PROFILE OF RANDOM RECURSIVE TREES AND RANDOM BINARY SEARCH TREES

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Academia Sinica, Taiwan

(joint with M. Drmota, M. Fuchs, R. Neininger)

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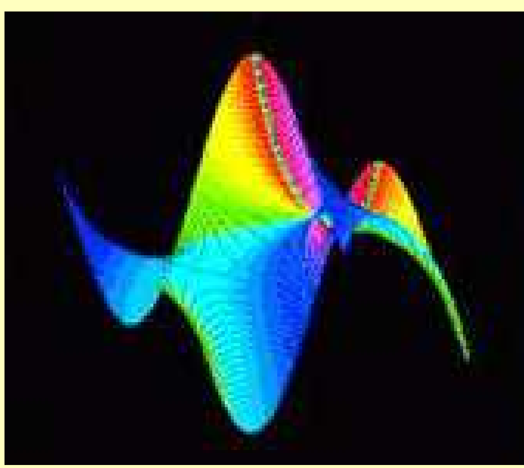


PROFILE OF TREES

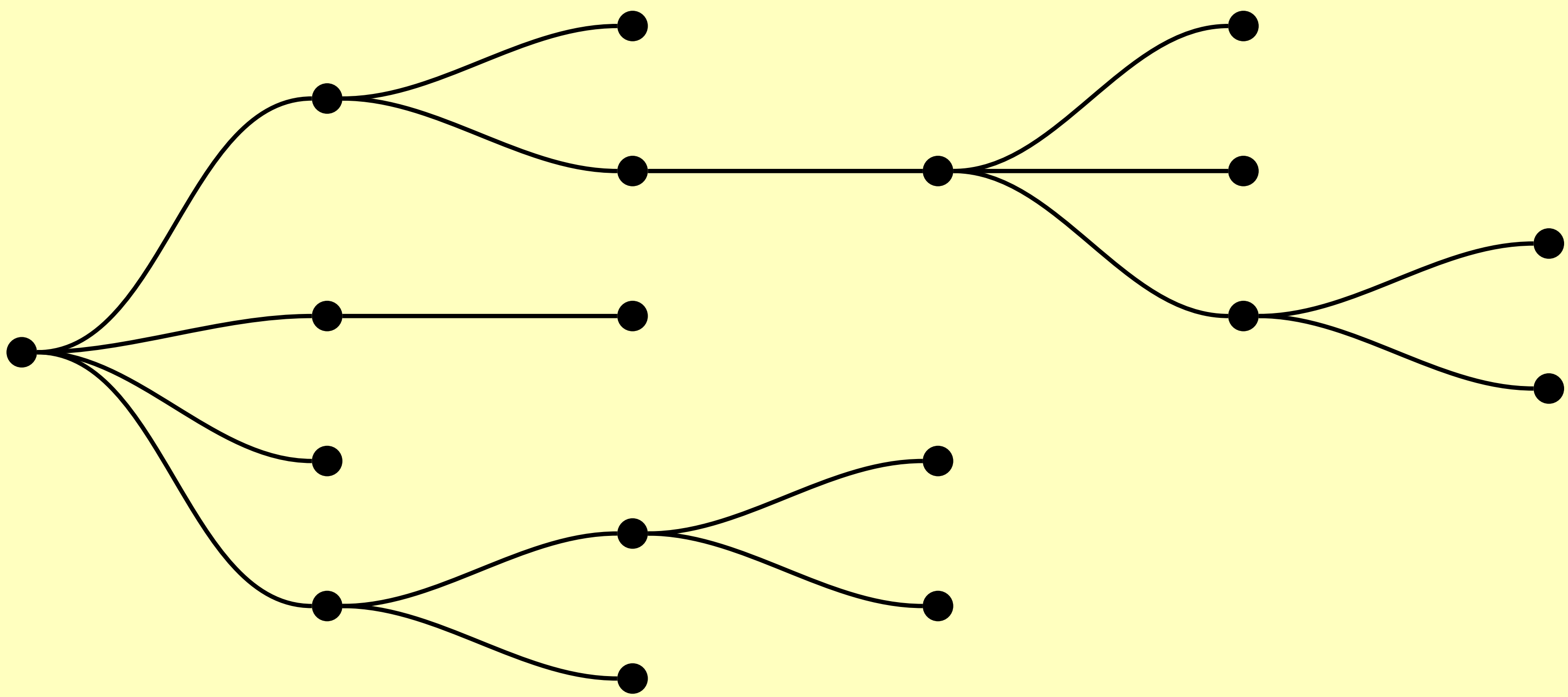

Figure 1: Profile $= \{$1,4,5,3,3,2$\}$

PROFILE: MOTIVATIONS

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- generation of random trees (Devroye, Robson)
- level-wise analysis of quicksort (Chern, H.)

PROFILE OF RANDOM TREES: A RICH SOURCE OF INTRIGUING PHENOMENA

RANDOM RECURSIVE TREES

1

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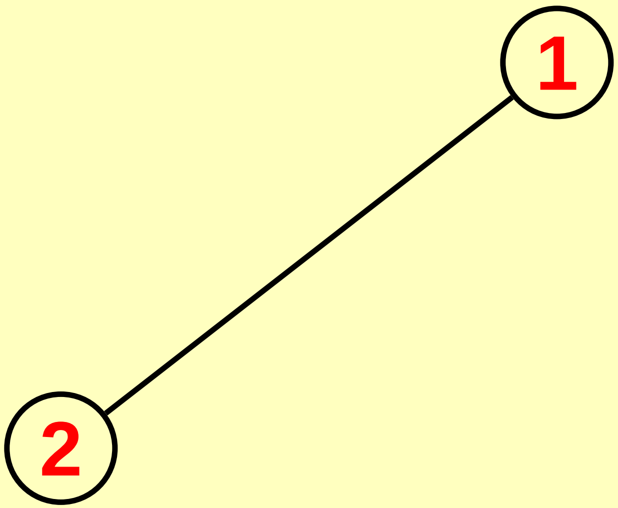

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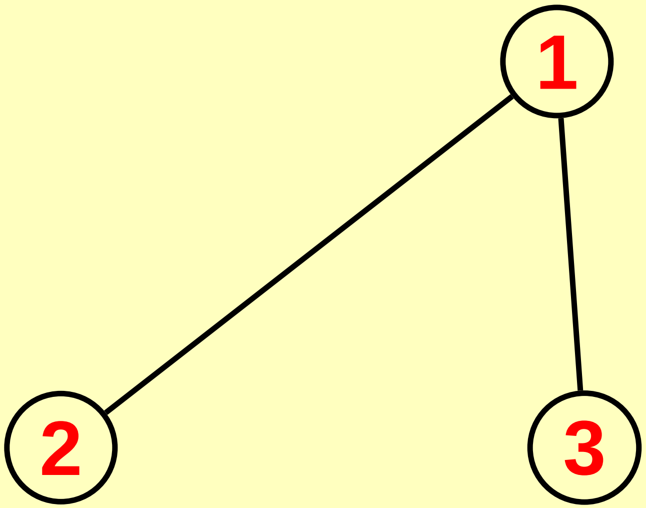

