# Analytic urns of triangular form

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## General problem

– One urn containing balls of one of $d$ colors
– Composition at $t = 0$ fixed
– At time $n$, a ball is randomly chosen in the urn, its color is inspected and the ball is placed back into the urn with some new balls. The number of added balls of each color only depends on the color of the drawn one.

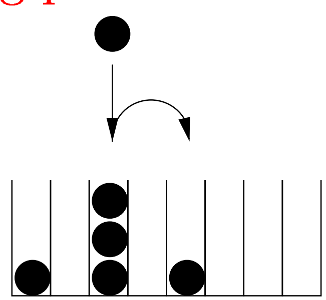
| Drawn | Added | | | |
|---|---|---|---|---|
| $\downarrow$ | $c_1$ | $c_2$ | $\cdots$ | $c_d$ |
| $c_1$ | $\delta_{1,1}$ | $\cdots$ | $\cdots$ | $\delta_{1,d}$ |
| $\vdots$ | $\vdots$ | | | $\vdots$ |
| $\vdots$ | $\vdots$ | | | $\vdots$ |
| $c_d$ | $\delta_{d,1}$ | $\cdots$ | $\cdots$ | $\delta_{d,d}$ |

## Examples

### Classical examples with two balls :

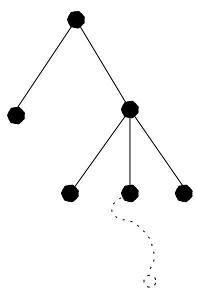
$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \quad \begin{bmatrix} 0 & s \\ s & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

### Hashing problems :



$$\begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & 0 \end{bmatrix}$$

### Random generation of $n$-trees :



$$\begin{bmatrix} 0 & 1 & & & \\ 1 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 1 & & & -1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

## Probabilistic approach

$$\text{Matrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Introduce $X_n$, the r.v. giving the number of balls of type $I$. At time $n$, there are exactly $t_0 + 2n$ balls in the urn.

$$\underbrace{\Pr(X_{n+1} = k)}_{p_{n+1,k}} = \underbrace{\Pr(X_n = k)}_{p_{n,k}} \cdot \frac{k}{t_0 + 2n} + \underbrace{\Pr(X_n = k-1)}_{p_{n,k-1}} \cdot \left(1 - \frac{k-1}{t_0 + 2n}\right) \qquad (\mathbf{1})$$

Set $p_n(u) = \sum_k p_{n,k} u^k$ and $F(z,u) = \sum_n p_n(u) z^n$.

$$\text{Eq } (\mathbf{1}) \quad \overset{\text{Translate}}{\Longrightarrow} \quad \text{PDE satisfied by } F$$

## Enumerative approach (balanced urns only !)

Idea : Enumerate all possible "histories" starting from the initial urn

$$M = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad U_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

| Time 0 | Draw | Time 1 | Draw | Time 2 |
|---|---|---|---|---|
| B : 1 W : 2 | 1 | B : 1 3 4 W : 2 | 1 | B : 1 3 4 5 6 W : 2 |
| | | | 2 | B : 1 3 4 6 W : 2 5 |
| | | | 3 | B : 1 3 4 5 6 W : 2 |
| | | | 4 | B : 1 3 4 5 6 W : 2 |
| | 2 | B : 1 4 W : 2 3 | 1 | B : 1 4 5 6 W : 2 3 |
| | | | 2 | B : 1 4 6 W : 2 3 5 |
| | | | 3 | B : 1 4 6 W : 2 3 5 |
| | | | 4 | B : 1 4 5 6 W : 2 3 |

Represent urn by monomials. We have

$$f_0(u,v) = uv \quad f_1(u,v) = u^3v + u^2v^2 \quad f_2(u,v) = 3u^5v + 3u^4v^2 + 2u^3v^3$$

and each history of length $n$ has the same probability $2 \times 4 \times \cdots \times 2n$.

