

Algorithms for the Construction of the Minimal Telescopers

H.Q. Le

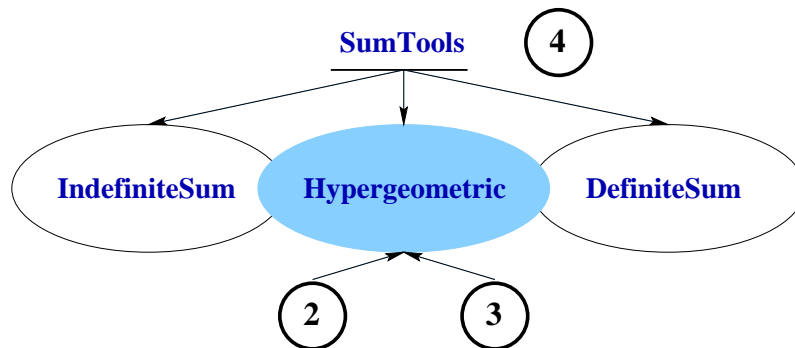
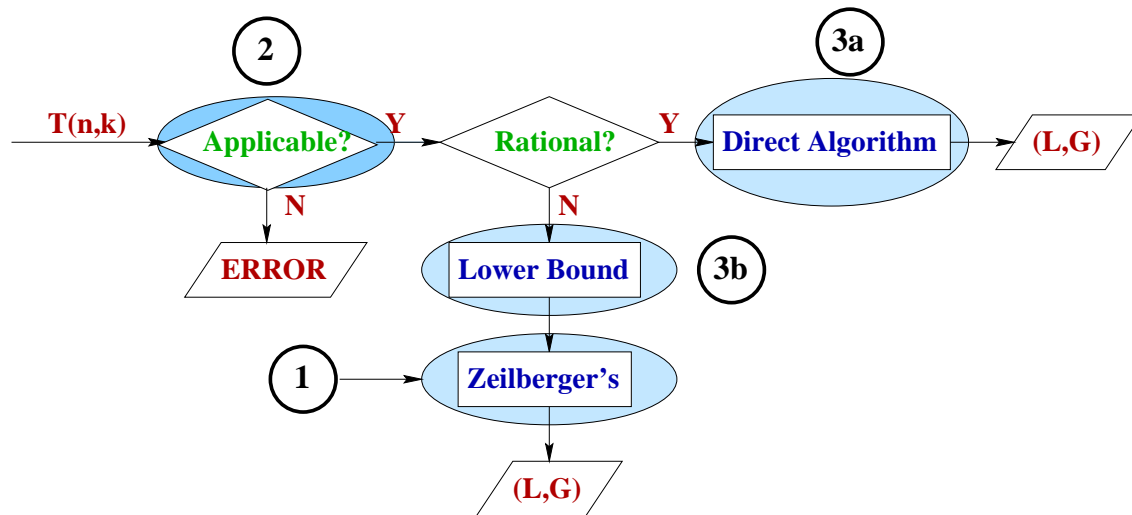
Algorithms Project

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Summary



Definitions and Notation

\mathbb{K} a field of characteristic 0

$T(k)$ a sequence, or a function over \mathbb{K}

$E_k T(n, k) = T(n, k + 1)$ shift operator w.r.t. k

$E_n T(n, k) = T(n + 1, k)$ shift operator w.r.t. n

Creative Telescoping

input: a function $T(n, k)$.

output: a pair (L, G) where $L \in \mathbb{K}[n, E_n]$, i.e.,

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \quad a_i \in \mathbb{K}[n],$$

and a function $G(n, k)$ such that

$$LT(n, k) = (E_k - 1)G(n, k).$$

Remark. The theory of creative telescoping was initially designed by Zeilberger (Zeilberger, 1991) for the case when the function $T(n, k)$ is a **hypergeometric term**. It was later adapted to **holonomic functions** (Chyzak & Salvy, 1998), (Chyzak, 2000).

Definitions and Notation

Definition. A sequence $T(k)$ over \mathbb{K} is a **hypergeometric term** if there are $f, g \in \mathbb{K}[k] \setminus \{0\}$ such that

$$f(k)T(k+1) + g(k)T(k) = 0.$$

$\mathcal{C}_k(T) = T(k+1)/T(k) = -g(k)/f(k)$: the **certificate** of T .

Example 1.

| $T(k)$ | $\mathcal{C}_k(T) = \frac{E_k T(k)}{T(k)}$ |
|-----------------|--|
| 2^k | 2 |
| $k!$ | $k+1$ |
| $\binom{2k}{k}$ | $2 \frac{2k+1}{k+1}$ |

Definition. A sequence $T(n, k)$ is a **hypergeometric term** if there are $f_0, f_1, g_0, g_1 \in \mathbb{K}[n, k] \setminus \{0\}$ such that

$$f_1(n, k)T(n + 1, k) + f_0(n, k)T(n, k) = 0,$$

$$g_1(n, k)T(n, k + 1) + g_0(n, k)T(n, k) = 0.$$

$\mathcal{C}_n(T) = T(n + 1, k)/T(n, k)$, $\mathcal{C}_k(T) = T(n, k + 1)/T(n, k)$:
the **n -certificate** and the **k -certificate** of T , respectively.

