down, zig-zag) permutations (Désiré André, 1881).

[a] Given an algebraic curve $P(z,y) = 0$, an abelian integral is any integral $\int R(z,y)\,dz$, where R is a rational function.

The "complexity" of integral calculus over an algebraic curve depends on its (topological) genus.





Sphere with 3 holes, $g = 3$

For Fermat curve $\mathbf{F}_p$, genus is $\frac{1}{2}(p-1)(-2)$.

- $\mathbf{F}_2 \Longrightarrow g = 0$;
- $\mathbf{F}_3 \Longrightarrow g = 1$; Normal forms of Weierstraß and Jacobi;
- $\mathbf{F}_4 \Longrightarrow g = 3, \ldots$

A clever generalization of $\sin, \cos$: the *nonlinear* system

$$\boxed{s' = c^2, \qquad c' = -s^2} \qquad \text{with } s(0) = 0,\ c(0) = 1.$$

We have: $s(z)^3 + c(z)^3 = 1$: the pair $\langle s(z), c(z) \rangle$ parametrizes $\mathbf{F}_3$.
Follow Dixon and set: $\mathrm{sm}(z) \equiv s(z), \qquad \mathrm{cm}(z) \equiv c(z)$.
(See sn, cn by Jacobi, sl, cl for lemniscate.)

$$\begin{cases} \mathrm{sm}(z) &= z - 4\dfrac{z^4}{4!} + 160\dfrac{z^7}{7!} - 20800\dfrac{z^{10}}{10!} + 6476800\dfrac{z^{13}}{13!} - \cdots \\ \mathrm{cm}(z) &= 1 - 2\dfrac{z^3}{3!} + 40\dfrac{z^6}{6!} - 3680\dfrac{z^9}{9!} + 8880000\dfrac{z^{12}}{12!} - \cdots. \end{cases}$$

2.1 A hypergeometric connection.

One can make $s \equiv \mathrm{sm}$ and $c \equiv \mathrm{cm}$ somehow "explicit".
Start from the defining system and differentiate

$$s' = c^2 \quad \stackrel{\partial}{\Longrightarrow} \quad s'' = 2cc' \quad \stackrel{E}{\Longrightarrow} \quad s'' = -2cs^2 \quad \stackrel{E}{\Longrightarrow} \quad s'' = -2c\sqrt{s'}.$$

Then "cleverly" multiply by $\sqrt{s'}$ to integrate ($\int$):

$$s''\sqrt{s'} = -2s^2 s' \quad \stackrel{\int}{\Longrightarrow} \quad \frac{2}{3}(s')^{3/2} = -\frac{2}{3}s^3 + K.$$

$$\int_0^{\mathrm{sm}(z)} \frac{dt}{(1-t^3)^{2/3}} = z,$$

at the same time an incomplete Beta integral and an Abelian integral over the Fermat curve.

Classical hypergeometric function:

$$ {}_2F_1[\alpha, \beta, \gamma; z] := 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \cdots . $$

Let $\mathrm{Inv}(f)$ denote the inverse of f w.r.t. composition (i.e., $\mathrm{Inv}(f) = g$ if $f \circ g = g \circ f = Id$)

Proposition: *Function* sm *is defined by inversion,*

$$ \mathrm{sm}(z) = \mathrm{Inv} \int_0^z \frac{dt}{(1-t^3)^{2/3}} = \mathrm{Inv}\, z \cdot {}_2F_1\left[\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z^3\right]. $$

The function cm *is then defined near* 0 *by* $\mathrm{cm}(z) = \sqrt[3]{1 - \mathrm{sm}^3(z)}$*.*

$\leadsto$ Regard sm, cm as "known" functions.

Alfred Cardew Dixon

Born: 22 May 1865 in Northallerton, Yorkshire, England
Died: 4 May 1936 in Northwood, Middlesex, England
(Him or his brother Arthur?)