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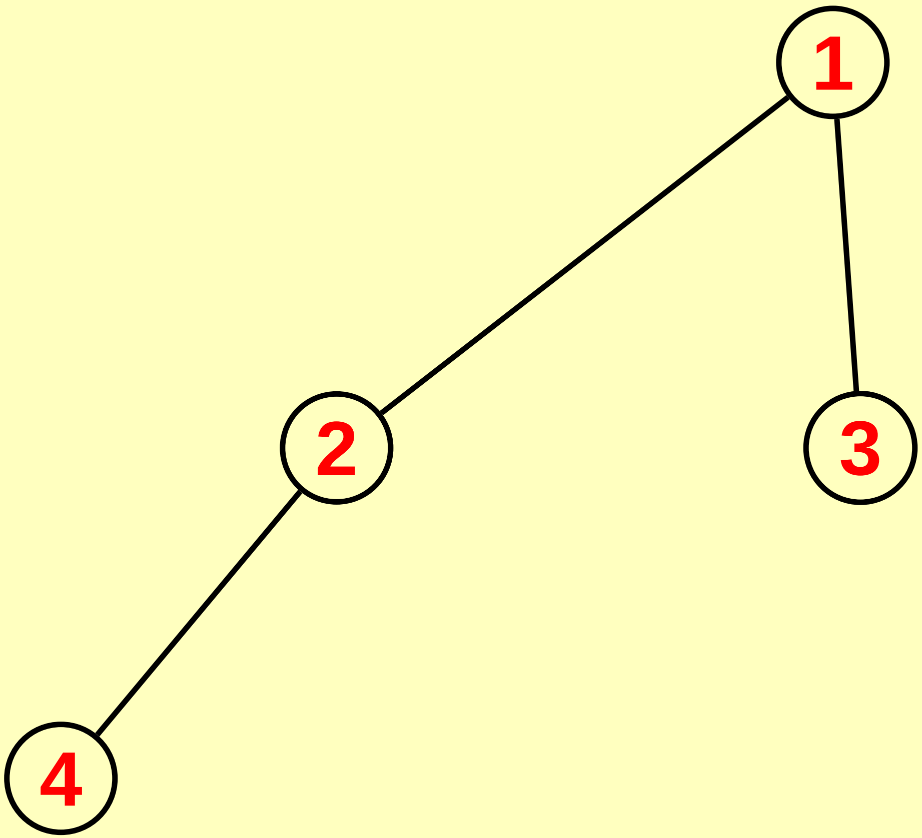

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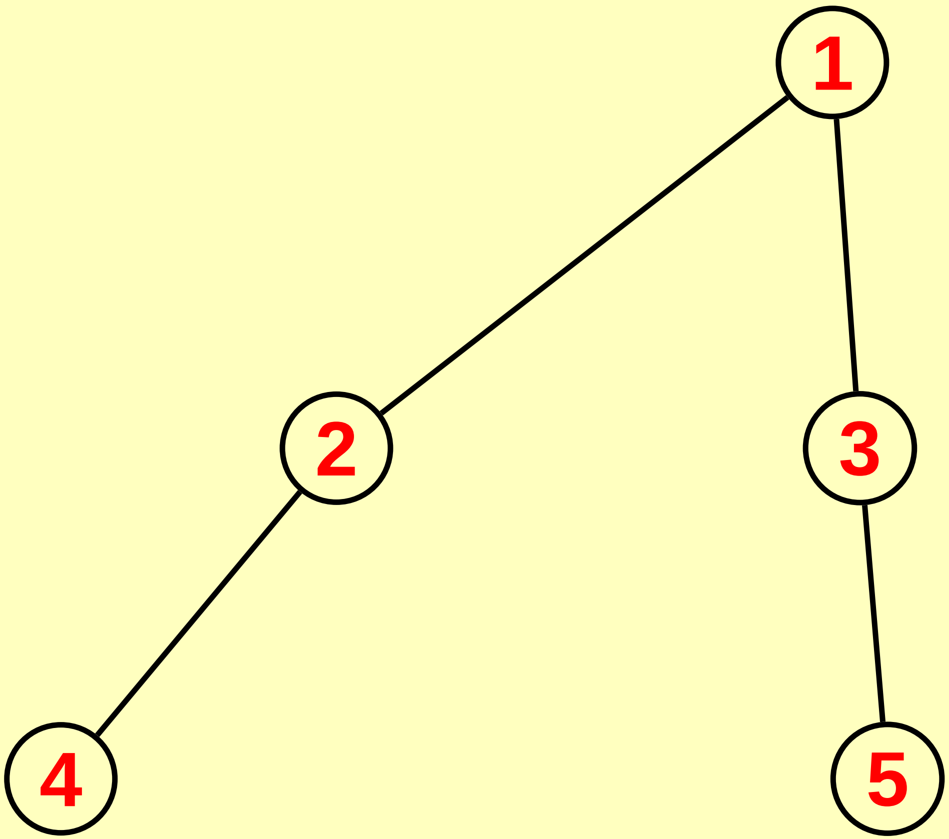

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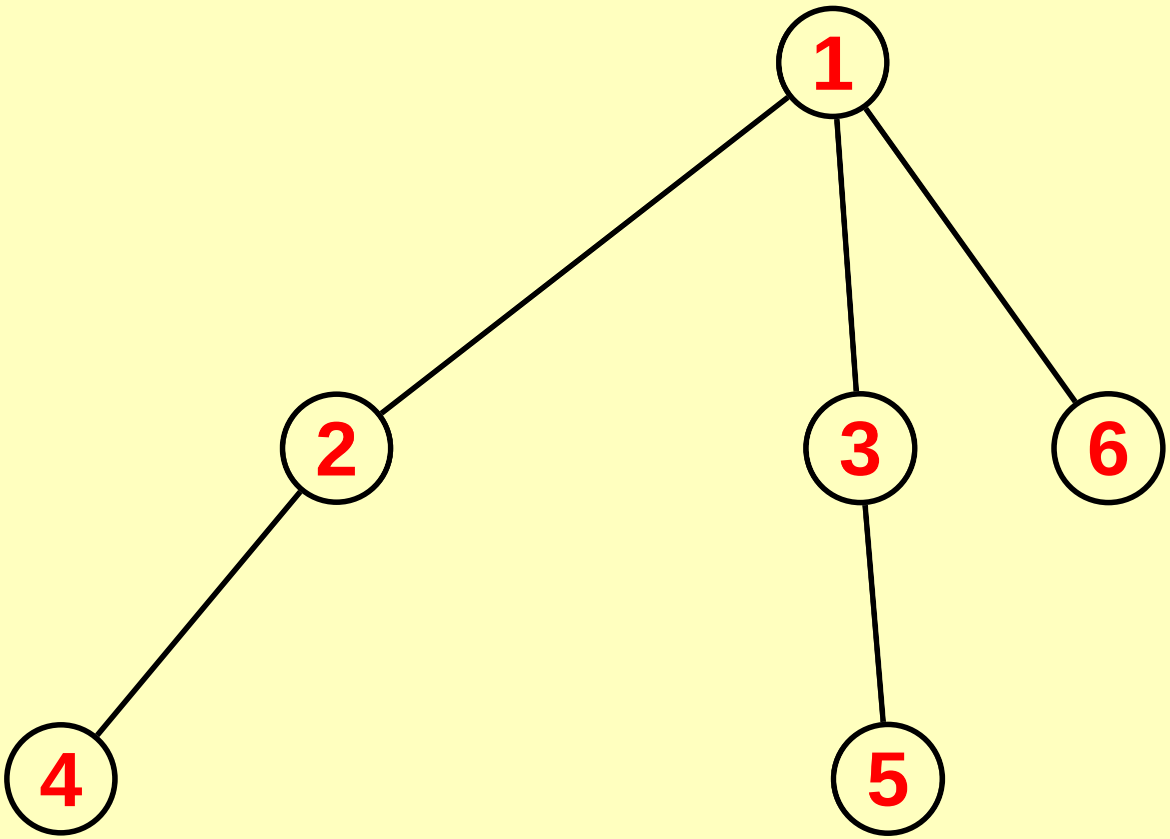