Starting from one urn with $(r, s)$ balls, the operator

$$u^r v^s \longrightarrow r\ u^{r+2} v^s + s\ u^{r+1} v^{s+1} = \left( u^2 \cdot u \frac{\partial}{\partial u} + uv \cdot v \frac{\partial}{\partial v} \right) u^r v^s$$

describes all possible events that can occur from that urn. By linearity

$$f_n = \Gamma f_{n-1} = \Gamma^n\ uv \quad \text{where} \quad \Gamma = u^2 \cdot u \frac{\partial}{\partial u} + uv \cdot v \frac{\partial}{\partial v}$$

Introduce $H = \sum f_n(u, v) \frac{z^n}{n!}$. $H$ satisfies linear PDE

$$\frac{\partial H}{\partial z} = \Gamma H \qquad H(0, u, v) = uv$$

General case :

$$\frac{\partial H}{\partial z} = x_1^{\delta_{1,1}} \cdots x_d^{\delta_{1,d}} x_1 \frac{\partial H}{\partial x_1} + \cdots + x_1^{\delta_{d,1}} \cdots x_d^{\delta_{d,d}} x_d \frac{\partial H}{\partial x_d}$$

$$H(0, x_1, \ldots, x_d) = {x_1}^{a_1} \cdots {x_d}^{a_d}$$

## Relations between the two methods

Let $\ell$ be the constant sum on the lines and $t_0$ the initial number of balls. With the previous notations, one has

$$p_n = \frac{1}{t_0(t_0+\ell)\cdots(t_0+\ell(n-1))} f_n$$

Recall that $F = \sum p_n z^n$ and $H = \sum f_n \frac{z^n}{n!}$. Let $\mathcal{L}_\alpha$ defined for any $\alpha > 0$ as

$$\mathcal{L}_\alpha : g \longmapsto \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} g(st) e^{-t} \mathrm{d}t$$

dfThen $\mathcal{L}_\alpha(\sum a_n z^n) = \sum a_n \ \alpha(\alpha+1)\cdots(\alpha+n-1) \ s^n$. (Note : $\mathcal{L}_1$ is the classic Laplace transform). Then set $\alpha = t_0/\ell$. $F$ and $H$ satisfy

$$\mathcal{L}_1(H)(s) = \mathcal{L}_\alpha(F)(s\ell)$$

## Solving linear PDE

Pb : find all functions that satisfy

$$(\mathbf{2}) \quad P_1(x_1,\dots,x_d)\frac{\partial H}{\partial x_1} + \cdots + P_d(x_1,\dots,x_d)\frac{\partial H}{\partial x_d} = 0$$

Consider the system :

$$(\mathbf{3}) \qquad x_i'(t) = P_i(x_1,\dots,x_d) \qquad (1 \le i \le d)$$

Then $H$ is a solution iff it is constant on each integral curve of (**3**).

Pb : If $H_1,\dots,H_l$ are constants, then , $\phi(H_1,\dots,H_l)$ is also constant for any $\phi$. How do we get them all ?

**Theorem 1** *There exists $d-1$ independent functions $\psi_1,\dots,\psi_{d-1}$ such that the solutions of (**2**) are given by $\phi(\psi_1,\dots,\psi_{d-1})$ for an arbitrary function $\phi$.*

*Independence means here that the dimension of $(\overrightarrow{grad}(\psi_i))_{1\le i\le d-1}$ is $d-1$ in at least one point. This implies that there is no relation $\psi_i = \varphi(\psi_1,\dots,\psi_{i-1},\psi_{i+1},\dots,\psi_{d-1})$.*

## Finding the first integrals of a system of differential equations

Usual notations

$$\frac{\mathrm{d}x_1}{P_1} = \cdots = \frac{\mathrm{d}x_d}{P_d} = \mathrm{d}t$$

No general method!!

"Solvable" cases : Suppose $P_1$ is a function of the only variable $x_1$, $P_2$ is a function of $x_1$ and $x_2$ , etc ...

- Solve system $\frac{\mathrm{d}x_1}{P_1(x_1)} = \frac{\mathrm{d}x_2}{P_2(x_1,x_2)}$. This gives a relation of the form $G_1(x_1, x_2) = C_1$ where $G_1$ is a first integral.
- Invert this relation to get $x_1 = F_1(x_2, C_1)$. Use this relation to solve $\frac{\mathrm{d}x_2}{P_2(x_1,x_2)} = \frac{\mathrm{d}x_3}{P_3(x_1,x_2,x_3)}$.
- Repeat the process till the last equation ...

For triangular urns, our PDE is of this type. What a coincidence!!!!!

