01/04

Matrix Models and Knot Theory

P. Zinn-Justin

References:
◇ P. Zinn-Justin, J.-B. Zuber, math-ph/9904019, math-ph/0002020, math-ph/0303049.
◇ P. Zinn-Justin, math-ph/9910010, math-ph/0106005.
◇ J. Jacobsen, P. Zinn-Justin, math-ph/0102015, math-ph/0104009.
◇ G. Schaeffer, P. Zinn-Justin, math-ph/0304034.

- Classification and Enumeration of Knots, Links, Tangles.
- Feynman diagrams. $O(n)$ matrix model and renormalization.
- Universality and conjectures on asymptotic counting.
- Algorithms: (i) Transfer Matrix (ii) Random Sampling

A bit of History…

- Knots represented by their projection: **diagrams** (Tait, 1876):

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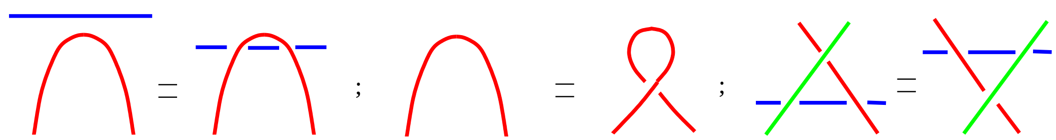
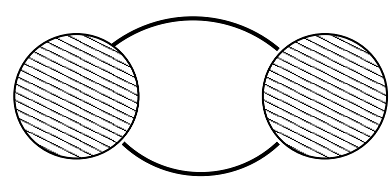
- Two diagrams represent the knot/link/tangle iff they are related by a sequence of Reidemeister moves: (Reidemeister, 1932)



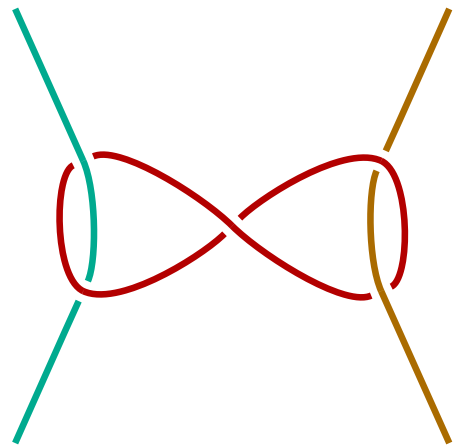
- All knots are connected sums of **prime knots** (Schubert, 1949):



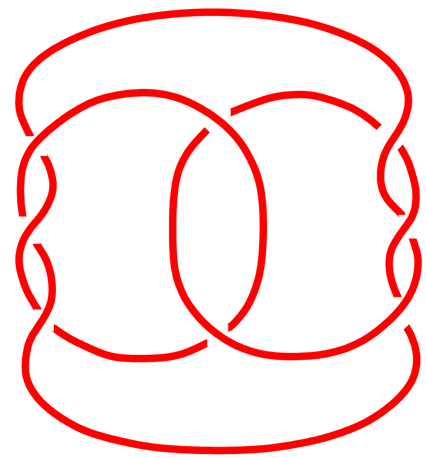
Knots, links and tangles

Links are collections of knots:

Tangles have strings coming out:

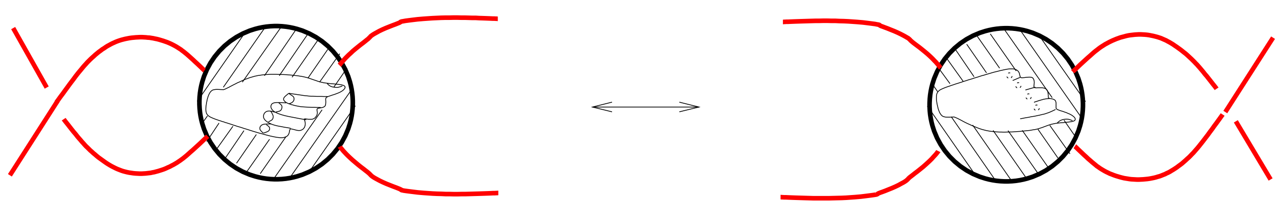


Alternating vs non-alternating:

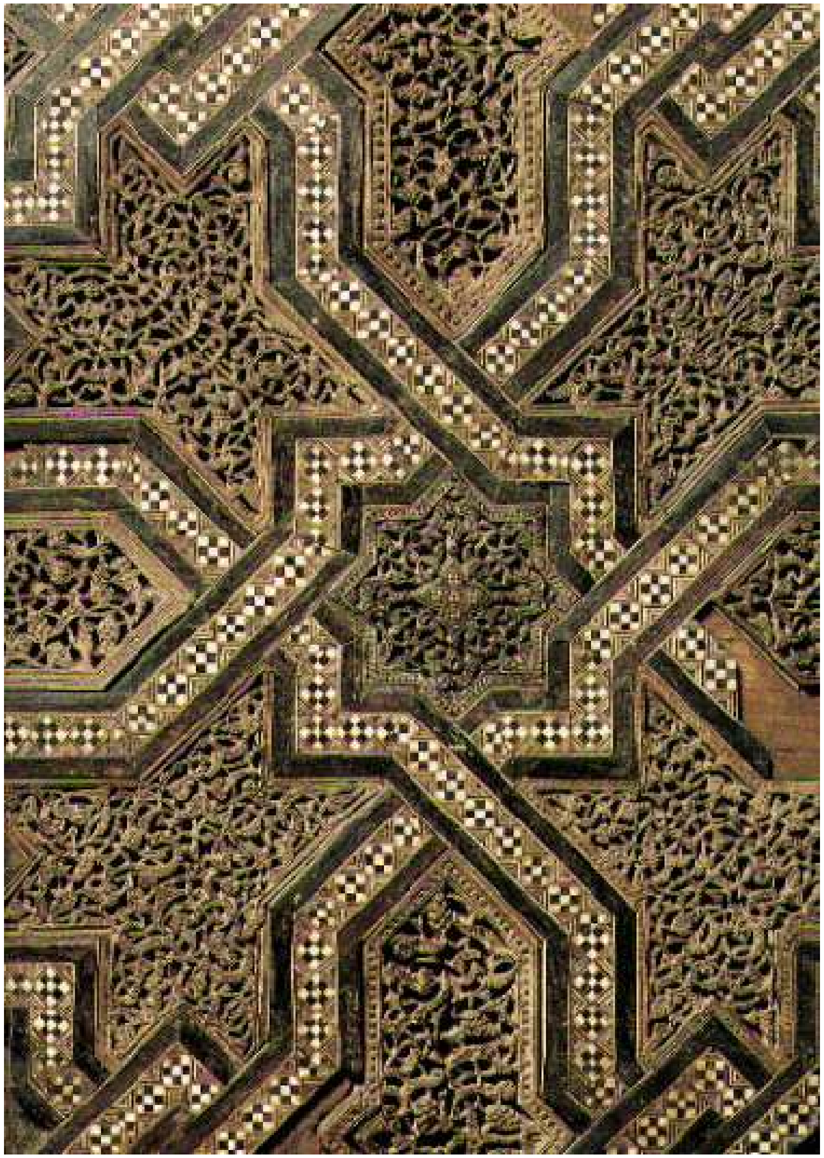


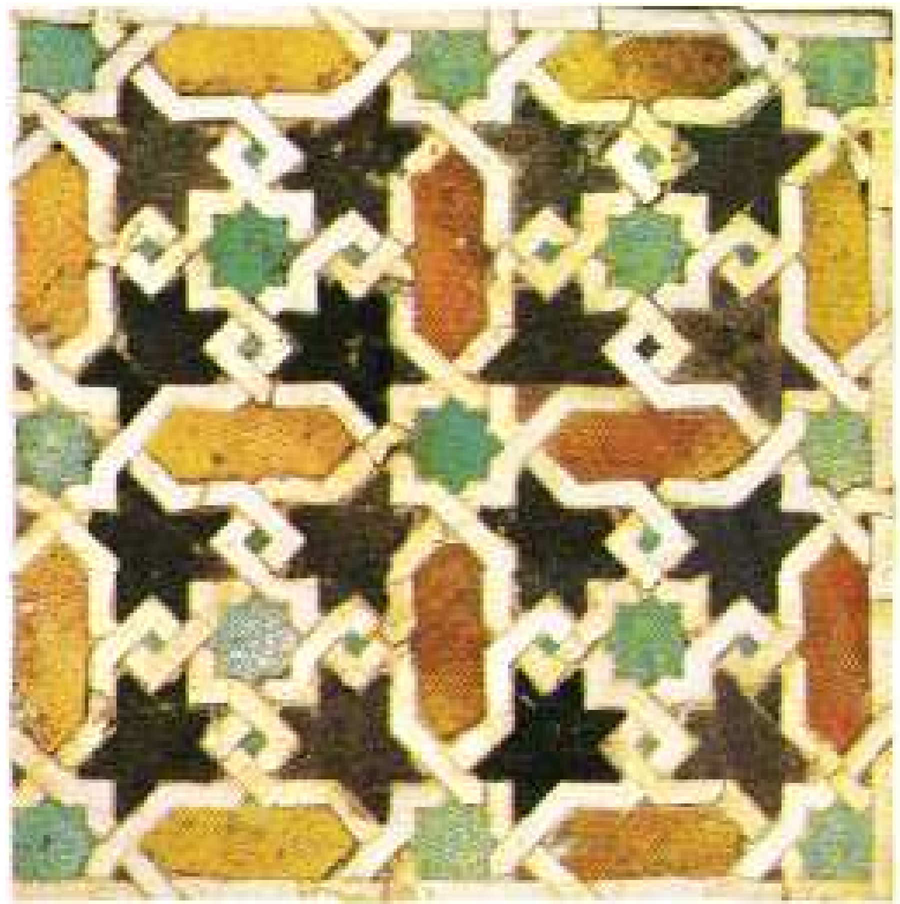
Tait's flyping conjecture: (Tait, 1898)

Two reduced alternating diagrams represent the same object iff they are related by a sequence of flypes:



Proved by Menasco and Thisthlethwaite ('91).