Zeilberger's Algorithm

D. Zeilberger. The method of creative telescoping. *Journal of Symbolic Computation* **11**, 195–204, 1991.

input : a hypergeometric term $T(n, k)$.

output: a Z -pair (L, G) where $L \in \mathbb{C}[n, E_n]$, i.e.,

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \quad a_i \in \mathbb{C}[n],$$

and a hypergeometric term $G(n, k)$ such that

$$LT(n, k) = (E_k - 1)G(n, k).$$

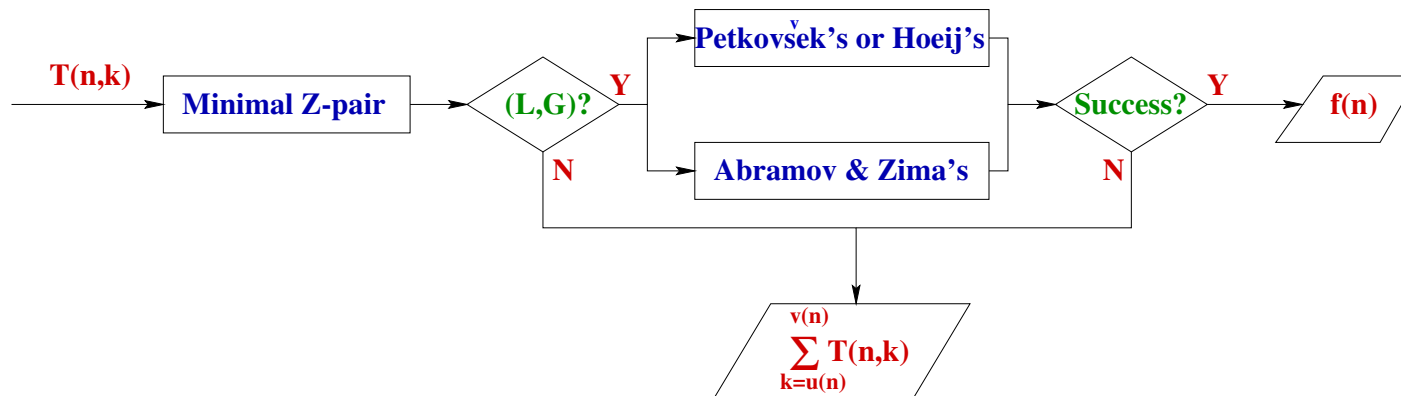
Remark. The telescoper L in the returned Z -pair is of minimal possible order, and is called **the minimal telescoper**.

Applications

- finding closed forms of definite sums of hypergeometric terms (\mathcal{ZP} method),
- certifying combinatorial identities (\mathcal{WZ} method),
- proving transformation formulas for hypergeometric series (Paule, Strehl, 1995).

Closed Forms of Definite Sums

Goal: Compute a closed form of $f(n) = \sum_{k=u(n)}^{v(n)} T(n, k)$, where $T(n, k)$ is a hypergeometric term in n and k .



Example 2 (Riordan, 1968)

$$f(n) = \sum_{k=0}^n T(n, k) = \sum_{k=0}^n \binom{2n}{2k}^2.$$

> req := ZeilbergerRecurrence(T, n, k, f, 0..n);

$$\begin{aligned} req := & (10n^5 + 55n^4 + 112n^3 + 104n^2 + 43n + 6)f(n+2) \\ & - (120n^5 + 600n^4 + 1124n^3 + 972n^2 + 376n + 48)f(n+1) \\ & - (640n^5 + 2880n^4 + 4728n^3 + 3492n^2 + 1148n + 132)f(n) = 0 \end{aligned}$$

> LREtools['hypergeomsols'](req, {f(n)}, {f(0)=1, f(1)=2});

$$\sum_{k=0}^n \binom{2n}{2k}^2 = \frac{1}{2} \frac{4^n \left(\Gamma\left(2n + \frac{1}{2}\right) \sqrt{\pi} + (-1)^n \Gamma\left(n + \frac{1}{2}\right)^2 \right)}{\sqrt{\pi} \Gamma\left(n + \frac{1}{2}\right) \Gamma(n+1)}$$

Zeilberger's Algorithm: the Strategy

The algorithm uses an **item-by-item examination** on the order ρ of L . It starts with the value of 0 for ρ , and increases ρ until it is successful in finding a Z -pair (L, G) .

Example. Consider the hypergeometric term

$$T(n, k) = \frac{1}{n^2 + 9nk - 4n - 22k^2 + 21k - 5}.$$

> Zpair := Zeilberger(T,n,k,E_n):

"applying Zeilberger's algorithm for order 0"

...

"applying Zeilberger's algorithm for order 12"

> L := Zpair[1];

$$L = (13n + 157)E_n^{12} + (13n + 144)E_n^{11} - (13n + 14)E_n - (13n + 1)$$

Problem 1: Applicability of \mathcal{Z}

Problem. \mathcal{Z} tries to compute a Z -pair for $T(n, k)$ when such a pair might not exist. In other words, \mathcal{Z} might not terminate.

Side effect.

- An upper bound, e.g. 6, for the **guessed order** of a telescoper needs to be set in advance.
- It is not always possible to set the order of telescopers L to a “high enough” value. e.g.,

$$F_m(n, k) = \frac{1}{n + (m + 1)k}, \quad m \in \mathbb{N}.$$

Additive Decomposition Problem (ADP)

input: a function $T(k)$ over \mathbb{K} .

output: two functions $T_1(k), T_2(k)$ over \mathbb{K} such that

$$T(k) = (E_k - 1)T_1(k) + T_2(k)$$

where $T_2(k)$ is “**simpler**” than T in some sense.