Do it yourself

$$M = \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,d} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{d-1,d} \\ 0 & \cdots & 0 & \delta_{d,d} \end{bmatrix} \qquad U_0 = \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_d \end{bmatrix}$$

Step $(\mathbf{1})$ : Find the PDE :

$$\frac{\partial H}{\partial z} = {x_1}^{\delta_{1,1}} \cdots {x_d}^{\delta_{1,d}} \cdot x_1 \frac{\partial H}{\partial x_1} + \cdots + {x_d}^{\delta_{d,d}} \cdot x_d \frac{\partial H}{\partial x_d}$$

Step $(\mathbf{2})$ : Solve the system of $d$ differential equations

$$-\mathrm{d}z = \frac{\mathrm{d}x_d}{x_d^*} = \frac{\mathrm{d}x_{d-1}}{x_{d-1}^* x_d^*} = \frac{\mathrm{d}x_{d-2}}{x_{d-2}^* x_{d-1}^* x_d^*} \cdots$$

The general solution of the PDE is given by

$$H = \phi(c_1, c_2, \ldots, c_d)$$

for any arbitrary function $\phi$.

Step (**3**) : Set $z = 0$. One has $H[z \leftarrow 0] = {x_1}^{a_1} \cdots {x_d}^{a_d}$. Solve the system

- $x_d = f_d(c_d[z \leftarrow 0])$
- $x_{d-1} = f_{d-1}(c_{d-1}[z \leftarrow 0], c_d[z \leftarrow 0])$
- $\ldots$

The generating function is then given by

$$H(x_1, \ldots, x_d, z) = [f_1(c_1, \ldots, c_d)]^{a_1} \cdots [f_d(c_d)]^{a_d}$$

## Triangular urns of size 2

We consider the matrix $\binom{a \quad 0}{a-b \; b}$, with initial conditions $(a_0, b_0)$ (also introduce $t_0 = a_0 + b_0$). Following the previous methodology, one has

$$H(z,u,v) = u^{a_0} v^{b_0} (1 - azu^a)^{-a_0/a} \left(1 - v^{b_0}(1 - (1 - azu^a)^{b/a})\right)^{-b_0/b}$$

Let $\Delta_n = \frac{n!}{t_0 \cdots (t_0 + (n-1)a)}$. Let $X_n$ be the random variable giving the number of balls of the second type. As a classical application of bivariate generating functions, one has

$$E[X_n(X_n - 1) \cdots (X_n - \ell + 1)] = \Delta_n [z^n] \frac{\partial^\ell H}{\partial v^\ell}_{|u,v=1}$$

By a small recurrence, we find that

$$E[(X_n)^\ell] = c_{\ell,0}(1 - az)^{-(t_0 + a\ell)\cdot b/a} + c_{\ell,1}(1 - az)^{-(t_0 + a(\ell - 1))\cdot b/a} + \ldots$$

with $c_{\ell,0} = b_0 \cdots (b_0 + (\ell - 1)b)$.

Since for $\alpha < \beta$ one has $[z^n](1-z)^{-\alpha} = o([z^n](1-z)^{-\beta})$, only the first term gives the asymptotics and

$$E[(X_n)^\ell] = \frac{1}{b^\ell} \frac{\Gamma\left(\frac{b_0+\ell\, b}{b}\right)\Gamma\left(\frac{t_0}{a}\right)}{\Gamma\left(\frac{b_0}{b}\right)\Gamma\left(\frac{t_0+l\, b}{a}\right)}\, n^{lb/a} + O(n^{(l-1)b/a}).$$

$\Longrightarrow$ Standard deviation and expectation of the same order of growth.

Non gaussian law !