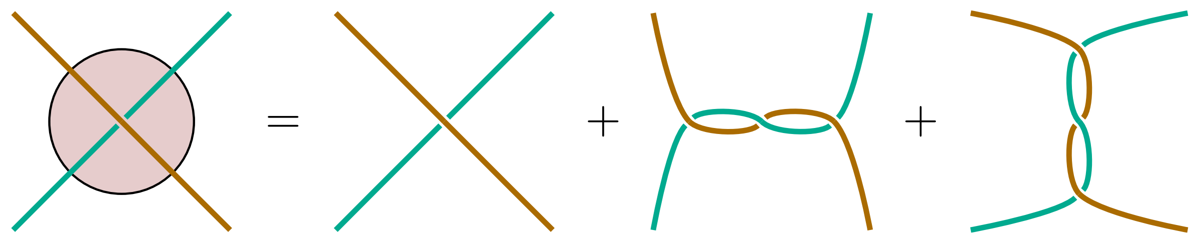




What is the problem?

We want to enumerate prime alternating tangles with given number of components and crossings:

$$\Gamma(n,g)=\sum_{k,p=1}^{\infty} a_{k;p}\, g^p n^k$$

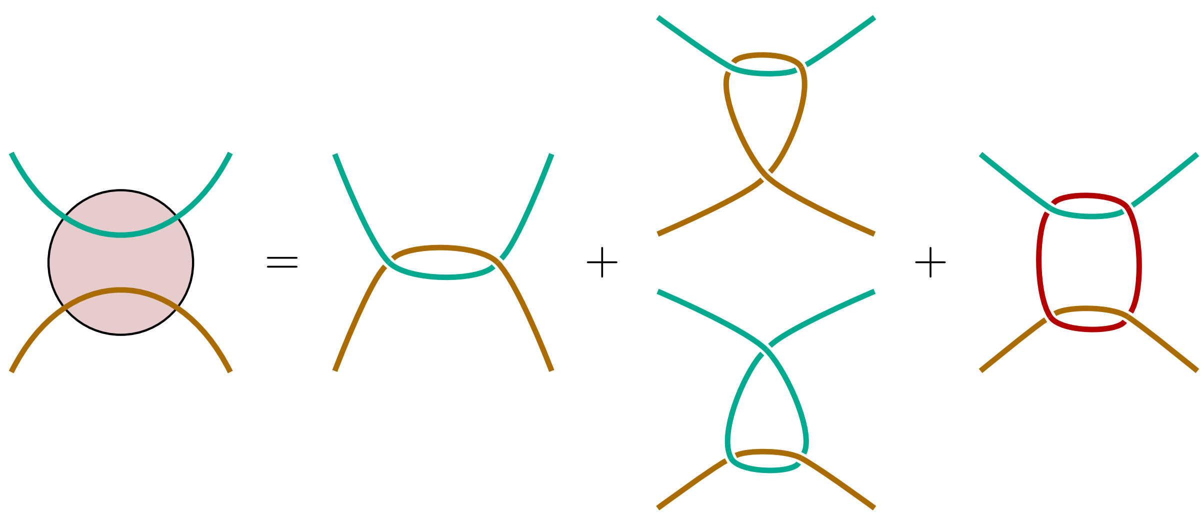


Example:

$$\Gamma_1(n,g) \quad = \quad g \quad + \quad g^3 \quad + \quad g^3 \quad + \cdots$$

tangles with

four external legs:



$$\Gamma_2(n,g) \quad = \quad g^2 \quad + \quad g^3 \quad + \quad ng^4 \quad + \cdots$$

Matrix Integrals: Feynman Rules (JBZ)

$N \times N$ Hermitean matrices M,

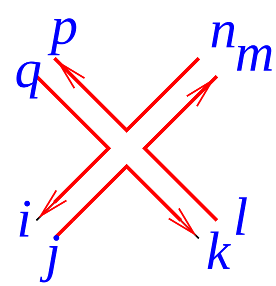
$dM = \prod_i dM_{ii} \prod_{i<j} d\Re e M_{ij}\, d\Im m M_{ij}$

$$Z = \int dM \mathrm{e}^{N[-\frac{1}{2}\operatorname{tr} M^2 + \frac{g}{4}\operatorname{tr} M^4]}$$

Feynman rules:

propagator 

$= \frac{1}{N}\delta_{i\ell}\delta_{jk}$

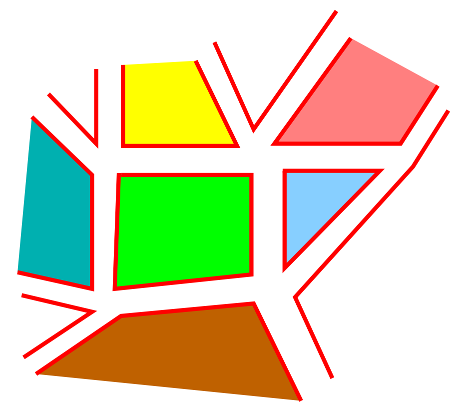
4-valent vertex 

$= gN\delta_{jk}\delta_{\ell m}\delta_{np}\delta_{qi}$

Count powers of N in a connected diagram:

- each vertex $\rightarrow N$;
- each double line $\rightarrow N^{-1}$;
- each loop $\rightarrow N$.

$$\#\text{vert.} - \#\text{lines} + \#\text{loops} = \chi_{\text{Euler}}(\Sigma)$$

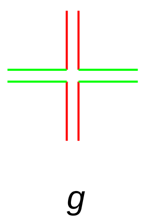


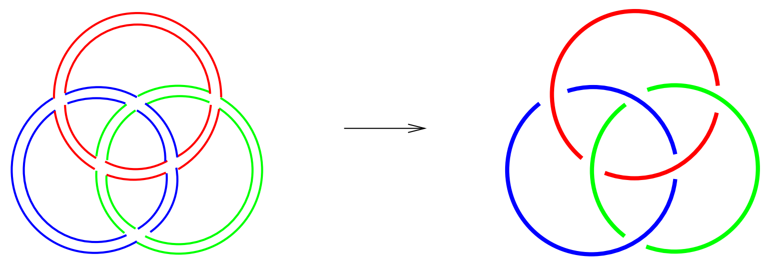
't Hooft (1974):

$$\log Z = \sum_{\text{conn. surf.}\Sigma} N^{2-2\text{genus}(\Sigma)} \frac{g^{\#\text{vert.}(\Sigma)}}{\text{symm. factor}}$$

A Matrix Model for Alternating Link Diagrams

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} \mathrm{d}M_a \, \mathrm{e}^{N \, \mathrm{tr}\left(-\frac{1}{2}M_a^2 + \frac{g}{4}(M_a M_b)^2\right)}$$





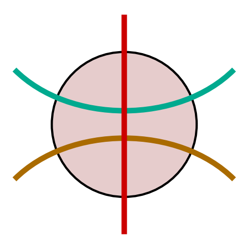
The large N free energy $F(n,g)$ and correlation functions are double generating series in n, g.

$F(n,g)$ counts link diagrams (weighted by their symmetry factors):

$$F(n,g) = \lim_{N\to\infty} \frac{\log Z^{(N)}(n,g)}{N^2} = \sum_{k,p=1}^{\infty} f_{k;p} g^p n^k$$

The correlation functions count tangle diagrams:

$$\lim_{N\to\infty} \left\langle \frac{1}{N} \mathrm{tr}(M_1 M_2 M_3 M_2 M_1 M_3) \right\rangle_c =$$

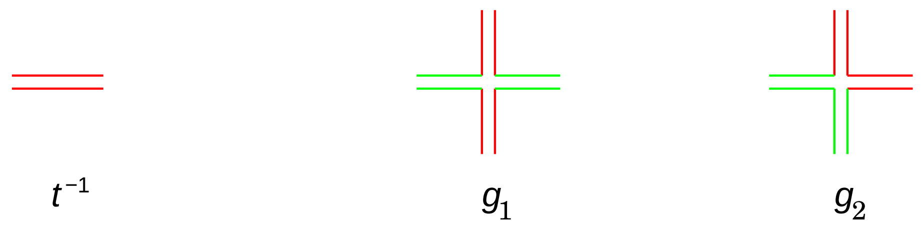


From tangle diagrams to tangles: Renormalization

General idea: removal of the redundancy associated to multiple equivalent diagrams acts like a "finite renormalization" on the model.

- Reduced diagrams $\Rightarrow$ renormalization of the quadratic term in the action.
- Taking into account the flyping equivalence renormalizes the quartic term. However, there are **two** four-vertex interactions compatible with the $O(n)$-symmetry $\rightarrow$ more general $O(n)$ model:

$$Z^{(N)}(n,t,g_1,g_2) = \int \prod_{a=1}^{n} \mathrm{d}M_a \mathrm{e}^{N \operatorname{tr}\left[-\frac{t}{2}M_a^2 + \left(\frac{g_1}{4}M_aM_bM_aM_b + \frac{g_2}{2}M_aM_aM_bM_b\right)\right]}$$



t, g_1 and g_2 are functions of the renormalized coupling constant g, chosen such that the correlation functions are the appropriate generating series in g of the number of alternating links.

Exactly solved cases

- $n = 1$: the counting of alternating tangles, and more

Usual one-matrix model:

$$Z^{(N)}(t, g_0) = \int \mathrm{d}M \, \mathrm{e}^{N \, \mathrm{tr} \left(-\frac{t}{2} M^2 + \frac{g_0}{4} M^4\right)}$$

with $g_0 = g_1 + 2g_2$.