3 A STARTLING FRACTION.

From Eric van Fossen CONRAD, PhD Columbus, OH, 2002.

$$\int_0^\infty \mathrm{sm}(u)e^{-u/x}\,du = \cfrac{x^2}{1+b_0x^3-\cfrac{1\cdot 2^2\cdot 3^2\cdot 4\,x^6}{1+b_1x^3-\cfrac{4\cdot 5^2\cdot 6^2\cdot 7\,x^6}{1+b_2x^3-\cfrac{7\cdot 8^2\cdot 9^2\cdot 10\,x^6}{\ddots}}}}$$

where $\quad b_n = 2(3n+1)((3n+1)^2+1).$

Proof: Follow in the steps of Stieltjes and Rogers.
Cleverly introduce the family of integrals

$$S_n := \int_0^\infty \mathrm{sm}^n(u) e^{-u/x}\, du.$$

Then integration by parts shows that

$$\frac{S_n}{S_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + 2n(n^2+1)x^3 - n(n+1)(n+2)x^3 \dfrac{S_{n+3}}{S_n}}.$$

This is enough to prime the continued fraction pump and obtain what is a standard J-fraction in the variable $\xi = x^3$.

$$\int_0^\infty \mathrm{sm}(u) e^{-u/x}\, du = \frac{x^2}{1 + 4x^3 +} \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{-a_n x^6}{1 + b_n x^3},$$

where $a_n = (3n-2)(3n-1)^2(3n)^2(3n+1)$, $b_n = 2(3n+1)((3n+1)^2+1)$.

4 BALLS GAMES

In *Théorie analytique des probabilités* Laplace (1812): "*Une urne A renfermant un très grand nombre n de boules blanches et noires; à chaque tirage, on en extrait une que l'on remplace par une boule noire; on demande la probabilité qu'après r tirages, le nombre des boules blanches sera x.*"

> **Pólya urn model.** *An urn is given that contains black and white balls. At each epoch, a ball in the urn is chosen at random. If it is black, then α black and β white balls are placed into the urn; else it is white and γ black and δ white balls are placed into the urn.*

Described by the "placement matrix", $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

The main character here is the special urn: $\mathcal{M}_{12} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$.

$$\text{x} \longrightarrow \text{yy}, \qquad \text{y} \longrightarrow \text{xx}.$$

A *history* of length n (Francon78) is any description of a legal sequence of n moves of the Pólya urn. For instance ($n = 5$):

$$\underline{\text{x}} \longrightarrow \text{y}\underline{\text{y}} \longrightarrow \text{y}\underline{\text{x}}\text{x} \longrightarrow \underline{\text{y}}\text{yyx} \longrightarrow \text{x}\underline{\text{x}}\text{yyx} \longrightarrow \text{xyyyyx},$$

We let $H_{n,k}^{(\beta,\gamma)}$ be the number of histories that start with a $\text{x}^\beta\text{y}^\gamma$ and, after n actions, result in a word having k occurrences of y (hence $n + \beta + \gamma - k$ occurrences of x).

What are the "history numbers"? The sequence $(H_n) \equiv (H_{n,0}^{(1,0)})$ starts as $1, 0, 0, 4$ for $n = 0, 1, 2, 3$.

4.1 Urns and Dixonian functions.

Consider the (autonomous, nonlinear) ordinary differential system

$$\Sigma : \qquad \frac{dx}{dt} = y^2, \quad \frac{dy}{dt} = x^2, \quad \text{with} \quad x(0) = x_0, \quad y(0) = y_0,$$

The pair $\langle x(t), y(t) \rangle$ parameterizes the "Fermat hyperbola", $y^3 - x^3 = 1$.

Solutions are trivial variants of $\mathrm{smh}, \mathrm{cmh}$ with

$$\mathrm{smh}(z) = -\,\mathrm{sm}(-z), \qquad \mathrm{cmh}(z) = \mathrm{cm}(-z).$$

Define a linear transformation δ acting on polynomials $\mathbb{C}[x,y]$:

$$\delta[x]=y^2, \qquad \delta[y]=x^2, \qquad \delta[u\cdot v]=\delta[u]\cdot v+u\cdot\delta[v],$$

(Cf the elegant presentation of Chen grammars by (Dumont96) and the "combinatorial integral calculus" of Leroux–Viennot.)