Remark. Any algorithm which solves the ADP should guarantee that if $T(k)$ is summable, then $T_2(k) \equiv 0$.

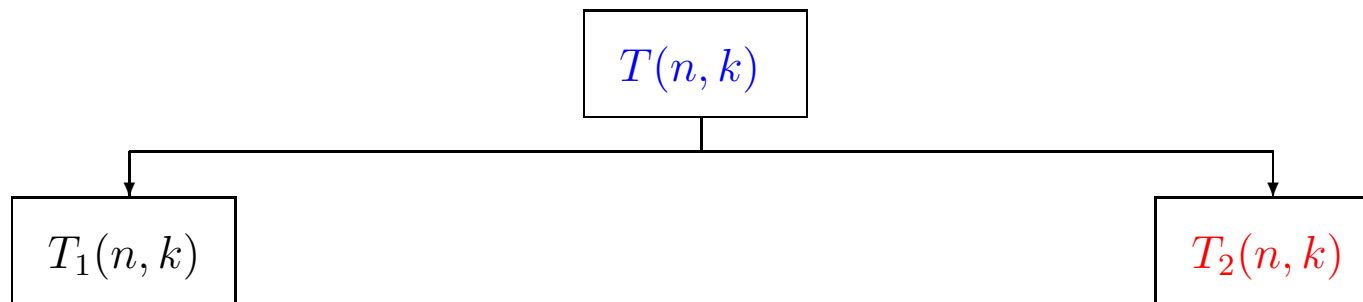
Applicability of \mathcal{Z} : the General Approach

For a given hypergeometric term $T(n, k)$, let $(T_1(n, k), T_2(n, k))$ be an additive decomposition of T w.r.t. k .

- If the non-summable part $T_2 \equiv 0$, then \mathcal{Z} is obviously applicable to T .
- Otherwise, it is easy to show that \mathcal{Z} is applicable to T iff it is applicable to T_2 . Hence, the investigation will be on the structure of the non-summable part T_2 .

Applicability of \mathcal{Z} to a Rational Function

S.A. Abramov, H.Q. Le. A criterion for the applicability of Zeilberger's algorithm to rational functions. *Discrete Mathematics* **259**, 1–17, 2002.



$$T(n, k) = (E_k - 1)T_1(n, k) + T_2(n, k)$$

additive decomposition w.r.t. k

$$T_2(n, k) = \frac{f(n, k)}{g(n, k)}, \quad f, g \in \mathbb{C}[n, k], \quad (g \text{ of minimal degree w.r.t. } k)$$

\mathcal{Z} is applicable to $T(n, k) \iff g(n, k) \in \mathcal{Z}_{n, k}$

$$g(n, k) = \prod_i (a_i n + b_i k + c_i), \quad a_i, b_i \in \mathbb{Z}, \quad c_i \in \mathbb{C}$$

ADP: Rational Function Case

S. A. Abramov, Rational component of the solutions of a first-order linear recurrence relation with a rational right-hand side. *USSR Comput. Math. Phys.* Transl. from *Zh. vychisl. mat. mat. fiz.* **14**, 1975, 1035–1039.

Given $T(k) \in \mathbb{K}(k)$, construct a pair of rational functions over \mathbb{K} $(T_1(k), T_2(k))$ such that

$$T(k) = (E_k - 1)T_1(k) + T_2(k)$$

where the denominator of T_2 has the minimal possible degree.

A Sketch of the Proof

$$T(n, k) = (E_k - 1)T_1(n, k) + \frac{f(n, k)}{g(n, k)}, \quad f, g \in \mathbb{K}[n, k].$$

$$\text{If } \exists(L, G) \text{ for } T_2, \text{ i.e., } LT_2 = (E_k - 1)G = \frac{a(n, k)}{b(n, k)}, \quad a, b \in \mathbb{C}[n, k].$$

Then

$$g(n, k) \text{ is shift-free w.r.t. } k, \quad b(n, k) \text{ is spread w.r.t. } k. \quad (1)$$

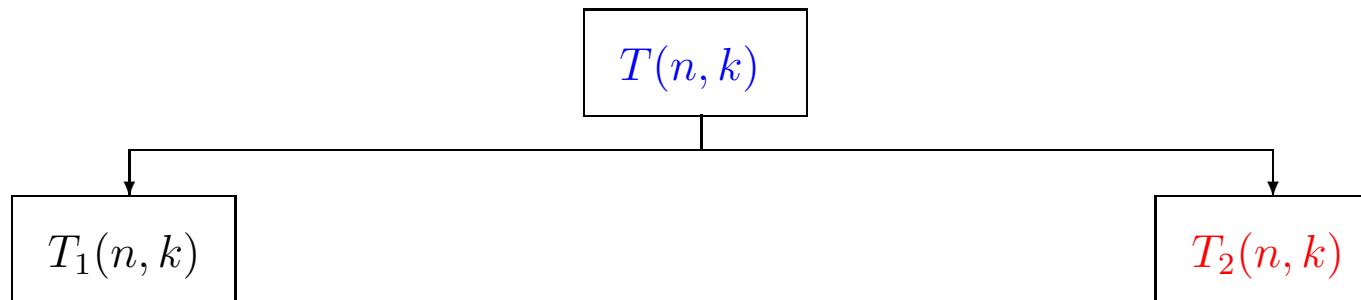
$$\xrightarrow{(1)} \quad \forall p_1 = (k - \alpha_i(n)), \quad p_1 \mid g, \quad \exists J, H \in \mathbb{Z}, \quad J > 0 : \quad (2)$$

$$p_1(n, k) = p_1(n + J, k + H)$$

$$\xrightarrow{(2)} \quad p_1(n, k) = \frac{1}{J} (Jk - Hn + Jc), \quad c \in \mathbb{C}$$