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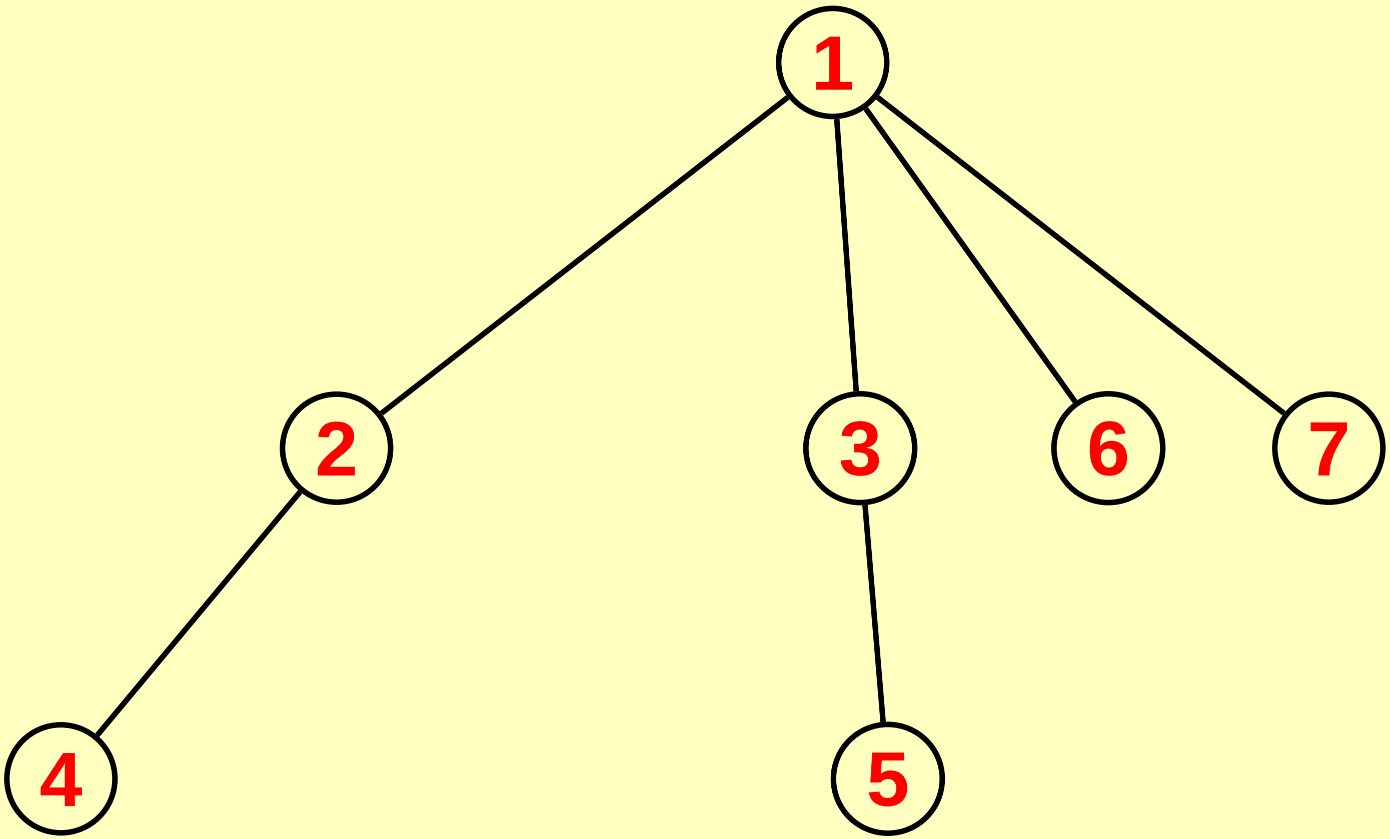

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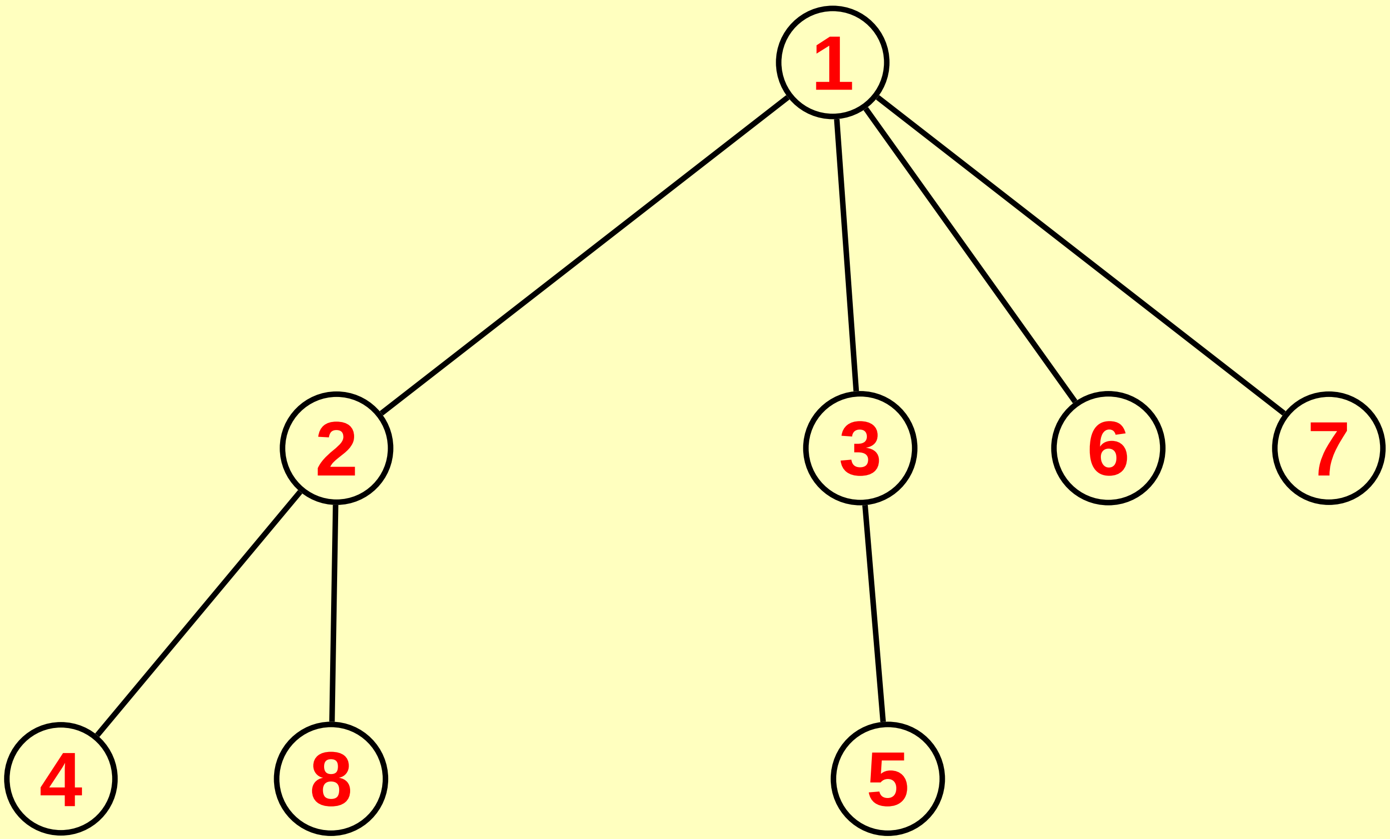