$xyy \mapsto \check{x}yy, x\check{y}y, xy\check{y} \xrightarrow{\delta} (yy)yy, x(xx)y, xy(xx),$ so that $\delta[xy^2]=y^4+2x^3y.$

(i) **Combinatorially**, the nth iterate $\delta^n[x^ay^b]$ is such that

$$H_{n,k}^{(a,b)}=[x^ky^\ell]\,\delta^n[x^ay^b], \qquad k+\ell=n+a+b,$$

(ii) **Algebraically**, the operator δ describes the "logical consequences" of the differential system Σ:

$$\delta^n[x^ay^b]=\frac{d^n}{dt^n}x(t)^ay(t)^b \quad \text{expressed in} \quad x(t),y(t),$$

Proposition: *The EGF of histories of the urn $\mathcal{M}_{12}$ that start with one ball and end with balls* all *of the other colour is*

$$\sum_{n\geq 0} H_{n,0}^{(1,0)} \frac{z^n}{n!} = \operatorname{smh}(z) = \frac{\operatorname{sm}(z)}{\operatorname{cm}(z)} = -\operatorname{sm}(-z).$$

The EGF of histories that end with balls that are all of the initial colour

$$\sum_{n\geq 0} H_{n,n+1}^{(1,0)} \frac{z^n}{n!} = \operatorname{cmh}(z) = \frac{1}{\operatorname{cm}(z)} = \operatorname{cm}(-z).$$

PROOF: This is nothing but Taylor's formula!

Note: this does not need the *method of characteristics* (FlGaPe05).

The knight's moves of Bousquet-Melou & Petkovšek.

```
    o----.             o----.
         |                  |
         |                  |
         .                  . multiplicity p
         |                  |
         |                  |
P=(p,q)  o----.----.      P o----.----.
                   |                  | multiplicity q
                   |                  |
                   o                  o
```

Note: OGF of walks that start at $(1,0)$ and end on the horizontal axis is

$$G(x) = \sum_{i \geq 0} (-1)^i \left(\xi^{\langle i \rangle}(x) \xi^{\langle i+1 \rangle}(x) \right)^2,$$

where ξ, a branch of the (genus 0) cubic $x\xi - x^3 - \xi^3 = 0$ is $\xi(x) = x^2 \sum_{m \geq 0} \binom{3m}{m} \frac{x^{3m}}{2m+1}$.

From a probabilistic standpoint, the number of black balls at time n is a random variable, X_n. Get *extreme large deviations*:

$$\mathbb{P}(X_n = 0) = \frac{H_{n,0}^{(1,0)}}{n!} = [z^n]\frac{\mathrm{sm}(z)}{\mathrm{cm}(z)}.$$

By an easy analysis of singularities:

$$\mathbb{P}(X_n = 0) \sim c\rho^{-n}, \qquad \rho = \frac{\sqrt{3}}{6\pi}\Gamma\left(\frac{1}{3}\right)^3, \quad n \equiv 1 \pmod{3}.$$

$$\left|\frac{[z^{31}]\,\mathrm{sm}(z)}{[z^{28}]\,\mathrm{sm}(z)}\right|^{1/3} \doteq 1.76663\,87\mathit{502}\cdots; \qquad \frac{\sqrt{3}}{6\pi}\Gamma\left(\frac{1}{3}\right)^3 \doteq 1.76663\,87490\cdots.$$

Note: Full analysis is doable.

4.2 Continuous-time branching = Yule process.

You have two types of particules, foatons and viennons. Any particle lives an amount of time T that is exponentially distributed ($\mathbb{P}(T \geq t) = e^{-t}$); then it disintegrates into two particles of the other type. (A foaton gives rise to two viennons and a viennon gives rise to two foatons.)