## Limit laws for the composition of the urn

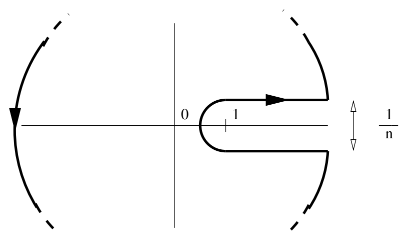
Let $Z_n$ be the r.v. giving the number of times a ball of type $II$ has been drawn ($X_n = b_0 + bZ_n$). Then

$$\Pr(Z_n = k) = a^n \Delta_n [v^k z^n](1-z)^{-a_0/a} \left(1 - v(1-(1-z)^{b/a})\right)^{-\frac{b_0}{b}}$$

Extending in $v$, then using Cauchy's formula yields

$$\Pr(Z_n = k) = C_{n,k} \oint (1-z)^{-a_0/a} \left(1-(1-z)^{b/a}\right)^k \frac{dz}{z^{n+1}}$$

The contour is extended as follows




From now, set $k = \lfloor x n^{b/a} \rfloor$ and $z = 1 - t/n$ :

$$\Pr\left(Z_n = \lfloor x n^{b/a} \rfloor\right) \sim C_{n,x} n^* \int_{\mathcal{H}} t^{-a_0/a} e^{-x\, t^{b/a}} e^t \, dt$$

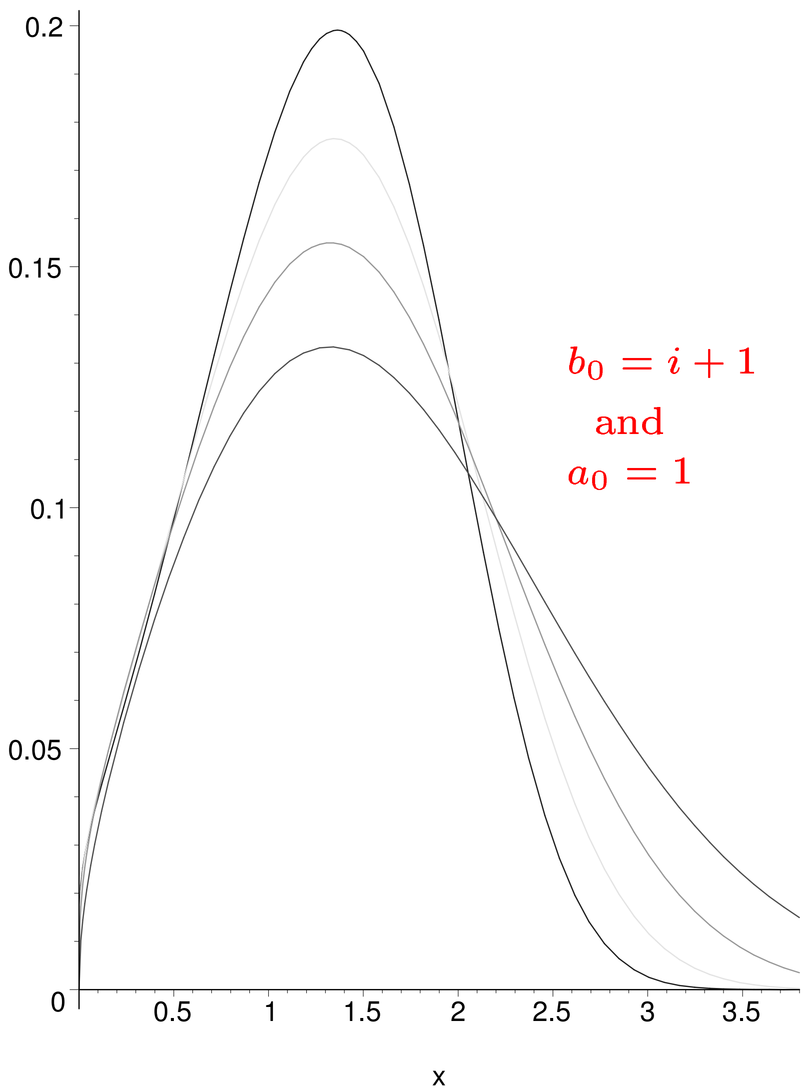
where $\mathcal{H}$ is a clockwise loop around the negative real axis.