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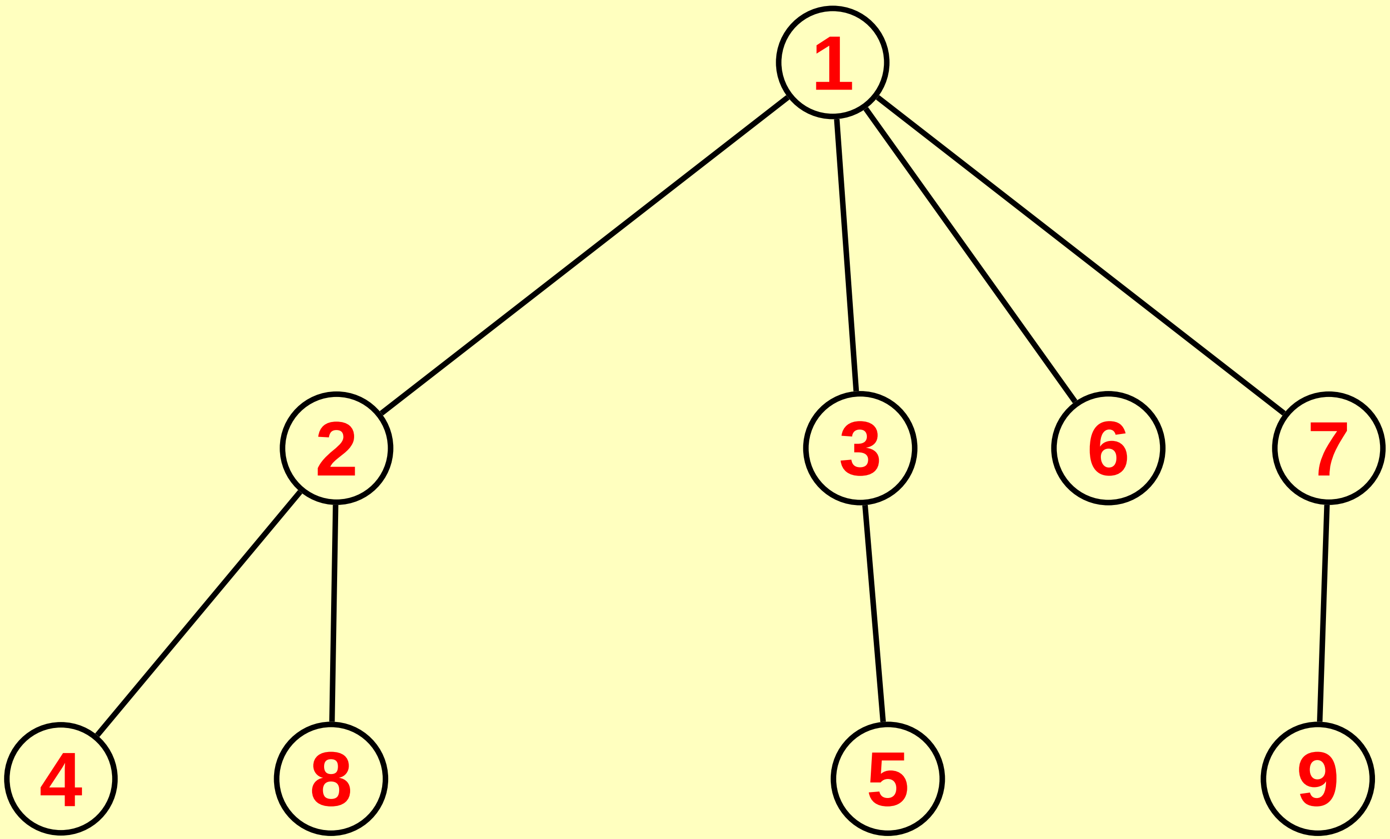

