

01/04

Matrix Models and Knot Theory

P. Zinn-Justin

References:

- ◇ P. Zinn-Justin, J.-B. Zuber, math-ph/9904019, math-ph/0002020, math-ph/0303049.
- ◇ P. Zinn-Justin, math-ph/9910010, math-ph/0106005.
- ◇ J. Jacobsen, P. Zinn-Justin, math-ph/0102015, math-ph/0104009.
- ◇ G. Schaeffer, P. Zinn-Justin, math-ph/0304034.

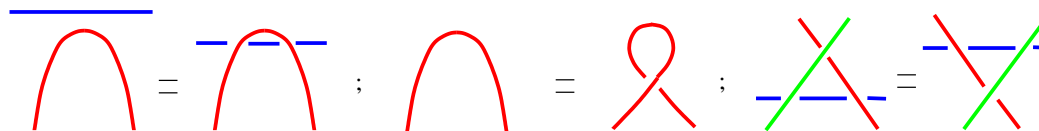
- Classification and Enumeration of Knots, Links, Tangles.
- Feynman diagrams. $O(n)$ matrix model and renormalization.
- Universality and conjectures on asymptotic counting.
- Algorithms: (i) Transfer Matrix (ii) Random Sampling

A bit of History...

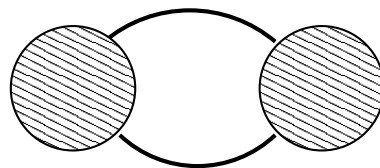
- Knots represented by their projection: **diagrams** (Tait, 1876):



- Two diagrams represent the knot/link/tangle iff they are related by a sequence of Reidemeister moves: (Reidemeister, 1932)



- All knots are connected sums of **prime knots** (Schubert, 1949):

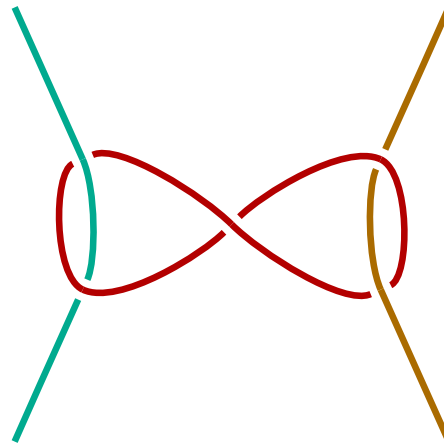


Knots, links and tangles

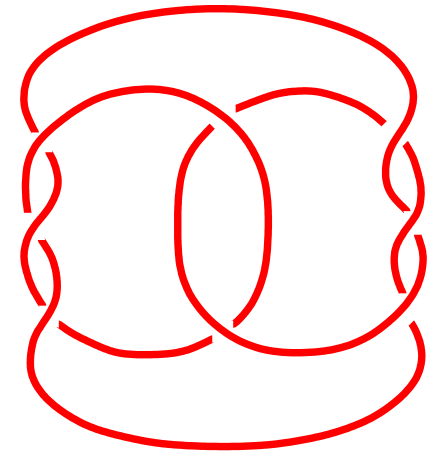
Links are collections of knots:



Tangles have strings coming out:

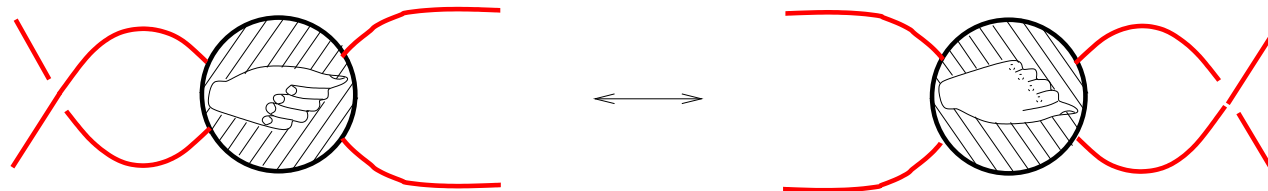


Alternating vs non-alternating:

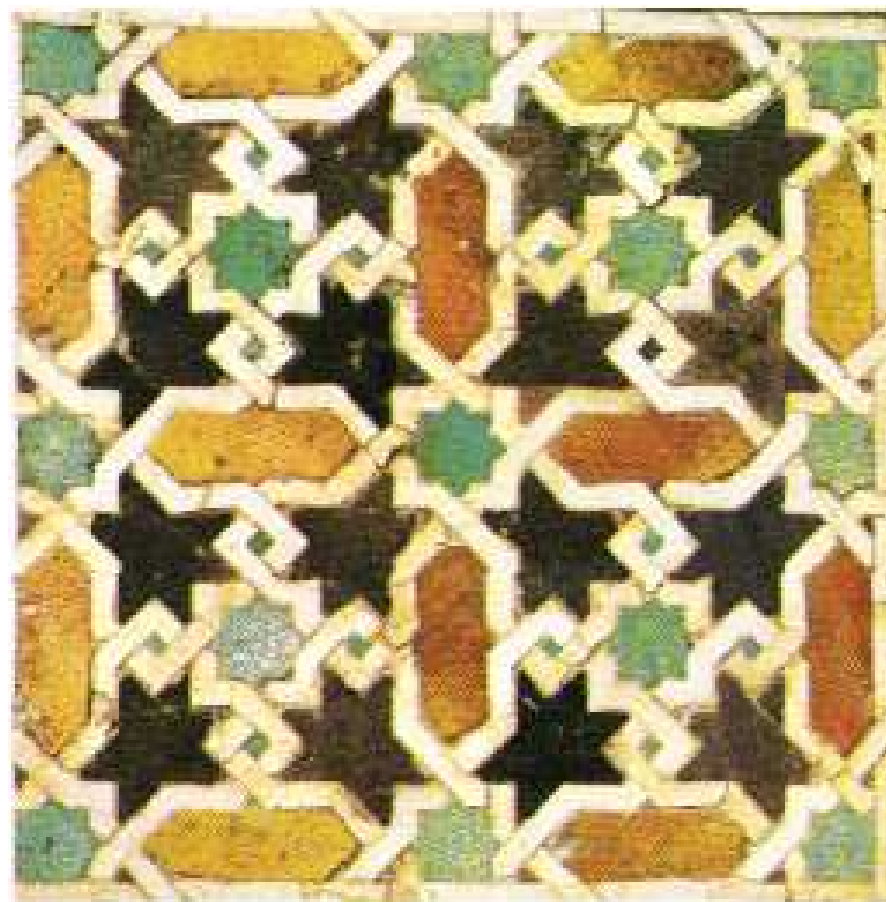
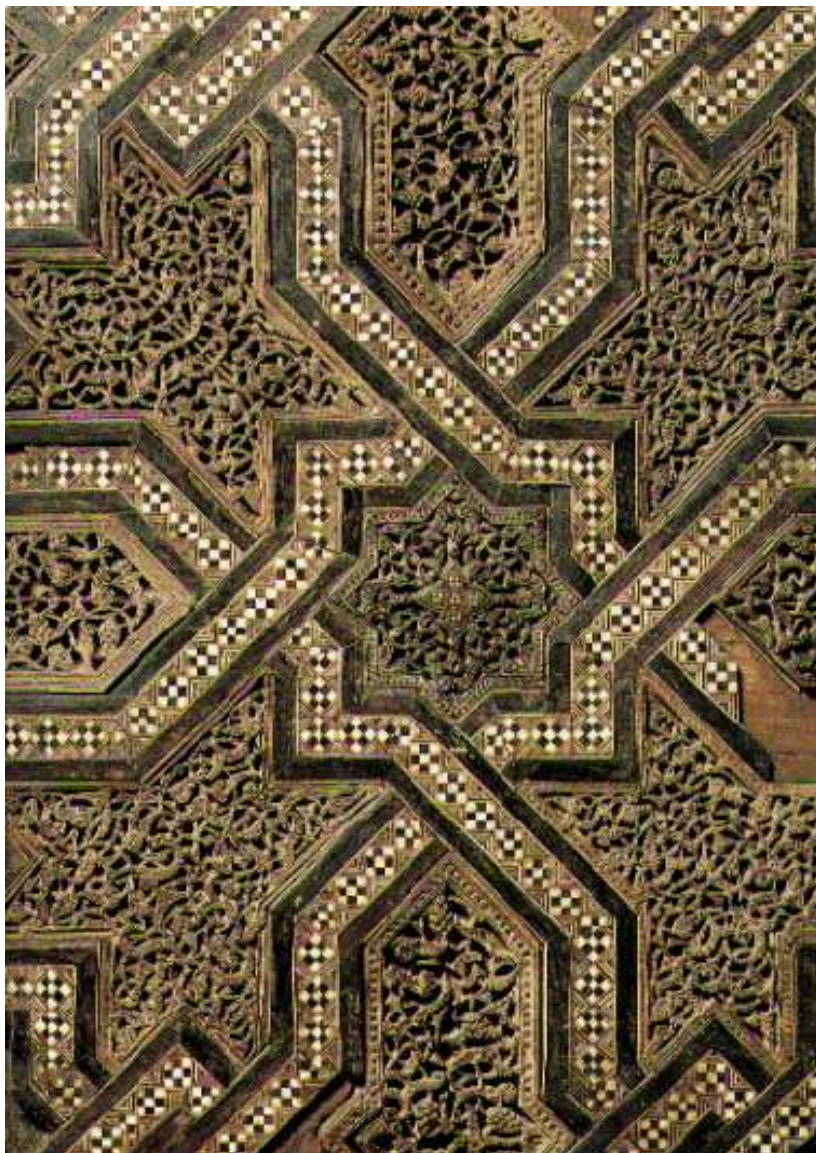


Tait's flying conjecture: (Tait, 1898)

Two reduced alternating diagrams represent the same object iff they are related by a sequence of flypes:



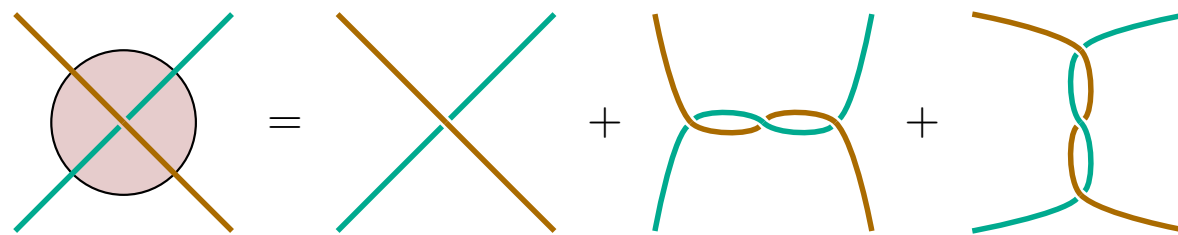
Proved by Menasco and Thistlethwaite ('91).



What is the problem?

We want to enumerate prime alternating tangles with given number of components and crossings:

$$\Gamma(n, g) = \sum_{k,p=1}^{\infty} a_{k;p} g^p n^k$$

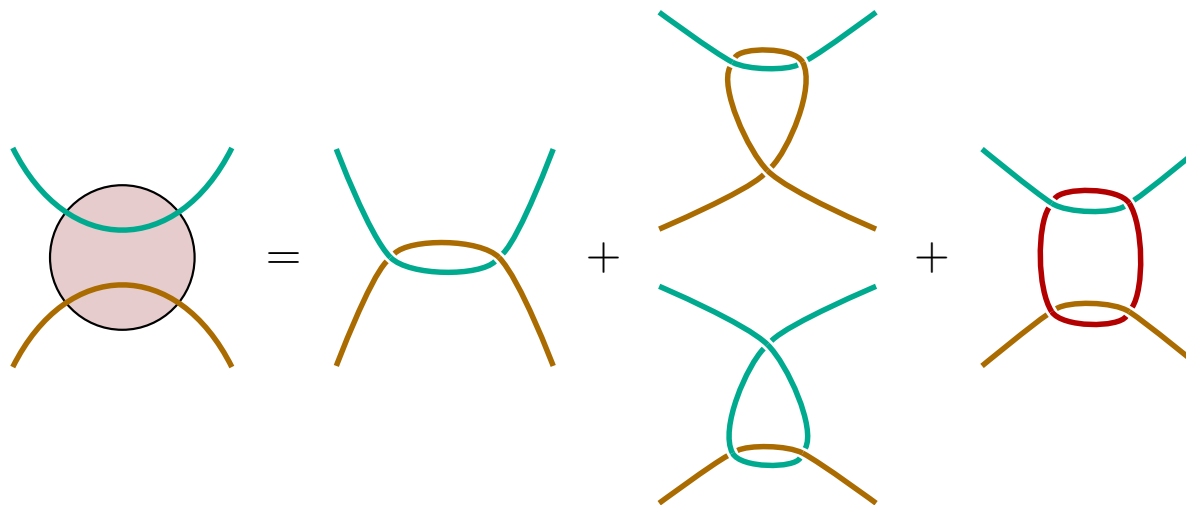


Example:

$$\Gamma_1(n, g) = g + g^3 + g^3 + \dots$$

tangles with

four external legs:



$$\Gamma_2(n, g) = g^2 + g^3 + ng^4 + \dots$$

Matrix Integrals: Feynman Rules

(JBZ)

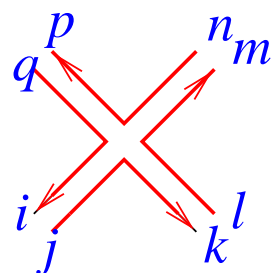
$N \times N$ Hermitean matrices M ,

$$dM = \prod_i dM_{ii} \prod_{i < j} d\Re M_{ij} d\Im M_{ij}$$

$$Z = \int dM e^{N[-\frac{1}{2} \text{tr} M^2 + \frac{g}{4} \text{tr} M^4]}$$

Feynman rules:

propagator  $= \frac{1}{N} \delta_{il} \delta_{jk}$

4-valent vertex  $= gN \delta_{jk} \delta_{lm} \delta_{np} \delta_{qi}$

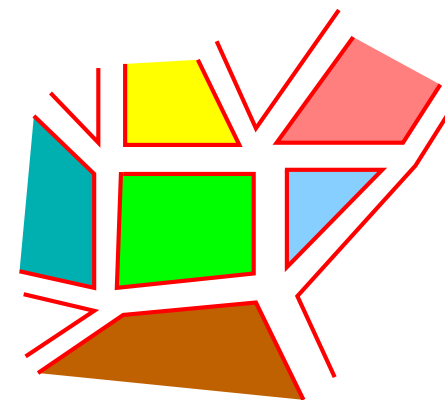
Count powers of N in a connected diagram:

- each vertex $\rightarrow N$;
- each double line $\rightarrow N^{-1}$;
- each loop $\rightarrow N$.