Applicability of \mathcal{Z} to a Hypergeometric Term

S.A. Abramov. When does Zeilberger's algorithm succeed? *Advances in Applied Mathematics* **30** (2003) 424–441.



$$T(n, k) = (E_k - 1)T_1(n, k) + T_2(n, k)$$

additive decomposition w.r.t. k

$$(z, f_1, f_2, v_1, v_2) := \text{RNF}_k(\mathcal{C}_k(T_2(n, k)))$$

$$\mathcal{Z} \text{ is applicable to } T(n, k) \iff v_2(n, k) \in Z_{n,k}$$

Example 3

$$T = \frac{nk}{(n+k-2)(nk+1)} \binom{n}{k} + \frac{1}{(nk+n+1)(n+k-1)} \binom{n}{k+1}.$$

An additive decomposition (T_1, T_2) of T w.r.t. k has

$$T_2 = \frac{33kn - 24k - 5kn^2 + 67n - 24 + 3n^3 - 22n^2}{24(n+k-1)} \prod_{w=0}^{k-1} \frac{n-w}{w+4}.$$

Since $v_2 \in Z_{n,k}$, \mathcal{Z} is applicable to $T(n, k)$.

Example 4

$$T(n, k) = \frac{-2nk^2 + (n^2 - 1)k + n + 1}{nk^2 + (n + 1)k + 1} \frac{\Gamma(n + 1)}{\Gamma(n - k + 2)\Gamma(k + 1)}.$$

An additive decomposition (T_1, T_2) of T w.r.t. k yields

$$T_1 = \frac{2nk + 2n - 1}{2n(n + 1)} \prod_{w=0}^{k-1} \frac{n - w + 1}{w + 2},$$

$$T_2 = \frac{(n^2 + 5n - 2)k + 5n - 1}{4n(n + 1)(nk + 1)} \prod_{w=0}^{k-1} \frac{n - w + 1}{w + 3}.$$

Since $v_2 = 4n(n + 1)(nk + 1) \notin Z_{n,k}$, \mathcal{Z} is not applicable to $T(n, k)$.

Factorization into Integer-Linear Polynomials

input : a polynomial $g(n, k) \in \mathbb{C}[n, k]$.

output: *true* if $g(n, k) \in Z_{n,k}$, *false* otherwise.

$$w(n, k) = \frac{g(n, k)}{\text{content}_k g(n, k) \cdot \text{content}_n g(n, k)}.$$

$$w(n, k) \stackrel{?}{=} \prod_{i=0}^m w_{c_i}(n, k), \quad c_i \in \mathbb{Q} \setminus \{0\}, \quad c_i \neq c_j \text{ for } i \neq j.$$

$$w_{c_i}(n, k) = \prod_j (k + c_i n + \gamma_j), \quad \gamma_j \in \mathbb{C}.$$

Remark. This procedure does not require a complete factorization of the input polynomial $g(n, k)$ into irreducible factors.

Problem 2: Efficiency of \mathcal{Z}

Due to the item-by-item examination strategy, let ρ be the order of the minimal telescoper for the input hypergeometric term $T(n, k)$. Then \mathcal{Z} wastes resources trying to compute a Z -pair for T where the guessed orders of the telescopers are less than ρ .

Example.

$$T(n, k) = \frac{1}{n + 100k + 1}$$

↓

Zeilberger's

↓
0, ..., 99, 100

$$\left(E_n^{100} - 1, \sum_{i=0}^{99} E_k^i T(n, k) \right)$$

Rational Function Case: a Direct Algorithm

A Direct Algorithm to Construct the Minimal Z-pairs for Rational Functions. *Advances in Applied Mathematics*, 30, 137–159, 2003.

An efficient algorithm which computes a Z-pair directly, without using an item-by-item examination. It is based on

- the construction of a special form of representation for the input rational function $T(n, k)$,
- a direct construction of the minimal telescopers for each member of this representation,
- the use of Least Common Left Multiple (lclm) computation.

Telescopers of a Sum of Rational Functions

Lemma.

$$\begin{array}{ccccccc}
 F(n, k) & = & F_1(n, k) & + & F_2(n, k) & + & \cdots & + & F_s(n, k) \\
 & & \Downarrow & & \Downarrow & & & & \Downarrow \\
 & & (L_1, G_1) & & (L_2, G_2) & & \cdots & & (L_s, G_s)
 \end{array}$$

$\text{lclm}(L_1, \dots, L_s)$ is a telescoper for $F = F_1 + \cdots + F_s$.