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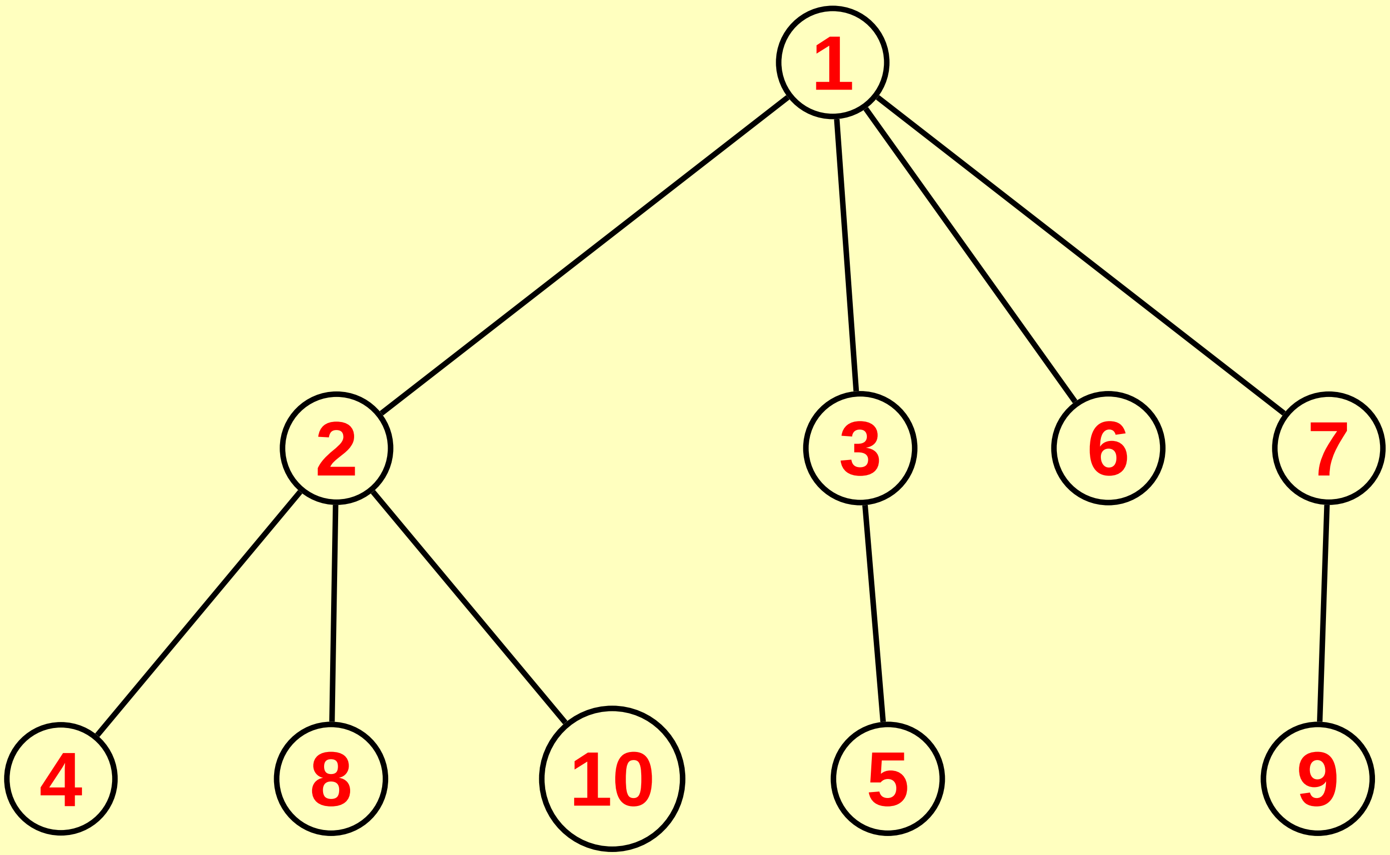

USEFULNESS OF RANDOM RECURSIVE TREES

Simple probability models for

- **system generation (Na, Rapoport, 1970)**
- **spread of contamination of organisms (Meir, Moon, 1974)**
- **pyramid scheme (Bhattacharya, Gastwirth, 1984)**
- **stemma construction of philology (Najock, Heyde, 1982)**
- **Internet interface map (Janic et al., 2002)**
- **stochastic growth of networks (Chan et al., 2003).**
- **Internet (van Mieghem et al., 2001; Devroye, McDiarmid, Reed, 2002)**
- **statistical physics (Tetzlaff, 2002)**

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They also appeared in Hopf algebra under the name of "heap-ordered trees"; see Grossman, Larsen (1989).

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Main range: $k \le K \log n$

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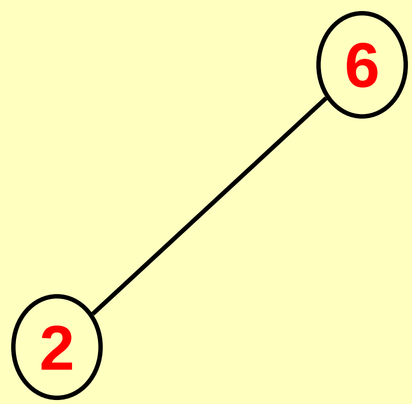

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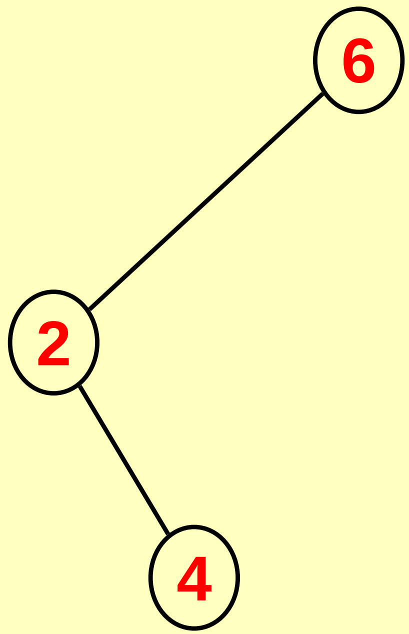

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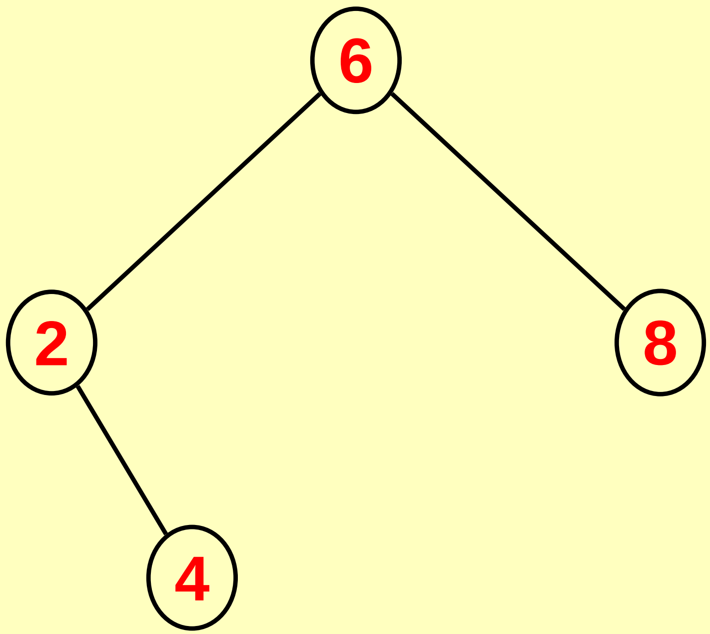

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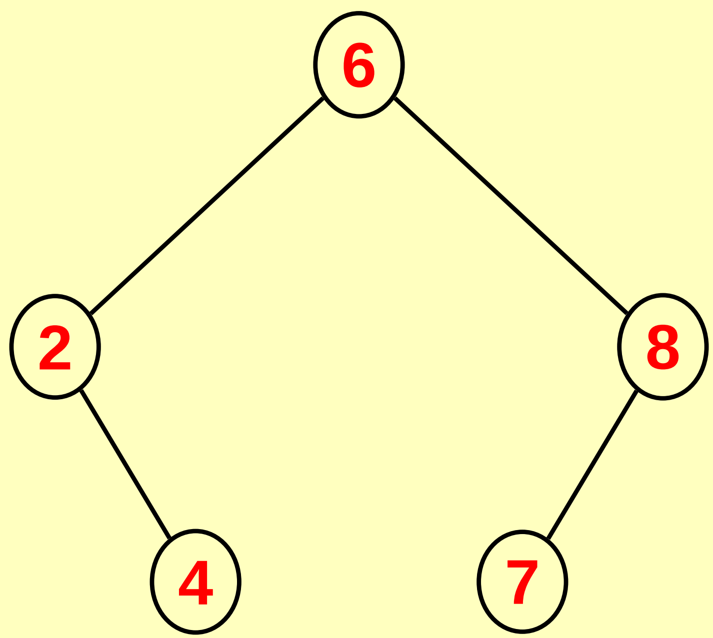