Let $S_k(t)$ be the probability that the total population at time t is of size k, and define

$$\Xi(t;w) := \sum_{k=1}^{\infty} S_k(t) w^k.$$

What happens between times 0 and dt (backwards equation)?

$$S_k(t+dt) = (1-dt)S_k(t) + dt \sum_{i+j=k} S_i(t)S_j(t),$$

Implies $S_k'(t) + S_k(t) = \sum_{i+j=k} S_i(t)S_j(t)$, hence a nonlinear ODE,

$$\Xi'(t;w) + \Xi(t;w) = \Xi(t;w)^2, \qquad \Xi(0,w) = w,$$

By separation of variables:

$$\Xi(t;w) = \frac{we^{-t}}{1-w(1-e^{-t})}, \qquad S_k(t) = e^{-t}\left(1-e^{-t}\right)^{k-1}, \qquad k \geq 1.$$

The size of the population at time t is 1 plus a geometric law of parameter $(1-e^{-t})$, with expectation e^t.

Also, we have an *Equivalence Principle*, discrete $\leftrightarrow$ continuous.

Proposition. *Consider the Yule process with two types of particles. The probabilities that particles are all of the second type at time t are*

$$X(t) = e^{-t}\,\mathrm{smh}(1 - e^{-t}), \qquad Y(t) = e^{-t}\,\mathrm{cmh}(1 - e^{-t}),$$

depending on whether the system at time 0 is initialized with one particle of the first type (X) or of the second type (Y).

Remarks. A partial differential operator:

$$\delta[f] = y^2 \frac{\partial}{\partial x} f + x^2 \frac{\partial}{\partial y} f.$$

(FlGaPe05) characterize all operators $\Gamma = x^{1-a} y^{s+a} \frac{\partial}{\partial x} + x^{s+b} y^{1-b} \frac{\partial}{\partial y}$, $(a, b, s > 0)$, such that $e^{z\Gamma}$ is expressible by elliptic functions.

Note: For $\mathbf{F}_3$ apparent genus = actual genus = 1. In some cases deal with higher apparent genus (e.g. $\mathbf{F}_6$).

$$A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$
$$D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}.$$

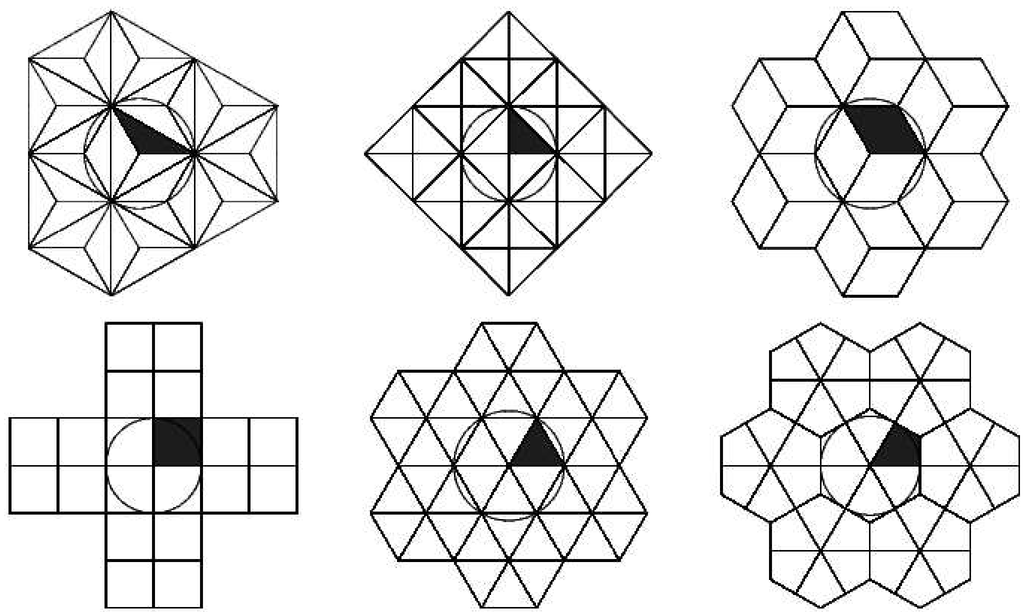
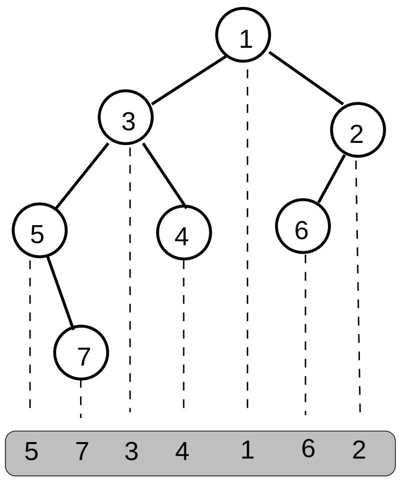