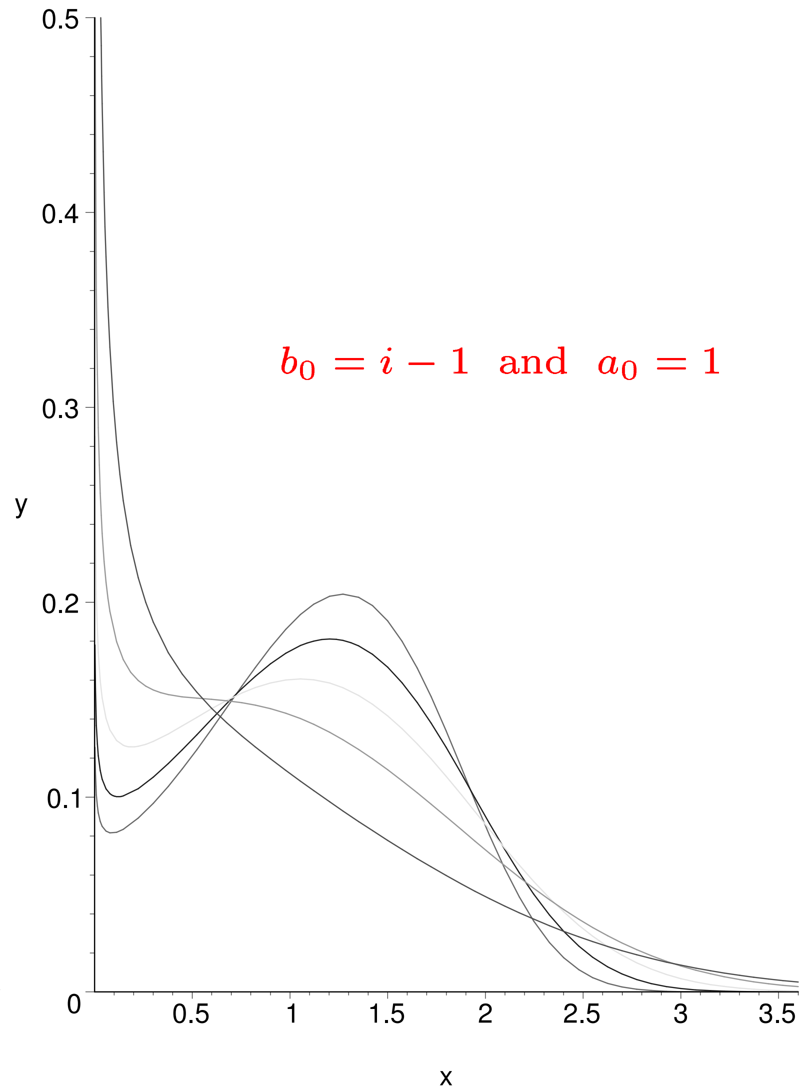
Finally, expanding $e^{-xt^{c/a}}$ and integrating termwise

**Theorem 2**

$$Pr\left(Z_n = \lfloor xn^{b/a} \rfloor\right) \sim n^{-b/a} \frac{\Gamma\left(\frac{t_0}{a}\right)}{\Gamma\left(\frac{b_0}{b}\right)} x^{b_0/b-1} \sum_{k\geq 0} \frac{(-1)^k}{\Gamma\left(\frac{a_0-kb}{a}\right)} \frac{x^k}{k!}$$

## Limit laws for Urn $\begin{pmatrix} i+2 & 0 \\ 2 & i \end{pmatrix}$ for $i$ from 2 to 6



## Mellin tranform of the limit law

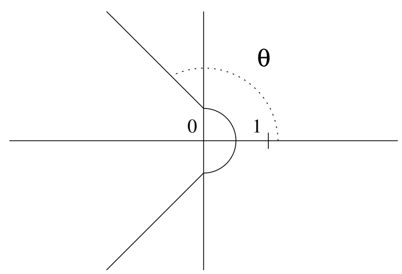
Why ? Strong relations with moments of the random variable.

$$f^*(s) = \int_0^{+\infty} x^{s-1} f(x) \mathrm{d}x$$

We start with the expression of the limit law as an integral

$$g(x) = \frac{\Gamma\left(\frac{t_0}{a}\right)}{\Gamma\left(\frac{b_0}{b}\right)} \int_{\mathcal{H}} G(x,t) \mathrm{d}t$$

with $G(x,t) = x^{b_0/b-1} e^{-x\ t^{b/a}}\ t^{-a_0/a} e^t$. We choose $\mathcal{H}$ as follows




where $\frac{\pi}{2} < \theta < \frac{\pi}{2} \cdot \frac{a}{b}$.