"Renormalization" equations recombine into a fifth degree equation:

$$32 - 64A + 32A^2 - 4\frac{1 + 2g - g^2}{1 - g} A^3 + 6gA^4 - gA^5 = 0$$

Correlation functions are given in terms of its solution. In particular, if $\left\langle \frac{1}{N} \mathrm{tr}\, M^{2\ell} \right\rangle_c = \sum_{p=0}^{\infty} a_p g^p$ is the generating function of **prime alternating tangles with 2ℓ legs**, then

$$a_p \overset{p \to \infty}{\sim} cst \, g_c{}^{-p} p^{-5/2}$$

with $g_c = \frac{\sqrt{21001} - 101}{270}$ ($g_c^{-1} \approx 6.1479$). ($\ell = 2$: Sundberg & Thistlethwaite '98)

$\Rightarrow$ The number f_p of prime alternating links grows like $f_p \sim cst \, g_c{}^{-p} p^{-7/2}$

(Schaeffer & Kunz-Jacques, '01)

- $n = -2$ ...

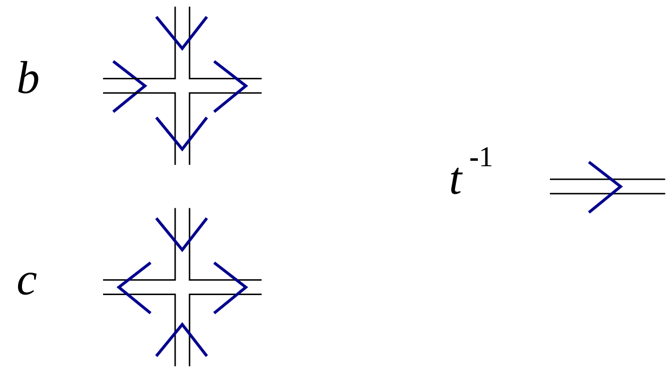
- $n = 2$: the counting of oriented alternating tangles (P.Z-J. & J.-B. Zuber)

$$Z^{(N)}(t, g_1, g_2) = \int dM_1 dM_2 \, e^{N \operatorname{tr} \left[-\frac{t}{2}(M_1^2 + M_2^2) + \frac{g_1 + 2g_2}{4}(M_1^4 + M_2^4) + \frac{g_1}{2}(M_1 M_2)^2 + g_2 M_1^2 M_2^2 \right]}$$

Introduce a complex matrix $X = \frac{1}{\sqrt{2}}(M_1 + iM_2)$:

$$Z^{(N)}(t, b, c) = \int dX dX^\dagger \, e^{N \operatorname{tr} \left(-tXX^\dagger + bX^2 X^{\dagger 2} + \frac{1}{2}c(XX^\dagger)^2 \right)}$$

with $b = g_1 + g_2$ and $c = 2g_2$. Feynman rules:



Six-vertex model on random lattices. This model has been exactly solved (P.Z-J.; I. Kostov).

$\Rightarrow$ Generating function of (prime, alternating) tangles given by transcendental equation. Asymptotics:

$$a_p \overset{p\to\infty}{\sim} \text{cst}\, g_c^{-p} p^{-2} (\log p)^{-2}$$

with $g_c^{-1} \approx 6.2832$.

Conjectures on the asymptotic behavior

Links $\sim$ discretized surfaces with random geometry $\rightarrow$ 2D quantum gravity...

Conjecture: For $|n| < 2$, the matrix model is in the universality class of a 2D field theory with spontaneously broken $O(n)$ symmetry, coupled to gravity.

The large size limit is described by a CFT with $c = n - 1 \Rightarrow$ (KPZ)

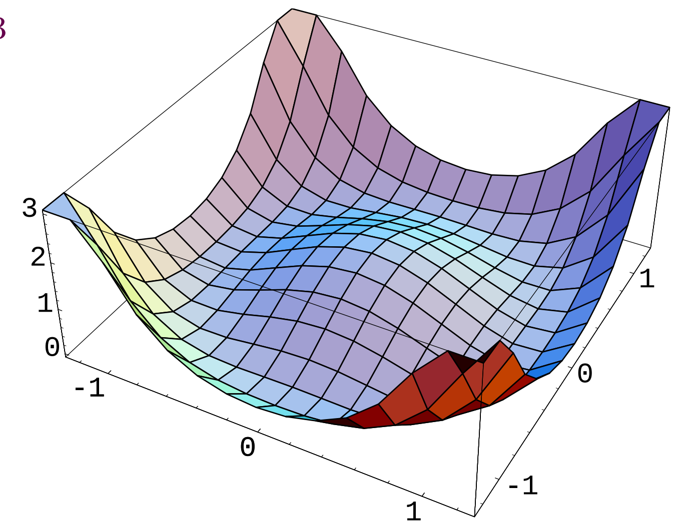
$$a_p(n) \sim \mathrm{cst(n)}\, g_c(n)^{-p}\, p^{\gamma(n)-2}$$