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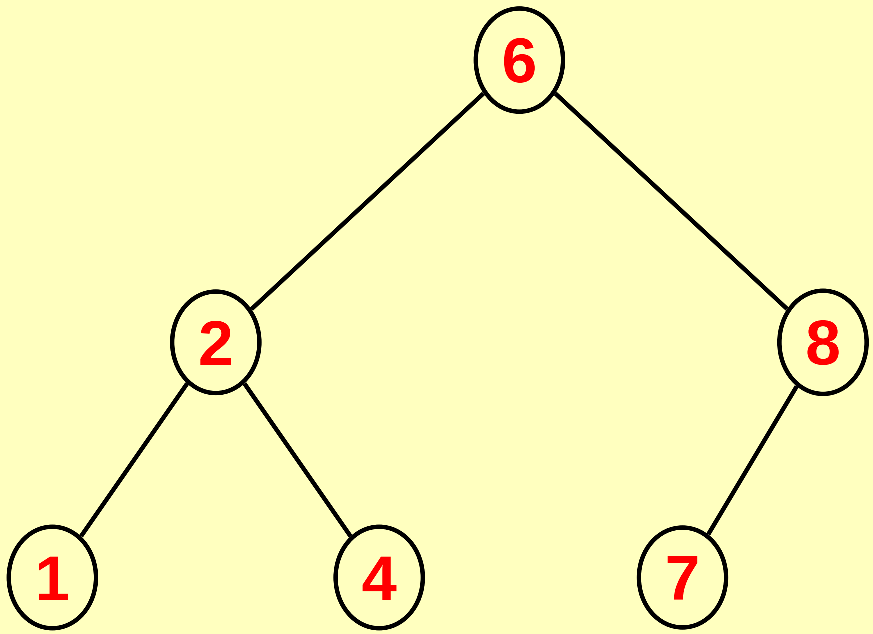

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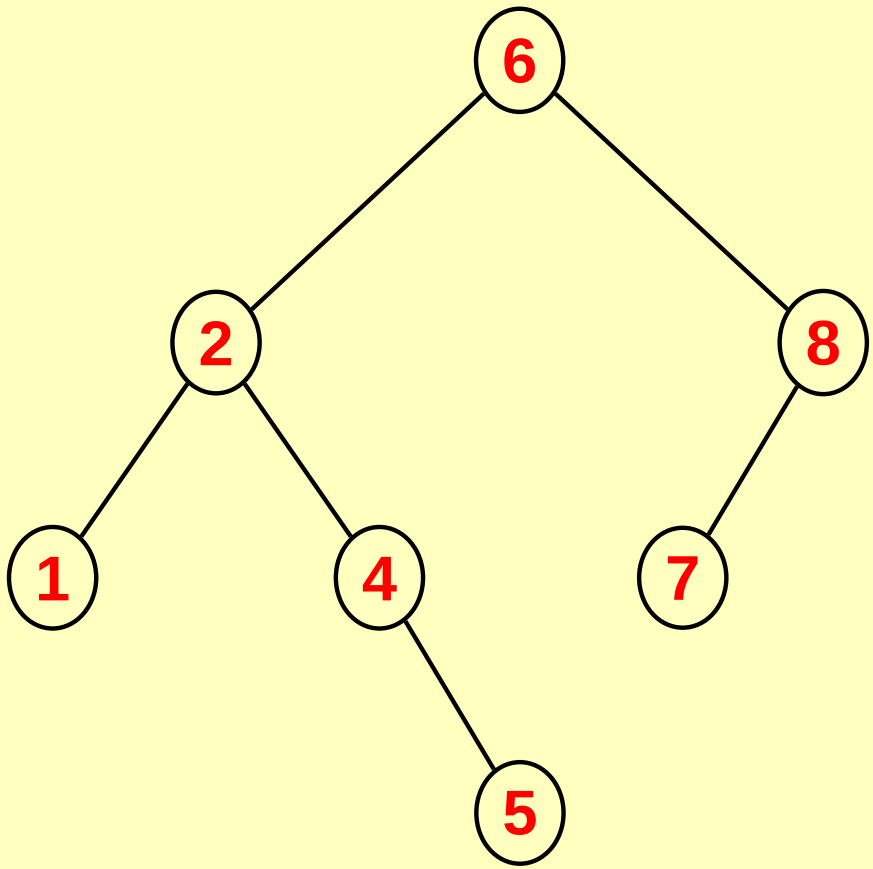

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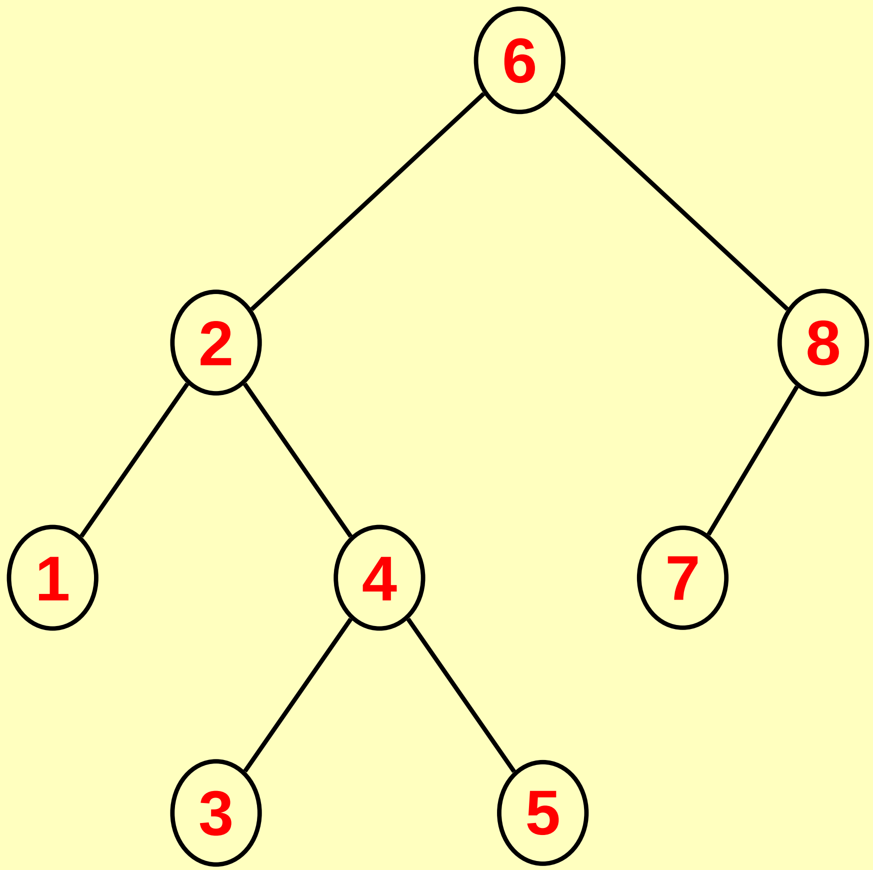