$$\#\text{vert.} - \#\text{lines} + \#\text{loops} = \chi_{\text{Euler}}(\Sigma)$$

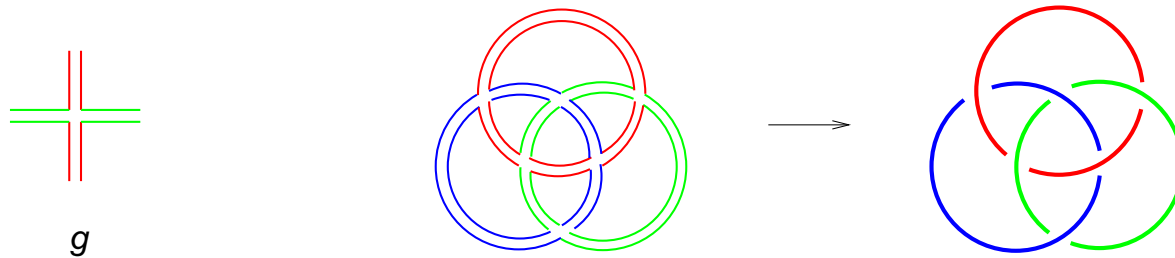
't Hooft (1974):

$$\log Z = \sum_{\text{conn. surf. } \Sigma} N^{2-2\text{genus}(\Sigma)} \frac{g^{\#\text{vert.}(\Sigma)}}{\text{symm. factor}}$$



A Matrix Model for Alternating Link Diagrams

$$Z^{(N)}(n, g) = \int \prod_{a=1}^n dM_a e^{N \operatorname{tr} \left(-\frac{1}{2} M_a^2 + \frac{g}{4} (M_a M_b)^2 \right)}$$



The large N free energy $F(n, g)$ and correlation functions are double generating series in n, g .

$F(n, g)$ counts link diagrams (weighted by their symmetry factors):

$$F(n, g) = \lim_{N \rightarrow \infty} \frac{\log Z^{(N)}(n, g)}{N^2} = \sum_{k,p=1}^{\infty} f_{k;p} g^p n^k$$

The correlation functions count tangle diagrams:

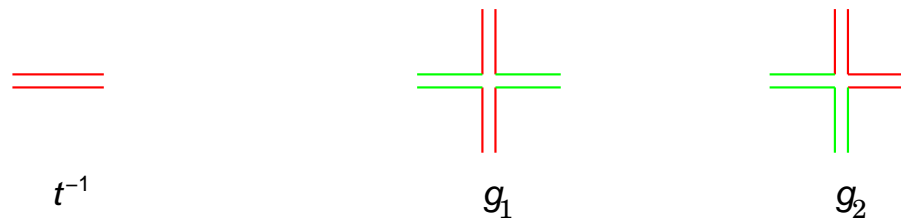
$$\lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \operatorname{tr}(M_1 M_2 M_3 M_2 M_1 M_3) \right\rangle_c = \text{Diagram}$$

From tangle diagrams to tangles: Renormalization

General idea: removal of the redundancy associated to multiple equivalent diagrams acts like a “finite renormalization” on the model.

- Reduced diagrams \Rightarrow renormalization of the quadratic term in the action.
- Taking into account the flying equivalence renormalizes the quartic term. However, there are **two** four-vertex interactions compatible with the $O(n)$ -symmetry \rightarrow more general $O(n)$ model:

$$Z^{(N)}(n, t, g_1, g_2) = \int \prod_{a=1}^n dM_a e^{N \operatorname{tr} \left[-\frac{t}{2} M_a^2 + \left(\frac{g_1}{4} M_a M_b M_a M_b + \frac{g_2}{2} M_a M_a M_b M_b \right) \right]}$$



t , g_1 and g_2 are functions of the renormalized coupling constant g , chosen such that the correlation functions are the appropriate generating series in g of the number of alternating links.

Exactly solved cases

- $n = 1$: the counting of alternating tangles, and more

Usual one-matrix model:

$$Z^{(N)}(t, g_0) = \int dM e^N \operatorname{tr} \left(-\frac{t}{2} M^2 + \frac{g_0}{4} M^4 \right)$$

with $g_0 = g_1 + 2g_2$.

“Renormalization” equations recombine into a fifth degree equation:

$$32 - 64A + 32A^2 - 4 \frac{1 + 2g - g^2}{1 - g} A^3 + 6gA^4 - gA^5 = 0$$

Correlation functions are given in terms of its solution. In particular, if $\langle \frac{1}{N} \operatorname{tr} M^{2\ell} \rangle_c = \sum_{p=0}^{\infty} a_p g^p$ is the generating function of **prime alternating tangles with 2ℓ legs**, then

$$a_p \stackrel{p \rightarrow \infty}{\sim} \text{cst } g_c^{-p} p^{-5/2}$$

with $g_c = \frac{\sqrt{21001} - 101}{270}$ ($g_c^{-1} \approx 6.1479$). ($\ell = 2$: Sundberg & Thistlethwaite '98)