FIGURE 7. The six elliptic cases in order A, B, C, D, E, F: The diagrams formed by the fundamental polygon together with its rotated images. (The elementary kite is darkened.)

5 FIRST PERMUTATION MODEL

A permutation can always be represented as a tree, which is binary, rooted, and increasing.



$$\text{Tree}(w) = \langle \xi, \text{Tree}(w'), \text{Tree}(w'') \rangle$$

Level of node $\equiv$ distance to root. **Type of node** $\leadsto$ Peak, Valley, db-rise, db-fall.

Peaks	Valleys	Double rises	Double falls
$\sigma_{j-1} < \sigma_j > \sigma_{j+1}$	$\sigma_{j-1} > \sigma_j < \sigma_{j+1}$	$\sigma_{j-1} < \sigma_j < \sigma_{j+1}$	$\sigma_{j-1} > \sigma_j > \sigma_{j+1}$

In Yule process, particules are determined by the parity of level of the corresponding node in the tree.

Proposition: *Consider the class $\mathcal{X}$ (resp. $\mathcal{Y}$) of permutations such that elements at any odd (resp. even) level are valleys only. Then the exponential generating functions are*

$$X(z) = \mathrm{smh}(z) = -\,\mathrm{sm}(-z), \qquad Y(z) = \mathrm{cmh}(z) = \mathrm{cm}(-z).$$

(Also follows from standard combinatorics, reading off $X' = Y^2, Y' = X^2$.)

6 THE SECOND PERMUTATION MODEL

Definition: An r–*repeated* permutation of size rn is a permutation such that for each j with $0 < j < n$, the elements $jr+1, jr+2, \ldots, jr+r-1$ are all of the same ordinal type (P, V, DR, DF).

Fl-Françon (1989) = a model for Jacobi sn, cn when $r = 2$.
Here: get $r = 3$.
Based on reverse engineering of Conrad's fractions.

6.1 Combinatorial aspects of continued fractions.

Define a *lattice path* aka *Motzkin path*, as a sequence $s = (s_0, s_1, \ldots, s_n)$, satisfying

$$s_0 = s_n = 0, \quad s_j \in \mathbb{Z}_{\geq 0}, \qquad |s_{j+1} - s_j| \in \{-1, 0, +1\}.$$

Let $P(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be the infinite-variable generating function of lattice paths in indeterminates $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, $\mathbf{c} = (c_k)$, with a_k marking an ascent from level k and similarly for descents marked by b_k and for levels marked by c_k.