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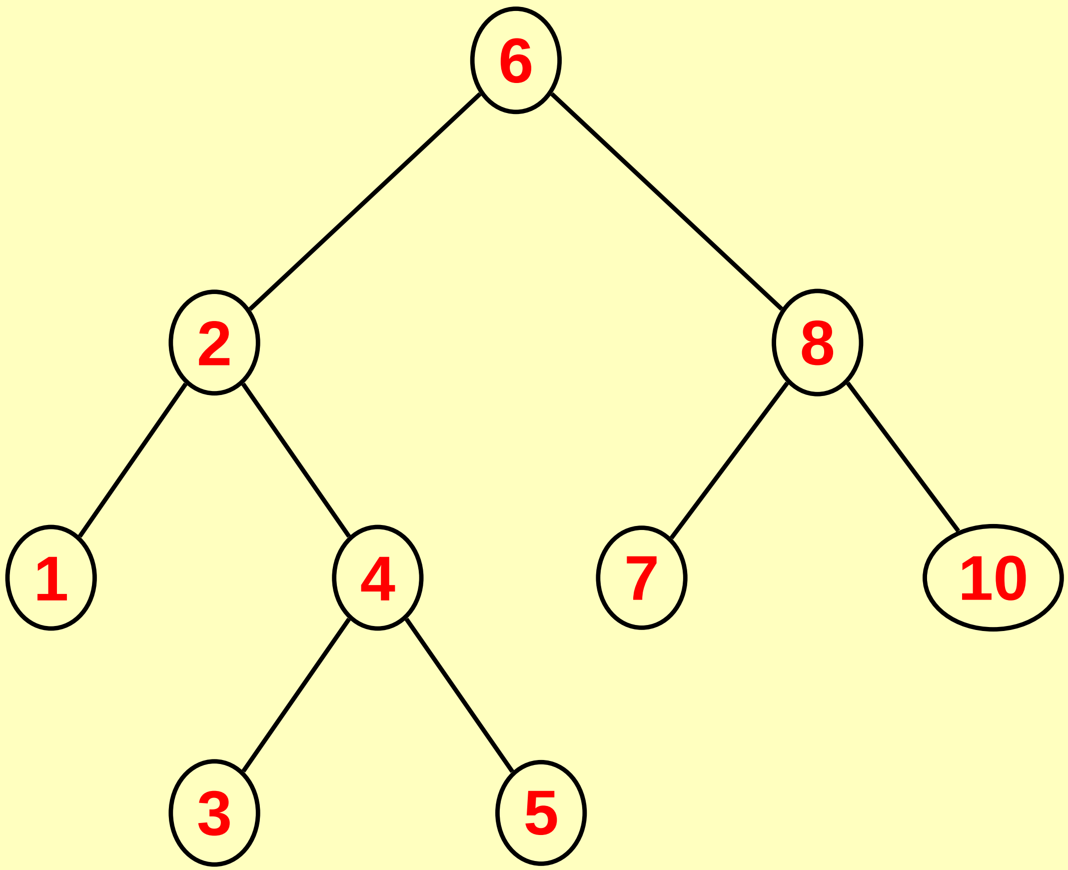

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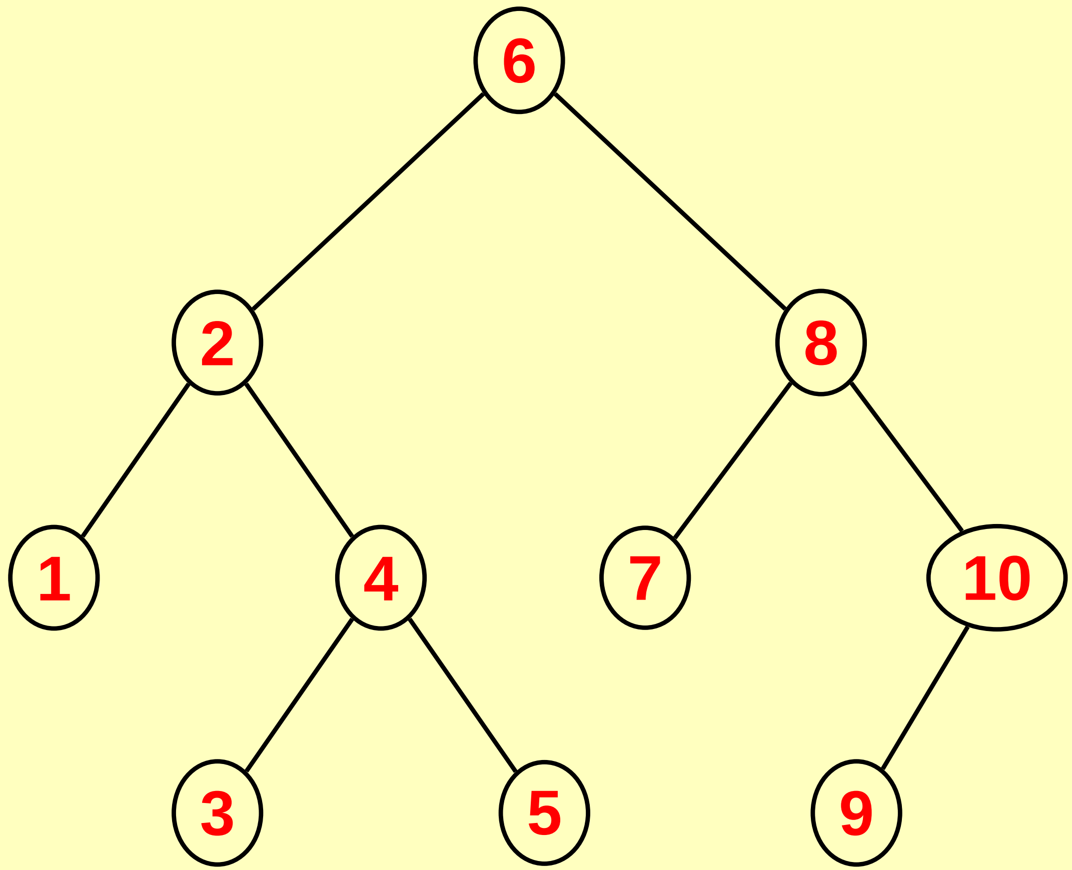
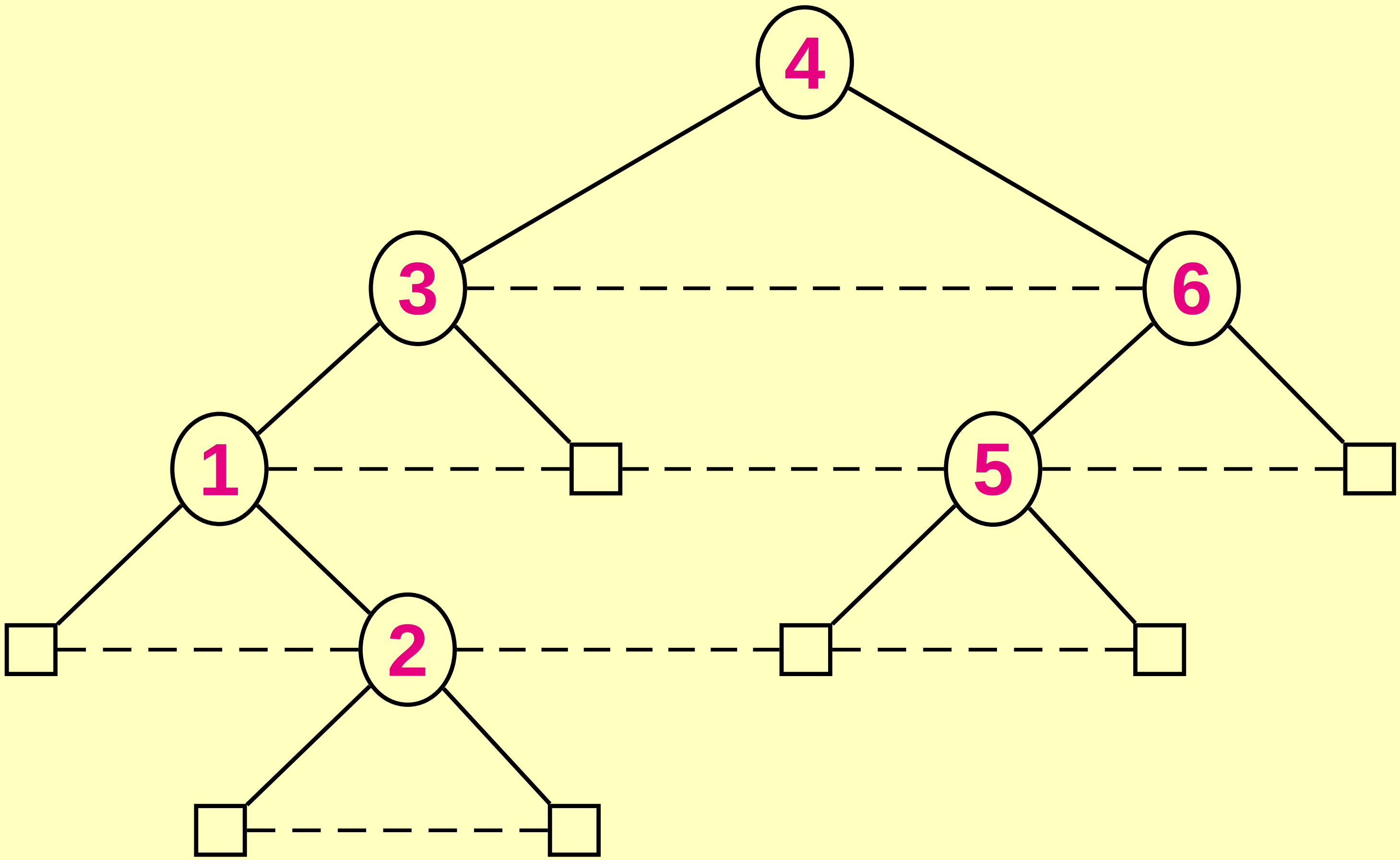