\Rightarrow The number f_p of prime alternating links grows like $f_p \sim \text{cst } g_c^{-p} p^{-7/2}$

(Schaeffer & Kunz-Jacques, '01)

- $n = -2 \dots$

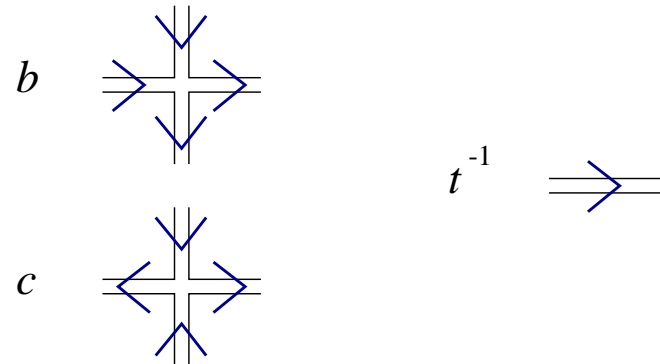
- $n = 2$: the counting of oriented alternating tangles (P.Z-J. & J.-B. Zuber)

$$Z^{(N)}(t, g_1, g_2) = \int dM_1 dM_2 e^{N \operatorname{tr} \left[-\frac{t}{2}(M_1^2 + M_2^2) + \frac{g_1 + 2g_2}{4}(M_1^4 + M_2^4) + \frac{g_1}{2}(M_1 M_2)^2 + g_2 M_1^2 M_2^2 \right]}$$

Introduce a complex matrix $X = \frac{1}{\sqrt{2}}(M_1 + iM_2)$:

$$Z^{(N)}(t, b, c) = \int dX dX^\dagger e^{N \operatorname{tr} \left(-t X X^\dagger + b X^2 X^{\dagger 2} + \frac{1}{2} c (X X^\dagger)^2 \right)}$$

with $b = g_1 + g_2$ and $c = 2g_2$. Feynman rules:



Six-vertex model on random lattices. This model has been exactly solved (P.Z-J.; I. Kostov).

⇒ Generating function of (prime, alternating) tangles given by transcendental equation. Asymptotics:

$$a_p \stackrel{p \rightarrow \infty}{\sim} \text{cst } g_c^{-p} p^{-2} (\log p)^{-2}$$

with $g_c^{-1} \approx 6.2832$.

Conjectures on the asymptotic behavior

Links \sim discretized surfaces with random geometry \rightarrow 2D quantum gravity...

Conjecture: For $|n| < 2$, the matrix model is in the universality class of a 2D field theory with spontaneously broken $O(n)$ symmetry, coupled to gravity.

The large size limit is described by a CFT with $c = n - 1 \Rightarrow$ (KPZ)

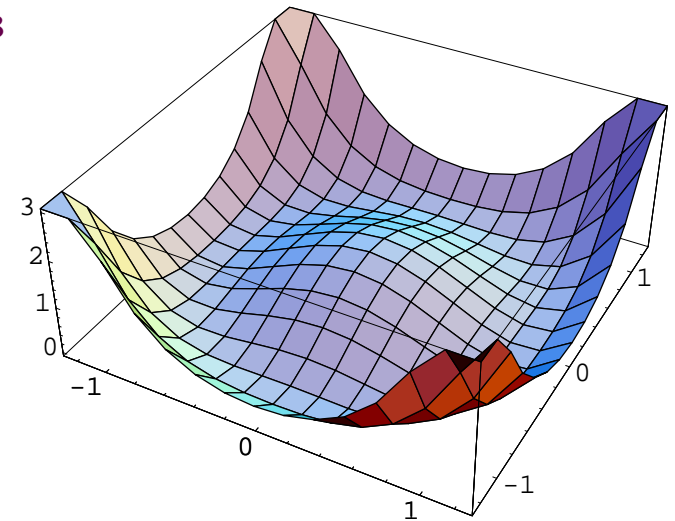
$$a_p(n) \sim \text{cst}(n) g_c(n)^{-p} p^{\gamma(n)-2}$$