Example 5.

$$F = F_1 + F_2, \quad F_1 = \frac{1}{5n - 4k - 4}, \quad F_2 = \frac{1}{5n - 4k + 6}.$$

$$\left. \begin{array}{l}
 F_1 \xrightarrow{z} L_1 = E_n^4 - 1 \\
 F_2 \xrightarrow{z} L_2 = E_n^4 - 1
 \end{array} \right\} \implies \text{lclm}(L_1, L_2) = E_n^4 - 1.$$

$$F_1 + F_2 \xrightarrow{z} L = E_n^2 - 1.$$

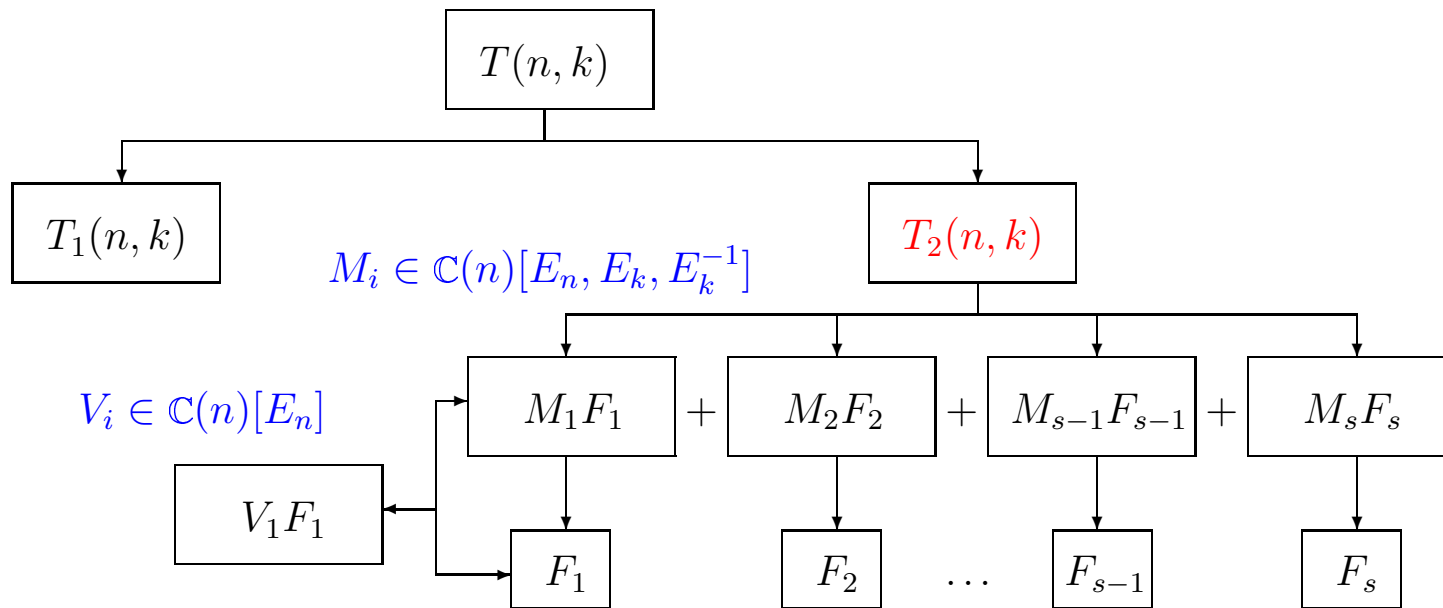
The Minimal Telescopier of a Sum

Lemma.

$$\begin{array}{ccccccccccc} F(n, k) & = & F_1(n, k) & + & F_2(n, k) & + & \cdots & + & F_s(n, k) \\ & & \Downarrow & & \Downarrow & & & & \Downarrow \\ & & (L_1, G_1) & & (L_2, G_2) & & \cdots & & (L_s, G_s) \end{array}$$

For any telescopier L^* of F , if L^* is also a telescopier for each F_i , $i = 1, \dots, s$, then $\text{lclm}(L_1, \dots, L_s)$ is the minimal telescopier for F .

The General Approach

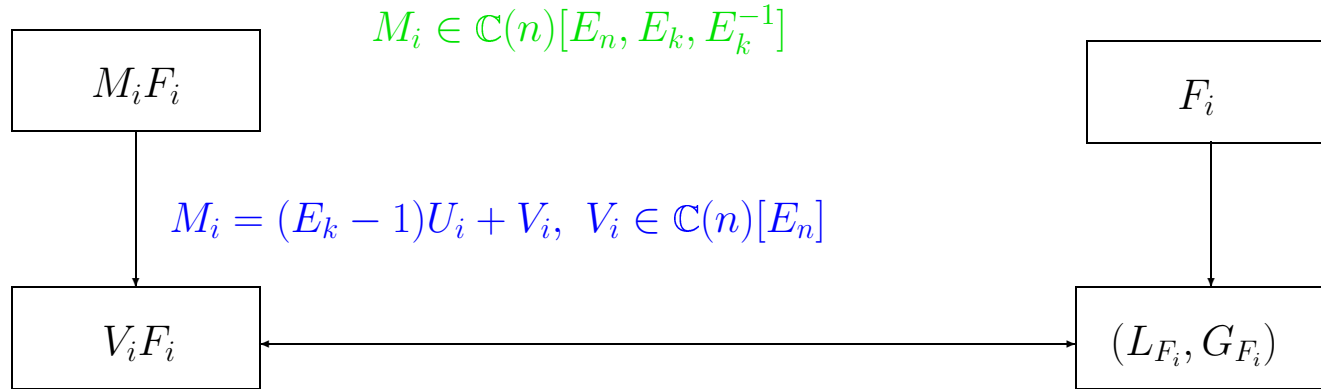


$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}}$$

$$a_i, b_i \in \mathbb{Z}, b_i > 0, \gcd(a_i, b_i) = 1, c_i \in \mathbb{C}, m_i \in \mathbb{N} \setminus \{0\}$$