Here is what Foata calls “the shallow Flajolet Theorem”:

Theorem: *The generating function in infinitely many variables of lattice paths according to step types and levels is*

$$P(\mathbf{a},\mathbf{b},\mathbf{c}) = \cfrac{1}{1-c_0-\cfrac{a_0b_1}{1-c_1-\cfrac{a_1b_2}{1-c_2-\cfrac{a_2b_3}{\ddots}}}}.$$

A *bona fide* generating function obtains by

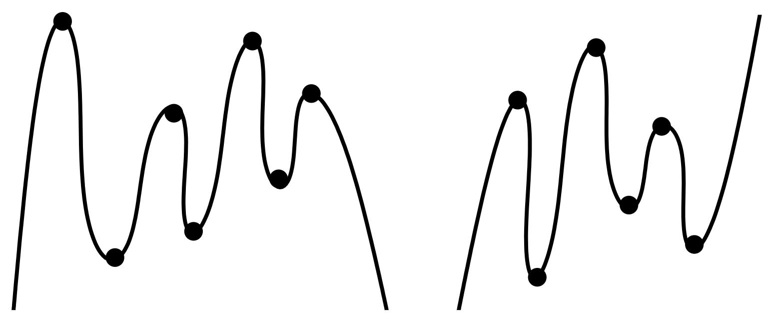
$$a_j \mapsto \alpha_j z, \qquad b_j \mapsto \beta_j z, \qquad c_j \mapsto \gamma_j z$$

The coefficients α, β, γ are referred to as possibilities.

6.2 Lattice paths and permutations.

• A bijection due to Françon-Viennot (1979);
• What V.I. Arnold (2000) calls *snakes*

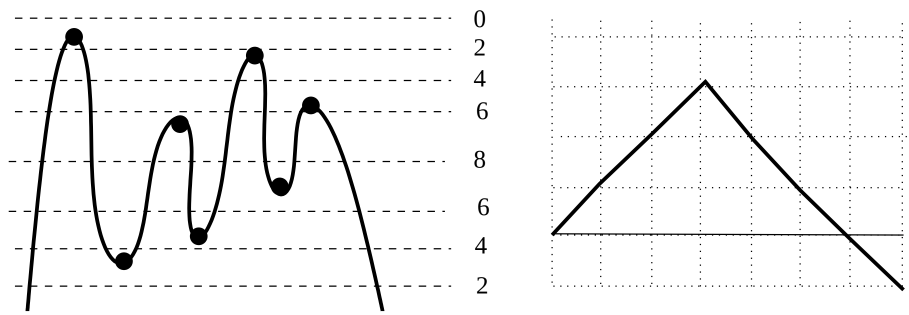
Consider piecewise monotonic smooth functions from $\mathbb{R}$ to $\mathbb{R}$, such that all the critical values are different, and take the equivalence classes up to orientation preserving maps of $\mathbb{R}^2$.



$(-\infty, -\infty)$ $(-\infty, +\infty)$

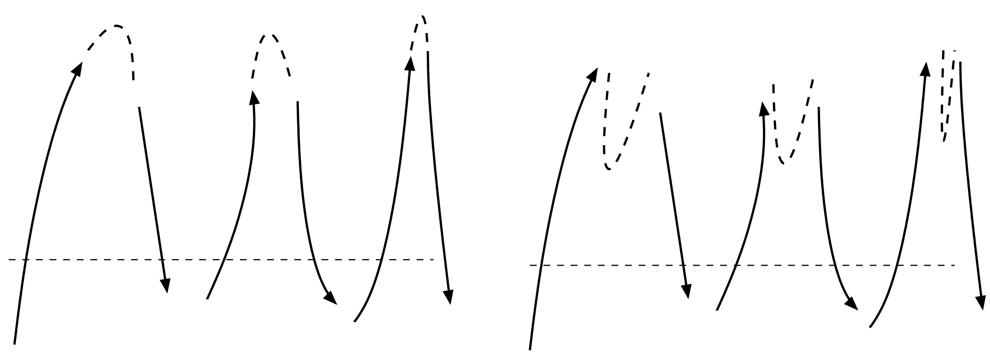
Clearly an equivalence class is an alternating permutation, and by André's theorem the EGFs are $\tan(z) = \dfrac{\sin(z)}{\cos(z)}$, $\sec(z) = \dfrac{1}{\cos(z)}$.

The sweepline algorithm: a snake and its associated Dyck path.