$$f_p(n) \sim \text{cst}(n) g_c(n)^{-p} p^{\gamma(n)-3}$$

$$\gamma = \frac{c - 1 - \sqrt{(1 - c)(25 - c)}}{12}$$

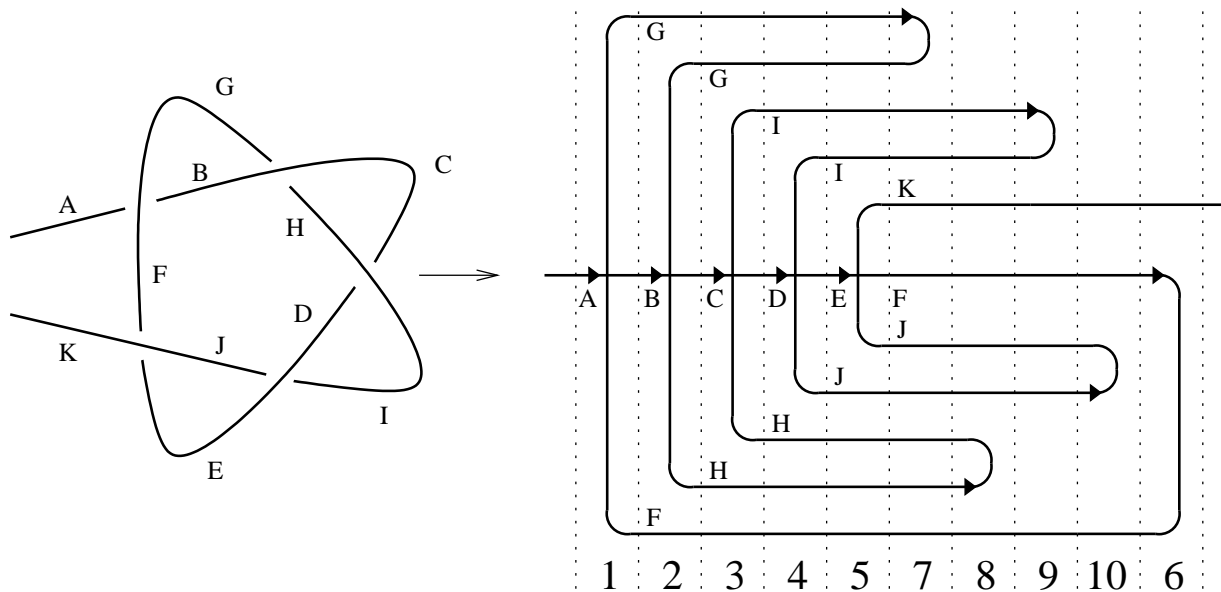
In particular, knots correspond to the limit $n \rightarrow 0$:

$$f_p(0) \sim \text{cst} g_c^{-p} p^{-\frac{19+\sqrt{13}}{6}}$$

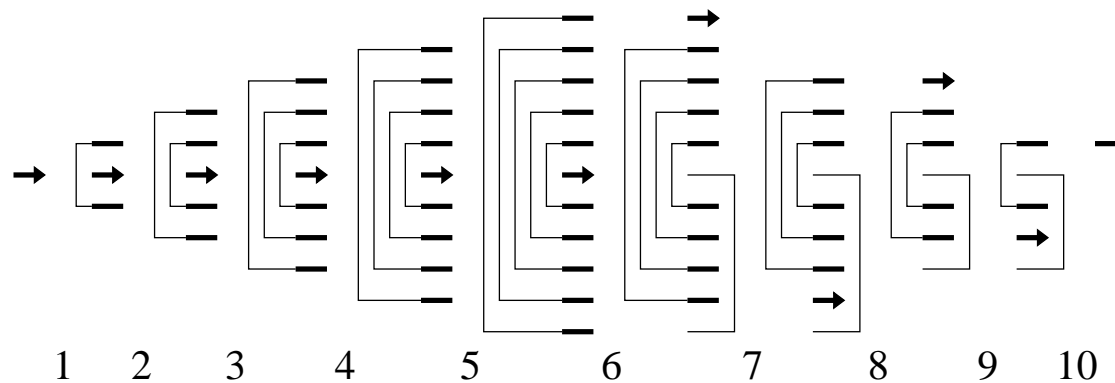


A Transfer Matrix for tangle diagrams

(J. Jacobsen and P. Z.-J., '01) For knots, follow the string as it winds around itself:



Structure of states:



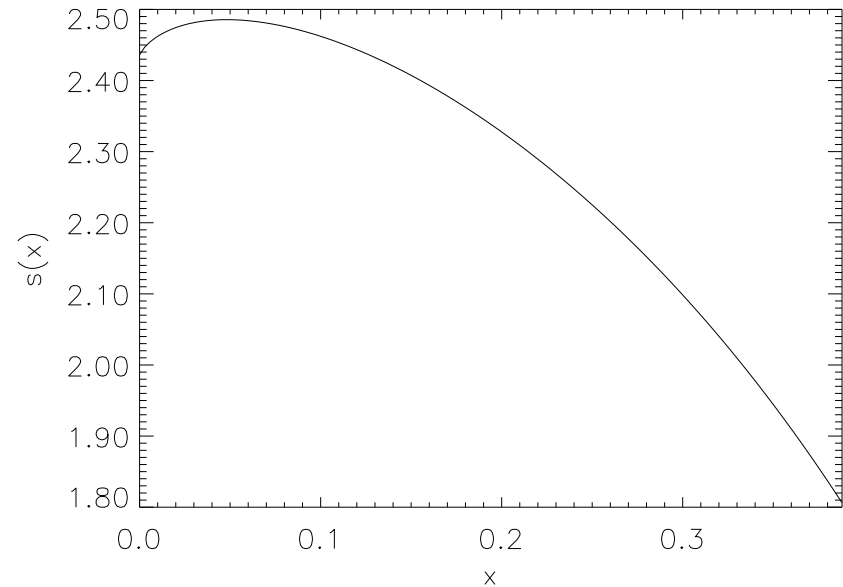
Similar but more complicated construction for links.