The Minimal Telescopier for $M_i F_i$

$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}}, \quad a_i, b_i \in \mathbb{Z}, b_i > 0, \gcd(a_i, b_i) = 1, c_i \in \mathbb{C}, m_i \in \mathbb{N} \setminus \{0\}$$



$$L_i \in \mathcal{T}(M_i F_i) \iff L_i \in \mathcal{T}(V_i F_i)$$

$$(L_i, G_i) \in \mathcal{P}(V_i F_i) \implies (L_i, G_i + L_i \circ U_i F_i) \in \mathcal{P}(M_i F_i)$$

$$\text{lclm}(V_i, L_{F_i}) = L_1 \circ V_i = L_2 \circ L_{F_i}, \quad L_1, L_2 \in \mathbb{C}(n)[E_n]$$

$(L_1, L_2 G_{F_i})$ is the minimal Z -pair for $V_i F_i$

The Minimal Telescopier for F_i

$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}},$$

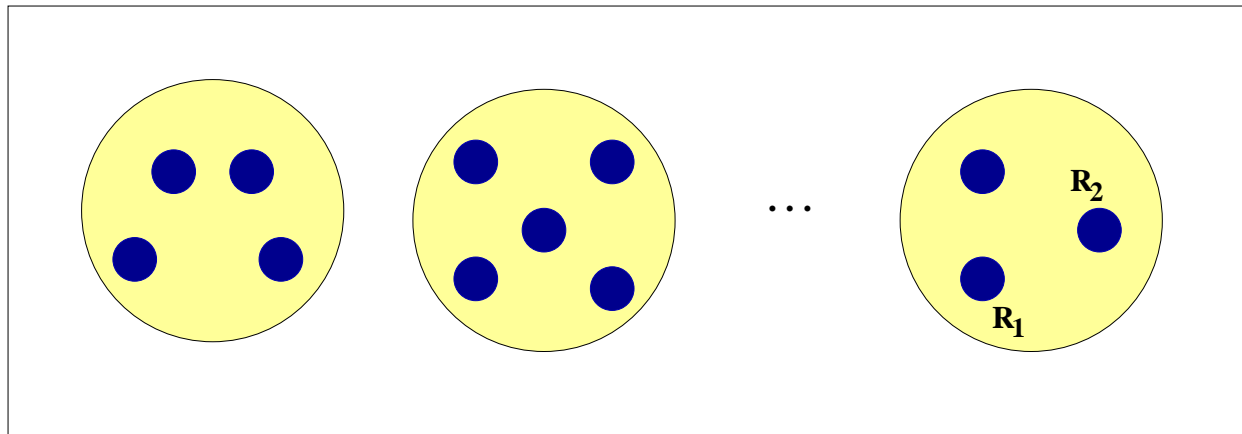
$$a_i, b_i \in \mathbb{Z}, b_i > 0, \gcd(a_i, b_i) = 1, c_i \in \mathbb{C}, m_i \in \mathbb{N} \setminus \{0\}.$$

$(E_n^{b_i} - 1, W_i F_i)$ is the minimal Z -pair for F_i where

$$W_i = \begin{cases} \sum_{j=0}^{a_i-1} E_k^j & a_i > 0 \\ \sum_{j=1}^{-a_i} -E_k^{-j} & \text{otherwise} . \end{cases}$$

A Special Form of Representation

The Equivalence Relation “ \sim ”



$$\begin{array}{l}
 R_1 = \frac{r_1(n)}{(a_1n + b_1k + c_1)^{m_1}} \\
 R_2 = \frac{r_2(n)}{(a_2n + b_2k + c_2)^{m_2}}
 \end{array}
 \left| \begin{array}{l}
 a_i, b_i \in \mathbb{Z}, b_i > 0, \gcd(a_i, b_i) = 1, \\
 c_i \in \mathbb{C}, m_i \in \mathbb{N} \setminus \{0\}, 1 \leq i \leq 2.
 \end{array} \right.$$

$$R_1 \sim R_2 \iff (m_1 = m_2, a_1 = a_2, b_1 = b_2, c_1 - c_2 \in \mathbb{Z} \setminus \{0\}).$$

Experiment

$$T = \sum_i M_i F_i, \quad M_i \in \mathbb{C}(n)[E_n, E_k, E_k^{-1}],$$

$$F_i = \frac{1}{(a_i n + b_i k + c_i)^{m_i}}, \quad a_i, b_i \in \mathbb{Z}, \gcd(a_i, b_i) = 1, b_i > 0, c_i \in \mathbb{C}, m_i \in \mathbb{N} \setminus \{0\}.$$

$$S_1 : T = M_1 F_1 + M_2 F_2,$$

$$M_l = (r_{l_1}(n)/r_{l_2}(n)) E_n^i E_k^j, \quad i \in \{1, 2, 3\}, j \in \{-3, \dots, 3\},$$

$$\deg r_{l_s}(n) = 1, \quad 1 \leq s \leq 2.$$

$$S_2 : T = M_1 F_1, M_1 = (r_1(n)/r_2(n)) E_n^i E_k^{-j} + (r_3(n)/r_4(n)) E_n^u E_k^v,$$

$$i, j, u, v \in \mathbb{N}, \quad i + j = u + v = 2,$$

$$\deg r_1(n) = \deg r_3(n) = 1, \quad \deg r_2(n) = \deg r_4(n) = 2.$$

$$S_3 : T = M_1 F_1 + M_2 F_2 + M_3 F_3.$$

We ran the tests using an implementation of the direct algorithm and the original form of Zeilberger's algorithm.