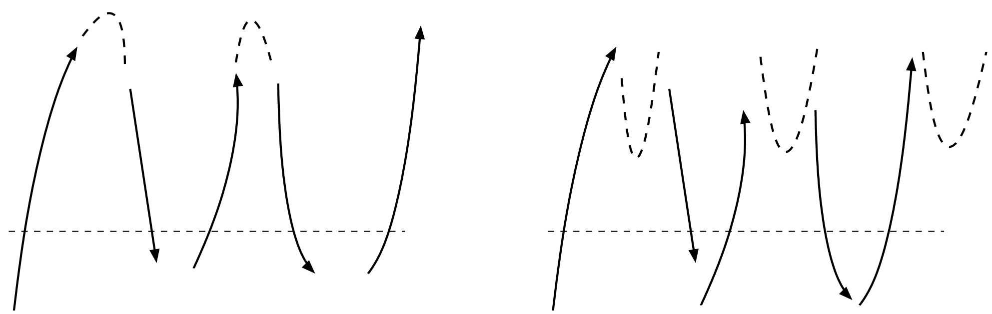
An encoding is obtained by the system of possibilities:

$$\Pi^{\mathrm{odd}}: \qquad \alpha_j = (j+1), \quad \beta_j = (j+1), \quad \gamma_j = 0.$$



For the $(-\infty,-\infty)$ case, possibilities are

$$\Pi^{\text{even}}: \qquad \alpha_j = (j+1), \quad \beta_j = j, \quad \gamma_j = 0.$$



$$\int_0^\infty \tan(zt)e^{-t}\,dt = \cfrac{z}{1-\cfrac{1\cdot 2\,z^2}{1-\cfrac{2\cdot 3\,z^2}{\ddots}}}, \quad \int_0^\infty \sec(zt)e^{-t}\,dt = \cfrac{1}{1-\cfrac{1^2\,z^2}{1-\cfrac{2^2\,z^2}{\ddots}}}.$$

All perms: Modify bijections and take into account all permutations, not just alternating ones: encode double rises and double falls by level steps.

$$\sum_{n=1}^{\infty} n! z^n = \cfrac{z}{1-2z-\cfrac{1\cdot 2\, z^2}{1-4z-\cfrac{2\cdot 3\, z^2}{\ddots}}}, \qquad \sum_{n=0}^{\infty} n! z^n = \cfrac{1}{1-z-\cfrac{1^2\, z^2}{1-3z-\cfrac{2^2\, z^2}{\ddots}}}.$$

These last two fractions are due to Euler.

++ The OGF of snakes of bounded width:

$$\frac{P_h(z)}{Q_h(z)}, \qquad z^h Q_h(1/z) = [t^h](1+t^2)^{-1/2} \exp(z \arctan t).$$

The polynomials are Meixner polynomials (Chihara78).

6.3 The model of 3–repeated permutations.

Proposition: The exponential generating function of 3–repeated permutations bordered by $(-\infty, -\infty)$ is

$$e^{-2z^3}\,\mathrm{smh}(z).$$

For 3–repeated permutations bordered by $(-\infty, +\infty)$, it is

$$e^{-2z^3}\,\mathrm{cmh}(z).$$

Note: By Fl-Françon, 2–repeated + recording rises (cf Eulerian #'s) gives Jacobian sn, cn, dn.

7 PERSPECTIVES & QUESTIONS

- Get full composition of Pólya urn and Yule process by same devices. Cf Aldous' "conceptual proofs".
- Works for all homogenous models with two types of balls. Three? (cf Schett-Dumont for a special elliptic case)
- **Q.** Anything to say about orthogonal polynomials (cf Carlitz for sn, cn)?
- **Q.** What about numerators like k^6 and such? Any combinatorics?
- **Q.** Any possibility of enumerating directly r–repeated perms for $r \geq 4$?
- **Q.** Anything (combinatorially) interesting regarding higher order systems associated to $\mathbf{F}_p$ for $p > 3$? (Not a global uniformization, though.)