Then for any $t \in \mathcal{H}$ and any $x > 1 - b_0/b$, $\int_0^{+\infty} \left| x^{s-1} G(x,t) \right| \mathrm{d}x$ is convergent and equal to

$$\int_0^{+\infty} x^{s-1} G(x,t) \mathrm{d}x = \Gamma\left(\frac{b_0}{b} + s - 1\right) t^{-\frac{t_0}{a} - (s-1)\frac{b}{a}} e^t$$

Integration with respect to variable $t$ then gives theorem

**Theorem 3** *The limit law admits a Mellin transform defined for any $s > 1 - b_0/b$ by*

$$g^*(s) = \frac{\Gamma\left(\frac{b_0 + (s-1)\ b}{b}\right) \Gamma\left(\frac{t_0}{a}\right)}{\Gamma\left(\frac{b_0}{b}\right) \Gamma\left(\frac{t_0 + (s-1)\ b}{a}\right)}$$

## Triangular urn of size 3

$$M = \begin{bmatrix} s_1 & \delta & s_3 - s_1 - \delta \\ 0 & s_2 & s_3 - s_2 \\ 0 & 0 & s_3 \end{bmatrix}$$

Diff. eq. $$\frac{\partial H}{\partial z} = u^{s_1+1} v^{\delta} w^{s_3-s_1-\delta} \frac{\partial H}{\partial u} + v^{s_2+1} w^{s_3-s_2} \frac{\partial H}{\partial z} + w^{s_3+1} \frac{\partial H}{\partial w}$$

Prime Integrals

$$-\mathrm{d}z = \frac{\mathrm{d}w}{w^{s_3+1}} \Rightarrow \boxed{c_1 = z - \frac{1}{s_3 w^{s_3}}}$$

$$\frac{\mathrm{d}v}{v^{s_2+1} w^{s_3-s_2}} = \frac{\mathrm{d}w}{w^{s_3+1}} \Rightarrow \boxed{c_2 = \frac{1}{v^{s_2}} - \frac{1}{w^{s_2}}}$$

$$\frac{\mathrm{d}u}{u^{s_1+1} v^{\delta} w^{s_3-s_1-\delta}} = \frac{\mathrm{d}v}{v^{s_2+1} w^{s_3-s_2}} \Rightarrow \frac{\mathrm{d}u}{u^{s_1+1}} = \frac{1}{v^{s_1+1}} (c_2 + v^{s_2})^{\frac{s_1+\delta-s_2}{s_2}} \mathrm{d}v$$

$$\boxed{c_3 = \frac{1}{u^{s_1}} + c_2^{\frac{s_1}{s_2}} g\left(v c_2^{\frac{1}{s_2}}\right) \quad \text{with} \quad g = s_1 \int \frac{1}{t^{s_1+1}} (1 + t^{s_2})^{\frac{s_1+\delta-s_2}{s_2}} \mathrm{d}t}$$

## Generating function

$$f_3 = w\left(1 - s_3 z w^{s_3}\right)^{-1/s_3}$$

$$f_2 = v\left(1 - v^{s_2}\left(\frac{1}{w^{s_2}} - \frac{1}{f_3^{s_2}}\right)\right)^{-1/s_2}$$

$$f_1 = u\left[1 - u^{s_1}\left[\left(\frac{1}{w^{s_2}} - \frac{1}{v^{s_2}}\right)^{s_1/s_2} g\left(v\left(\frac{1}{w^{s_2}} - \frac{1}{v^{s_2}}\right)^{1/s_2}\right)\right.\right.$$
$$\left.\left. - \left(\frac{1}{f_3^{s_2}} - \frac{1}{f_2^{s_2}}\right)^{s_1/s_2} g\left(f_2\left(\frac{1}{f_3^{s_2}} - \frac{1}{f_2^{s_2}}\right)^{1/s_2}\right)\right]\right]$$

$$H = {f_1}^{a_1} {f_2}^{a_2} {f_3}^{a_3}$$

## Examples with closed form generating functions

$Sum = 2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/2}$

$T_2 \sim \beta_0 \; n^{1/2}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/2}$

$T_2 \sim \alpha_0 \; n^{1/2} \ln \; n$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/2}$

$T_2 \sim \alpha_0 \; n^{1/2}$

$Sum = 3$

$s_1 < s_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/3}$

$T_2 \sim (\alpha_0 + \beta_0) \; n^{2/3}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/3}$