Note. We enforced a limit of 2,000 seconds on each input rational function in the tests.

Table 1: Total time and space requirements for \mathcal{D} and \mathcal{Z} .

| | Completed | | Timing (seconds) | | Memory (kilobytes) | |
|-------|---------------|---------------|------------------|---------------|--------------------|---------------|
| | \mathcal{D} | \mathcal{Z} | \mathcal{D} | \mathcal{Z} | \mathcal{D} | \mathcal{Z} |
| S_1 | 20 | 15 | 12.15 | 3127.84 | 54,159 | 8,095,930 |
| S_2 | 20 | 18 | 12.43 | 2635.94 | 54,653 | 7,873,146 |
| S_3 | 20 | 0 | 959.07 | — | 3,864,026 | — |

Note. All the reported timings were obtained on a 400Mhz SUN SPARC SOLARIS with 1Gb RAM.

Hypergeometric Case: a Lower Bound

S.A. Abramov, H.Q. Le. A Lower Bound for the Order of the Telescopers for a Hypergeometric Term. In O. Foda, Ed., *Formal Power Series and Algebraic Combinatorics*, on CD, 2002.

The approach is to define operators $M \in \mathbb{C}[n, E_n]$, called **crushing operators**, and show that if $L \in \mathbb{C}[n, E_n]$ is a telescoper for the input hypergeometric term $T(n, k)$, then L is also a crushing operator for $T(n, k)$. The problem is then reduced to **computing a lower bound for the order of the minimal crushing operator for T** .

Rational Normal Forms

S.A. Abramov, M. Petkovšek. Canonical representations of hypergeometric terms. *Proc. FPSAC 2001*, 1–10.

Definition. Let $R \in \mathbb{K}(k) \setminus \{0\}$. If there exist

$f_1, f_2, v_1, v_2 \in \mathbb{K}[k] \setminus \{0\}$ such that

(i) $R = F \cdot \frac{E_k V}{V}$ where $F = \frac{f_1}{f_2}$, $V = \frac{v_1}{v_2}$, and $\gcd(v_1, v_2) = 1$,

(ii) $\forall h \in \mathbb{Z}$, $\gcd(f_1, E_k^h f_2) = 1$ (F is *shift-reduced* w.r.t. k),

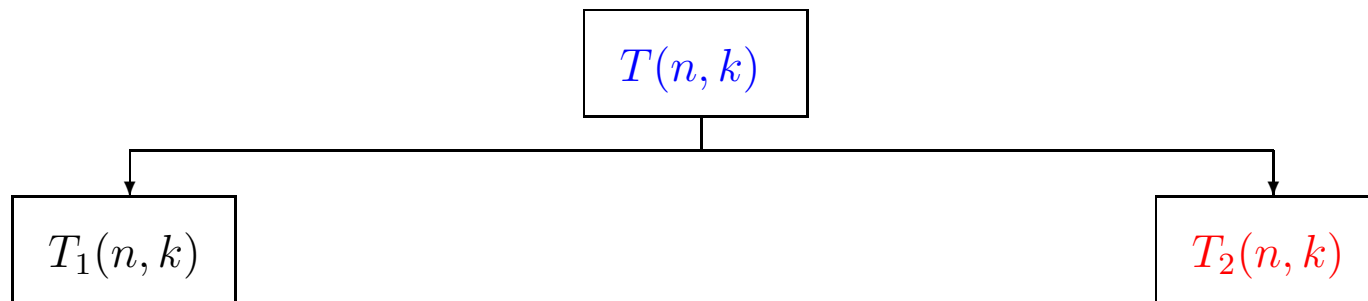
then

$$F \cdot \frac{E_k V}{V} \equiv (z, f_1, f_2, v_1, v_2) \text{ where } z = \text{lcoeff}(F)$$

is a **rational normal form (RNF)** of R .

ADP: Hypergeometric Case

S.A. Abramov, M. Petkovšek. Rational Normal Forms and Minimal Decompositions of Hypergeometric Terms. *Journal of Symbolic Computation* **33**, No. 5, 521–543, 2002.



$$T(n, k) = (E_k - 1)T_1(n, k) + T_2(n, k)$$

$$(z, f_1, f_2, v_1, v_2) := \text{RNF}_k(\mathcal{C}_k(T_2(n, k)))$$

Properties:

$$\forall p : p \mid v_2$$

$$H_1 : E_k^h p \mid v_2 \implies h = 0 \text{ (} v_2 \text{ is shift-free w.r.t. } k\text{)}$$

$$H_2 : E_k^h p \mid f_1 \implies h < 0, E_k^h p \mid f_2 \implies h > 0$$

A Structure Theorem of Hypergeometric Terms

S.A. Abramov, M. Petkovšek. Proof of a Conjecture of Wilf and Zeilberger.