$$f_p(n) \sim \mathrm{cst(n)}\, g_c(n)^{-p}\, p^{\gamma(n)-3}$$

$$\gamma = \frac{c - 1 - \sqrt{(1-c)(25-c)}}{12}$$

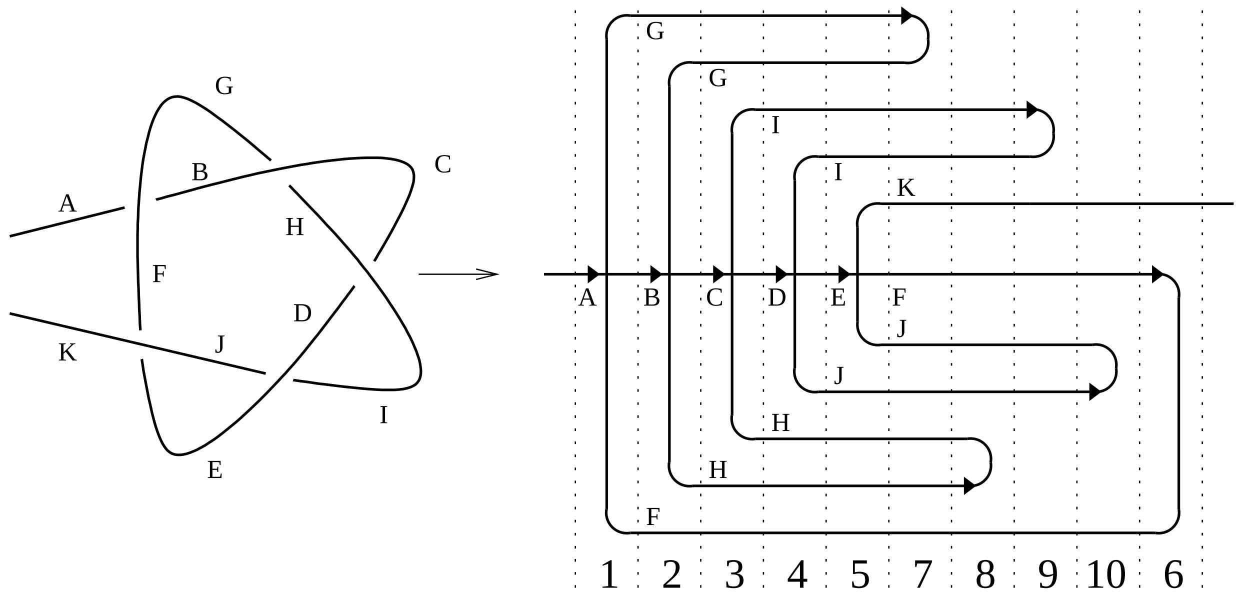
In particular, knots correspond to the limit $n \rightarrow 0$:

$$f_p(0) \sim \mathrm{cst}\, g_c^{-p} p^{-\frac{19+\sqrt{13}}{6}}$$

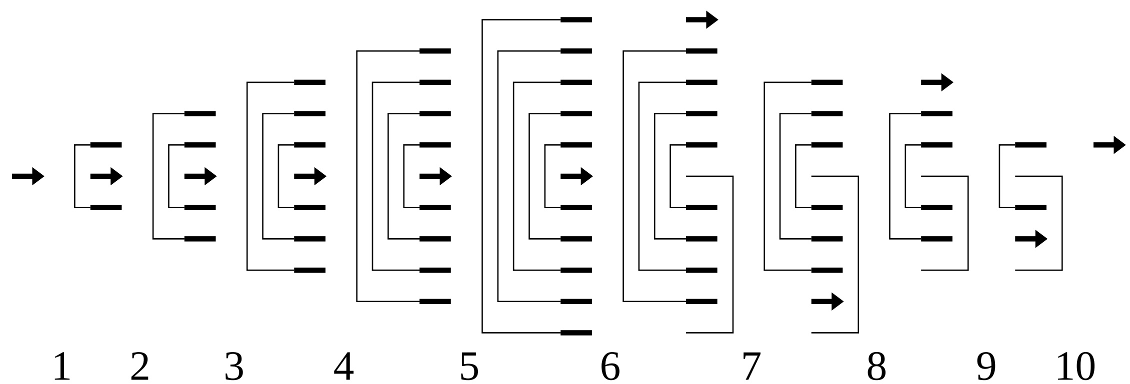


A Transfer Matrix for tangle diagrams

(J. Jacobsen and P. Z.-J., '01) For knots, follow the string as it winds around itself:



Structure of states:



Similar but more complicated construction for links.

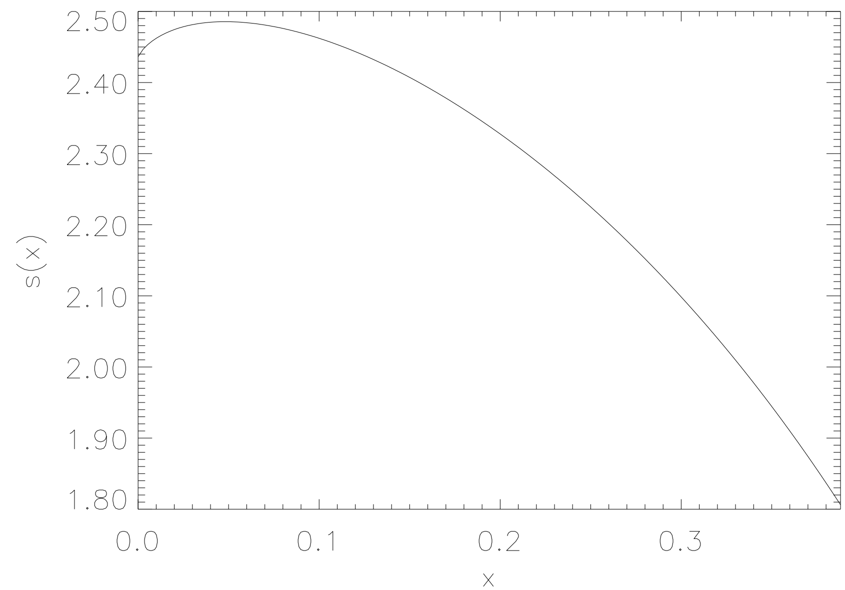
Numerical results

- Sample Table:

Γ_1

p \ k	0	1	2	3	4	5	6
1	1						
2	0						
3	2						
4	2						
5	6	3					
6	30	2					
7	62	40	2				
8	382	106	2				
9	1338	548	83	2			
10	6216	2968	194	2			
11	29656	11966	2160	124	2		
12	131316	71422	9554	316	2		
13	669138	328376	58985	5189	184	2	
14	3156172	1796974	347038	22454	478	2	
15	16032652	9298054	1864884	193658	10428	260	2

- Bulk entropy: $a_{k;p} \approx \mathrm{e}^{p\, s(x=k/p)}$



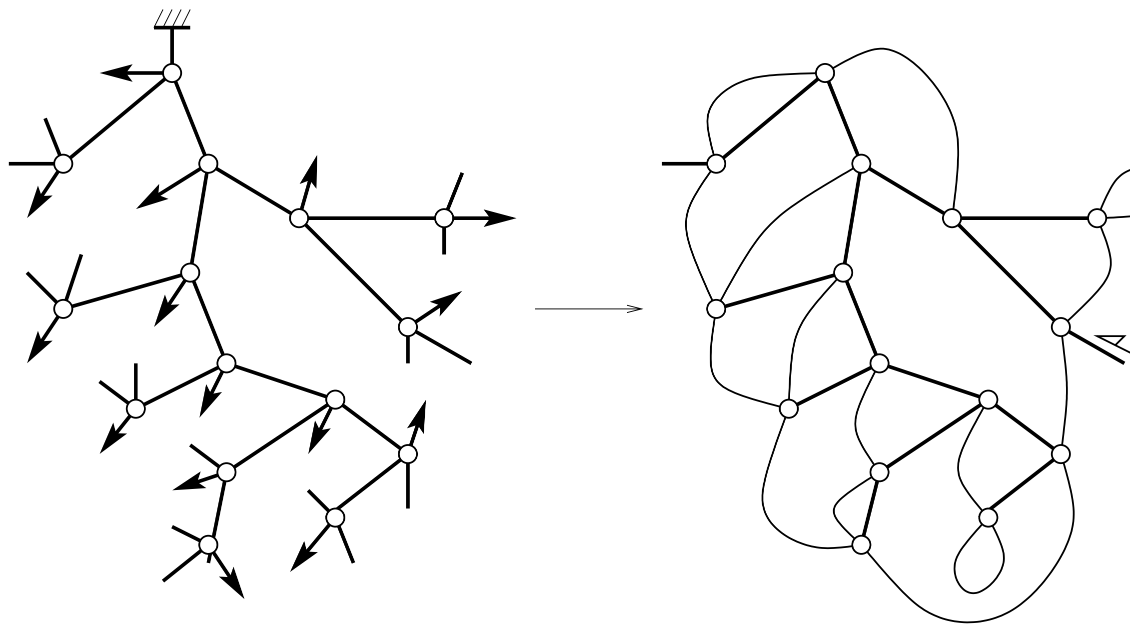
- Critical exponent γ: marginal agreement. low accuracy because of logarithmic corrections?

Monte Carlo: random sampling of planar maps

(G. Schaeffer and P. Z.-J., '03)

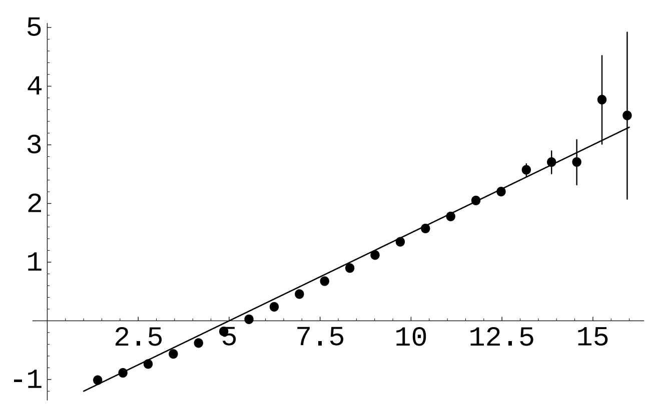
Schaeffer's bijection

between trees and planar maps:



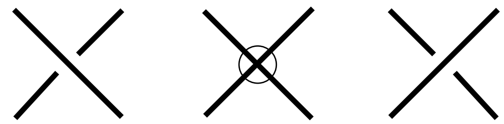
Results in an algorithm to produce random planar maps in linear time → up to $p = 10^7$ vertices.