$T_2 \sim (2\alpha_0 + \beta_0) \; n^{2/3}$

$Sum = 3$

$s_1 = s_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/3}$

$T_2 \sim \alpha_0 \; n^{1/3} \ln \; n$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{1/3}$

$T_2 \sim 2\alpha_0 \; n^{1/3}$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{2/3}$

$T_2 \sim \alpha_0 \; n^{2/3} \ln \; n$

$s_1 > s_2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$T_1 \sim \alpha_0 \; n^{2/3}$

$T_2 \sim \alpha_0 \; n^{2/3} \ln \; n$

## Special cases of size $d$

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 2 & \ddots & \vdots \\ & & \ddots & 1 \\ & & & d \end{bmatrix}$$

Diff. eq. $$\frac{\partial H}{\partial z} = x_1^2 x_2 \cdots x_d \frac{\partial H}{\partial x_1} + x_2^3 x_3 \ldots x_d \frac{\partial H}{\partial x_2} + \cdots + x_d^{d+1} \frac{\partial H}{\partial x_d}$$

Prime integrals

$$\frac{\mathrm{d}x_1}{x_1^2 x_2 \cdots x_d} = \frac{\mathrm{d}x_2}{x_2^3 x_3 \ldots x_d} = \cdots = \frac{\mathrm{d}x_d}{x_d^{d+1}} = -\mathrm{d}z$$

$$c_1 = \frac{1}{x_1} - \frac{1}{x_2} \quad c_2 = \frac{1}{x_2^2} - \frac{1}{x_3^2} \quad \cdots \quad c_d = z - \frac{1}{dx_d^d}$$

General solution

$$\Phi\left(\frac{1}{x_1}-\frac{1}{x_2};\frac{1}{x_2^2}-\frac{1}{x_3^2};\cdots;z-\frac{1}{dx_d^d}\right)$$

Initial conditions :

$$H\left(\frac{1}{x_1}-\frac{1}{x_2};\frac{1}{x_2^2}-\frac{1}{x_3^2};\cdots;\frac{1}{dx_d^d}\right)=x_1^{a_1}\cdots x_d^{a_d}$$

$\Rightarrow$ Define $f_d = x_d\left(1-dzx_d^d\right)^{-1/d}$

$$\vdots$$

$$f_i = x_i\left(1-x_i^i\left(\frac{1}{x_{i+1}^i}-\frac{1}{f_{i+1}^i}\right)\right)^{-1/i}$$

Then $H = f_1^{a_1}\cdots{}_d^{a_d}$

Finally, $\forall i \in [\![ 1 ; n-1 ]\!]$

$$E(X_i) \sim (a_1 + \cdots + a_i) \frac{\Gamma\left(\frac{t_0+i}{n}\right)}{\Gamma\left(\frac{t_0}{n}\right)} n^{i/n}$$

## Urn of the hashing problem

Let us begin with size 3

$$M = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad U_0 = \begin{bmatrix} n_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Diff. eq.

$$\frac{\partial H}{\partial z} = x_2 \frac{\partial H}{\partial x_1} + x_3 \frac{\partial H}{\partial x_2} + x_3 \frac{\partial H}{\partial x_3}$$

Prime integrals

$$-\mathrm{d}z = \frac{\mathrm{d}x_3}{x_3} \Rightarrow \boxed{c_3 = z - \ln\ x_3}$$

$$\frac{\mathrm{d}x_2}{x_3} = \frac{\mathrm{d}x_3}{x_3} \Rightarrow \boxed{c_2 = x_2 - x_3}$$

$$\frac{\mathrm{d}x_1}{x_2} = \frac{\mathrm{d}x_2}{x_3} \Rightarrow \mathrm{d}x_1 = \frac{x_2}{x_2 - c_2}\mathrm{d}x_2$$

$$\boxed{c_1 = x_1 - x_2 + (x_2 - x_3) \ln\ x_3}$$

$$\Rightarrow \qquad H(z, x_1, x_2, x_3) = (x_1 + x_2 z + x_3(e^z - z - 1))^{n_0}$$

General case :

$$H(z, x_1, \ldots, x_d) = \left(x_1 + \cdots + x_{d-1}\frac{z^d}{d!} + x_d(e^z - 1 - z - \cdots \frac{z^d}{d!})\right)^{n_0}$$