University of Ljubljana, Preprint series 39, 2001.

$$T(n, k) \left\{ \begin{array}{l} \text{RNF}_k(\mathcal{C}_k(T(n, k))) = F \cdot \frac{E_k V}{V} \\ \\ \mathcal{C}_n(T(n, k)) = D \cdot \frac{E_n V}{V} \end{array} \right.$$

for $D \in \mathbb{C}(n, k)$.

$$F, D \in Z_{n,k}$$

Crushing Operators

$T_2(n, k)$: non-summable part of the given hypergeometric term $T(n, k)$.

Consider $M \in \mathbb{C}[n, E_n] : MT_2 \neq 0$. Let

$$\text{RNF}_k(\mathcal{C}_k(MT_2)) = F' \cdot \frac{E_k V'}{V'}, \quad V' = \frac{v'_1}{v'_2}.$$

Then M is a **crushing operator** for T_2 (denoted by $CO(T_2)$) if each of the irreducible factors of v'_2 does not have at least one of the two Properties H_1, H_2 .

Proposition. If L is a telescoper for T_2 , then L is a crushing operator for T_2 .

A Lower Bound for $MCO(T_2)$

Theorem.

$$T_2(n, k) \left\{ \begin{array}{l} \text{RNF}_k(\mathcal{C}_k(T_2(n, k))) = F \cdot \frac{E_k V}{V}, \quad V = \frac{v_1}{v_2} \\ \mathcal{C}_n(T_2(n, k)) = D \cdot \frac{E_n V}{V}, \quad D = \frac{d_1}{d_2} \end{array} \right.$$

Let $\text{ord } MCO(T_2) = \rho$. Let p be an integer-linear factor of v_2 , $\deg_k p = 1$. Then

(i) $\exists h \in \mathbb{Z}$:

$$E_k^h p \mid E_n v_2 \cdot E_n^2 v_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2; \quad (3)$$

(ii) Let ρ_p be the minimal value of ρ such that (3) is satisfied. Then $\text{ord } MCO(T_2) \geq \max_{p \mid v_2} \rho_p$.

Algorithm Description

For each integer-linear factor p of v_2 , $\deg_k p = 1$, compute the minimal value of ρ in the pair (ρ, h) , $h \in \mathbb{Z}$, $\rho \in \mathbb{N} \setminus \{0\}$ such that

(i) $E_k^h p \mid E_n v_2 \cdot E_n^2 v_2 \cdots E_n^\rho v_2$, or

(ii) $E_k^h p \mid d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$.

Note. By the “structure theorem”, $d_2 \in Z_{n,k}$.

Example

Consider the class of hypergeometric terms of the form

$$T(n, k) = \frac{1}{(a_1n + b_1k + c_1)(a_2n + b_2k + c_2)!},$$

$a_1, b_1, a_2, b_2 \in \mathbb{Z}$, $\gcd(a_1, b_1) = 1$, $b_1 \neq 0$, $a_1 \neq a_2$ or $b_1 \neq b_2$.

One can prove that the computed lower bound for the orders of the telescopers for $T(n, k)$ is $|b_1|$.

Experiment

$$\begin{array}{ccc}
 T(n, k) = (E_k - 1) T_1(n, k) + T_2(n, k) & & \\
 \uparrow & & \downarrow \text{Zeilberger's} \\
 (L, LT_1 + G) & \longleftarrow & (L, G)
 \end{array}$$

For $b \in \mathbb{N} \setminus \{0\}$, $j \in \{1, 3\}$, let

$$T_1(n, k) = \frac{1}{(nk - 1)(n - bk - 2)^j (2n + k + 3)!},$$

$$T_2(n, k) = \frac{1}{(n - bk - 2)(2n + k + 3)!}.$$

We are interested in computing the minimal telescopers for

$$T(n, k) = (E_k - 1) T_1(n, k) + T_2(n, k).$$

Table 2: Time and space requirements for \mathcal{L} and \mathcal{Z} .

| | | Timing (seconds) | | Memory (kilobytes) | |
|-----|-----|------------------|---------------|--------------------|---------------|
| j | b | \mathcal{L} | \mathcal{Z} | \mathcal{L} | \mathcal{Z} |
| 1 | 1 | 6.49 | 5.35 | 27,838 | 24,702 |
| | 2 | 8.34 | 34.64 | 33,066 | 142,889 |
| | 3 | 11.13 | 124.53 | 44,233 | 535,736 |
| | 4 | 14.46 | 570.02 | 56,410 | 1,882,730 |
| | 5 | 25.79 | 2999.22 | 97,506 | 6,536,309 |
| 3 | 1 | 14.64 | 16.40 | 62,566 | 73,830 |
| | 2 | 17.24 | 228.59 | 68,304 | 770,529 |
| | 3 | 20.15 | 1286.51 | 78,701 | 3,074,051 |
| | 4 | 24.08 | 8771.08 | 91,844 | 10,766,646 |
| | 5 | 38.60 | 77663.68 | 139,823 | 33,423,168 |

Telescopers Package

S.A. Abramov, K.O. Geddes, H.Q. Le. Computer Algebra Library for the Construction of the Minimal Telescopers. *International Congress of Mathematical Software*, World Scientific, 319–329, 2002.