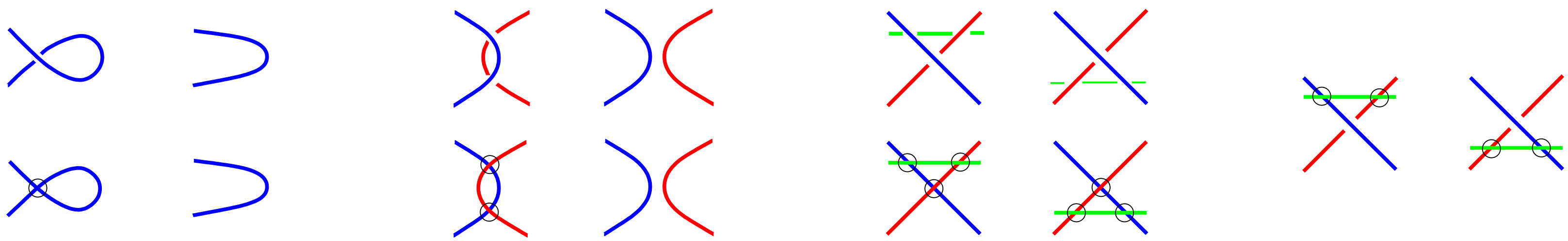
Test quantity: $\gamma' \equiv \frac{\mathrm{d}\gamma}{\mathrm{d}n}_{|n=1} = 3/10$ according to the conjecture. Very good agreement:



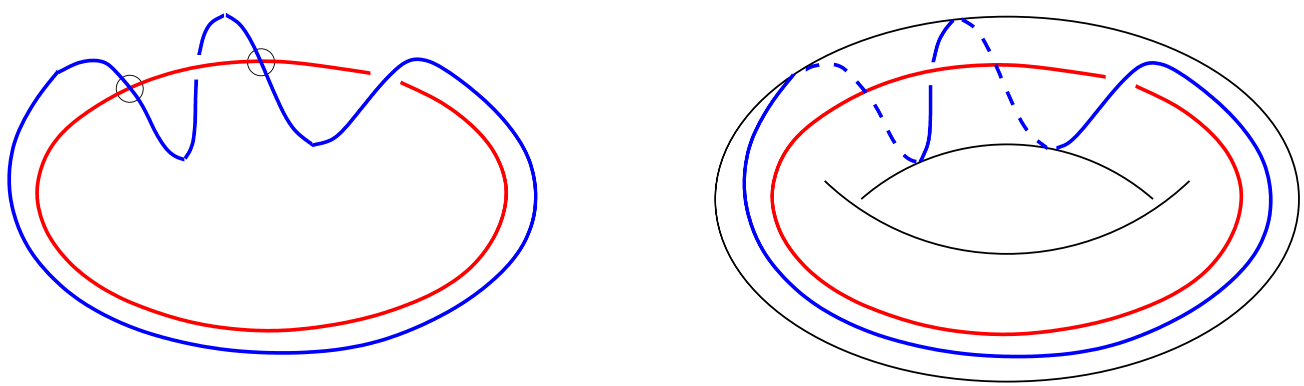
Virtual links

Kauffman’s definition via virtual diagrams and virtual Reidemeister moves:





Better to imagine links in **thickened surfaces** $\Sigma \times I$ (up to orientation-preserving homeomorphisms of the surface Σ)

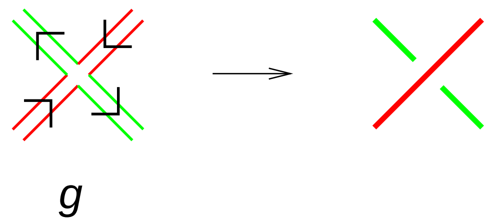


$\Rightarrow$ relation to matrix models!

• Virtual alternating links and tangles

NB: in genus > 0, not every quadrangulation is bipartite!!! $\Rightarrow$ complex matrix model:

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} \mathrm{d}M_a \mathrm{d}M_a^\dagger \, \mathrm{e}^{N \,\mathrm{tr}\left(-\frac{1}{2} M_a M_a^\dagger + \frac{g}{4}(M_a M_b^\dagger)^2\right)}$$



$$\log Z^{(N)}(n,g) = \sum_{h\geq 0, k\geq 1, p\geq 1} f_{k;p}^{(h)} N^{2-2h} g^p n^k$$

triple generating function of virtual alternating link diagrams.

"Renormalization" ?

Conjecture: Tait's flype conjecture also holds for virtual alternating links and tangles.

i.e. the only moves needed are *planar* flypes.

$\rightarrow$ Some exact results. Example: $n = 1$. The number of prime virtual alternating links of genus h

$$f_p^{(h)} \overset{p\to\infty}{\sim} c\, g_c^{-p} p^{5/2(h-1)-1}$$

Table of prime virtual alternating links

http://ipnweb.in2p3.fr/lptms/membres/pzinn/virtlinks (or google: Paul Zinn-Justin)

Generation and computation of invariants for low order tangles and links. The conjecture is checked for prime alternating tangles up to order 5 (first 13010 tangles).

Example: $p = 4$ links