Numerical results

• Sample Table:

p^k	Γ_1					
	0	1	2	3	4	5 6
1	1					
2	0					
3	2					
4	2					
5	6	3				
6	30	2				
7	62	40	2			
8	382	106	2			
9	1338	548	83	2		
10	6216	2968	194	2		
11	29656	11966	2160	124	2	
12	131316	71422	9554	316	2	
13	669138	328376	58985	5189	184	2
14	3156172	1796974	347038	22454	478	2
15	16032652	9298054	1864884	193658	10428	260 2

• Bulk entropy: $a_{k;p} \approx e^{p s(x=k/p)}$



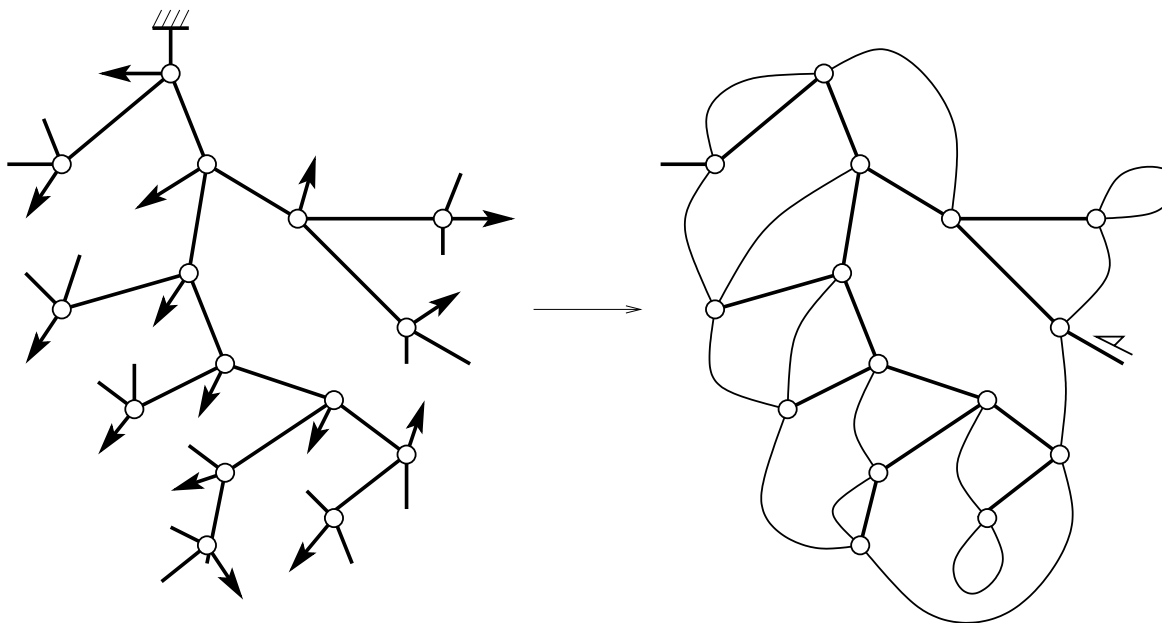
• Critical exponent γ : marginal agreement. low accuracy because of logarithmic corrections?

Monte Carlo: random sampling of planar maps

(G. Schaeffer and P. Z.-J., '03)

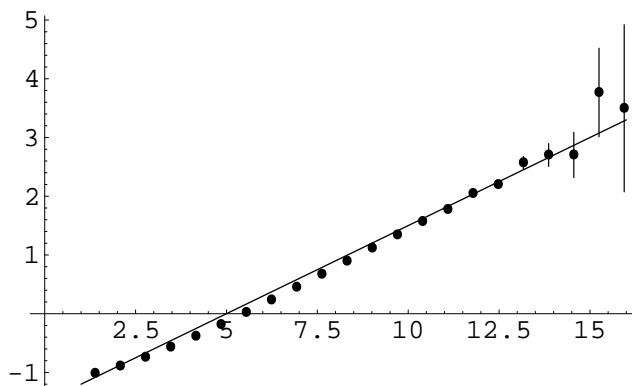
Schaeffer's bijection

between trees and planar maps:



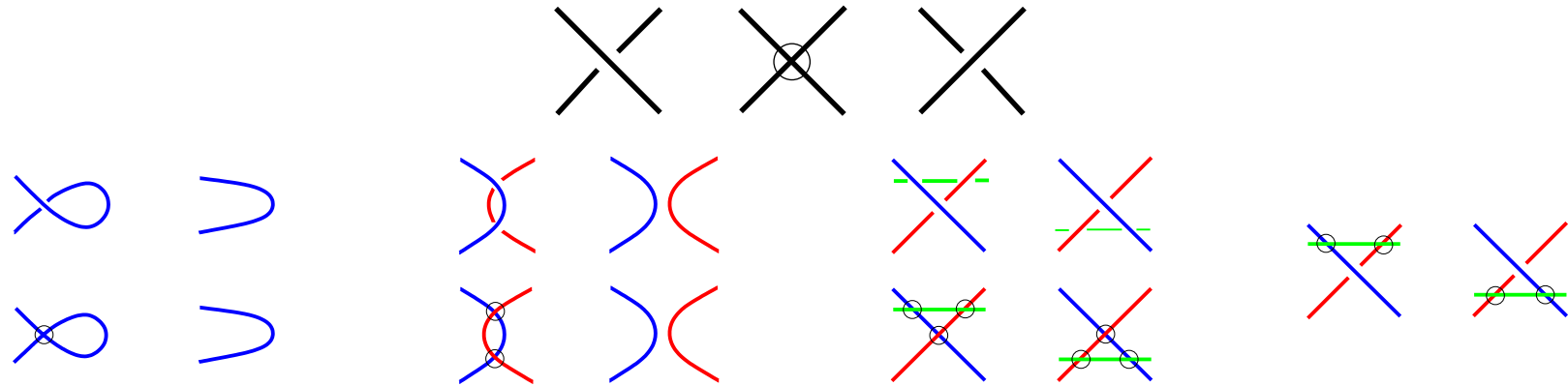
Results in an algorithm to produce random planar maps in linear time \rightarrow up to $p = 10^7$ vertices.

Test quantity: $\gamma' \equiv \frac{d\gamma}{dn}|_{n=1} = 3/10$ according to the conjecture. Very good agreement:

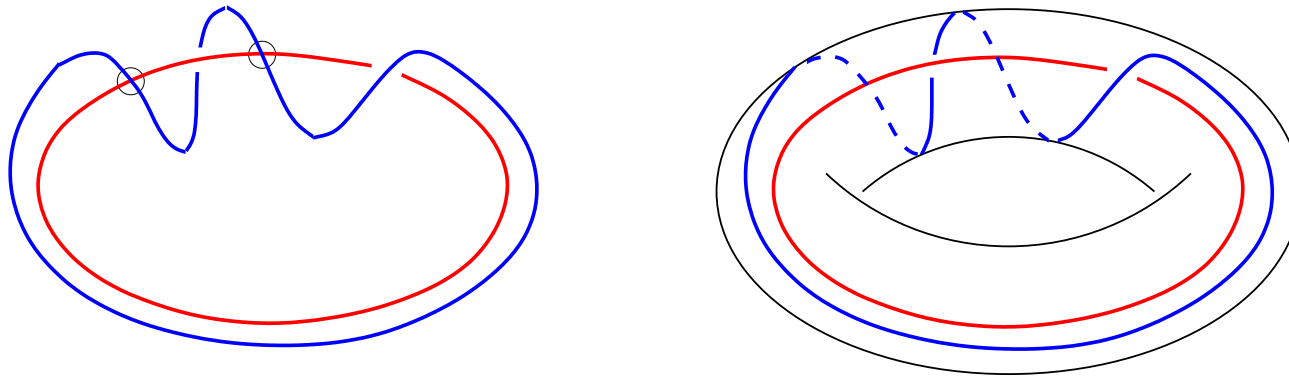


Virtual links

Kauffman's definition via virtual diagrams and virtual Reidemeister moves:



Better to imagine links in **thickened surfaces** $\Sigma \times I$ (up to orientation-preserving homeomorphisms of the surface Σ)

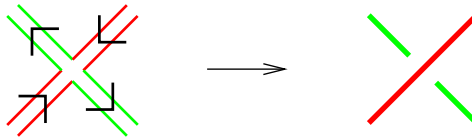


\Rightarrow relation to matrix models!

- Virtual alternating links and tangles

NB: in genus > 0 , not every quadrangulation is bipartite!!! \Rightarrow complex matrix model:

$$Z^{(N)}(n, g) = \int \prod_{a=1}^n dM_a dM_a^\dagger e^{N \operatorname{tr} \left(-\frac{1}{2} M_a M_a^\dagger + \frac{g}{4} (M_a M_b^\dagger)^2 \right)}$$



g

$$\log Z^{(N)}(n, g) = \sum_{h \geq 0, k \geq 1, p \geq 1} f_{k;p}^{(h)} N^{2-2h} g^p n^k$$

triple generating function of virtual alternating link diagrams.

“Renormalization” ?

Conjecture: Tait’s flype conjecture also holds for virtual alternating links and tangles.

i.e. the only moves needed are *planar* flypes.

\rightarrow Some exact results. Example: $n = 1$. The number of prime virtual alternating links of genus h

$$f_p^{(h)} \stackrel{p \rightarrow \infty}{\sim} c g_c^{-p} p^{5/2(h-1)-1}$$

Table of prime virtual alternating links

<http://ipnweb.in2p3.fr/lptms/membres/pzinn/virtlinks> (or google: Paul Zinn-Justin)

Generation and computation of invariants for low order tangles and links. The conjecture is checked for prime alternating tangles up to order 5 (first 13010 tangles).

Example: $p = 4$ links