INTERNAL NODES AND EXTERNAL NODES



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The model: Assume that all permutations of n elements are equally likely. Construct the BST from a random permutation.

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Main question: **probabilistic properties of $Y_{n,k}$ and $Z_{n,k}$?**

A CONNECTION

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But (i) it's much simpler to start from the better structured recursive trees; (ii) behaviors not identical in all ranges.

SUMMARY OF MAIN PHENOMENA

Write throughout $\alpha_{n,k} = \dfrac{k}{\log n}$ **and** $\lim_n \alpha_{n,k} = \alpha$**.**

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- **If** $0 \le \alpha \le 1$**, then** $\dfrac{X_{n,k}}{\mathbb{E}(X_{n,k})} \xrightarrow{m} X_\alpha$**.**

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- **For** $k = \log n + O(1)$, **the limit law of** $\dfrac{X_{n,k} - \mathbb{E}(X_{n,k})}{\sqrt{\mathbb{V}(X_{n,k})}}$ **does not exist.**

RECURRENCE OF $X_{n,k}$ ☯

$$X_{n,k} \stackrel{d}{=} X_{\mathsf{uniform}[1,n-1],k-1} + X^*_{n-\mathsf{uniform}[1,n-1],k}$$

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Three proofs: (i) **bijection conditioned on the size of the subtree rooted at 2;** (ii) **above-mentioned transformation;** (iii) **algebraic**

$$X_{n,k} \stackrel{d}{=} \sum_{s\geq 1} \frac{1}{s!} \sum_{j_1+\cdots+j_s=n-1} \underbrace{\binom{n-1}{j_1,\ldots,j_s}\frac{(j-1)!\cdots(j_s-1)!}{(n-1)!}}_{\mathbb{P}(\text{s subtrees have sizes } j_1,\ldots,j_s)}$$
$$\times\left(X_{j_1,k-1}+\cdots+X_{j_s,k-1}\right).$$

EXPECTED VALUE OF $X_{n,k}$

Meir, Moon (1978) (implicit): for $0 \le k < n$

$$\mu_{n,k} := \mathbb{E}(X_{n,k}) = \frac{\text{Stirling1}(n, k+1)}{(n-1)!};$$

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By known estimates for Stirling first numbers (H. 1995)

$$\mu_{n,k} = \frac{(\log n)^k}{\Gamma(1 + \frac{k}{\log n})k!}\left(1 + O\left(\frac{1}{\log n}\right)\right),$$

uniformly for $0 \le k \le K \log n$.

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Also $\dfrac{\log \mu_{n,k}}{\log n} \to \alpha(1-\log\alpha)$ **for** $0 \le \alpha \le K$**, and**

$\mu_{n,k} \to \infty$ **when** $1 \le k \le e\log n - \dfrac{1}{2}\log\log n + O(1)$**.**

TWO COROLLARIES

The estimate for $\mu_{n,k}$ also implies

☞ **an LLT for the depth; see Devroye (1988), Szymański (1990), Mahmoud (1991) for CLT, and Dobrow, Smythe (1996) for Poisson approximation;**

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- ☞ **an LLT for the depth; see Devroye (1988), Szymański (1990), Mahmoud (1991) for CLT, and Dobrow, Smythe (1996) for Poisson approximation;**
- ☞ **that the expected height is bounded above by**

$$\mathbb{E}(H_n) \le e \log n - \frac{1}{2} \log \log n + O(1).$$

(Roughly, the range when $\mu_{n,k} \to \infty$.)

SECOND MOMENT OF $X_{n,k}$

Meir, Moon (1978) (implicit)

$$\mathbb{E}(X_{n,k}^2) = \sum_{0 \le j \le k} \binom{2j}{j} \frac{\text{Stirling1}(n, k+j+1)}{(n-1)!};$$

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see also van der Hofstad et al. (2002); and for $k = O(1)$

$$\mathbb{V}(X_{n,k}) \sim \frac{(\log n)^{2k-1}}{(2k-1)(k-1)!^2}.$$

VARIANCE OF $X_{n,k}$: MIDDLE RANGE

Uniformly for $1 \leq k \leq 2\log n - K\sqrt{\log n}$

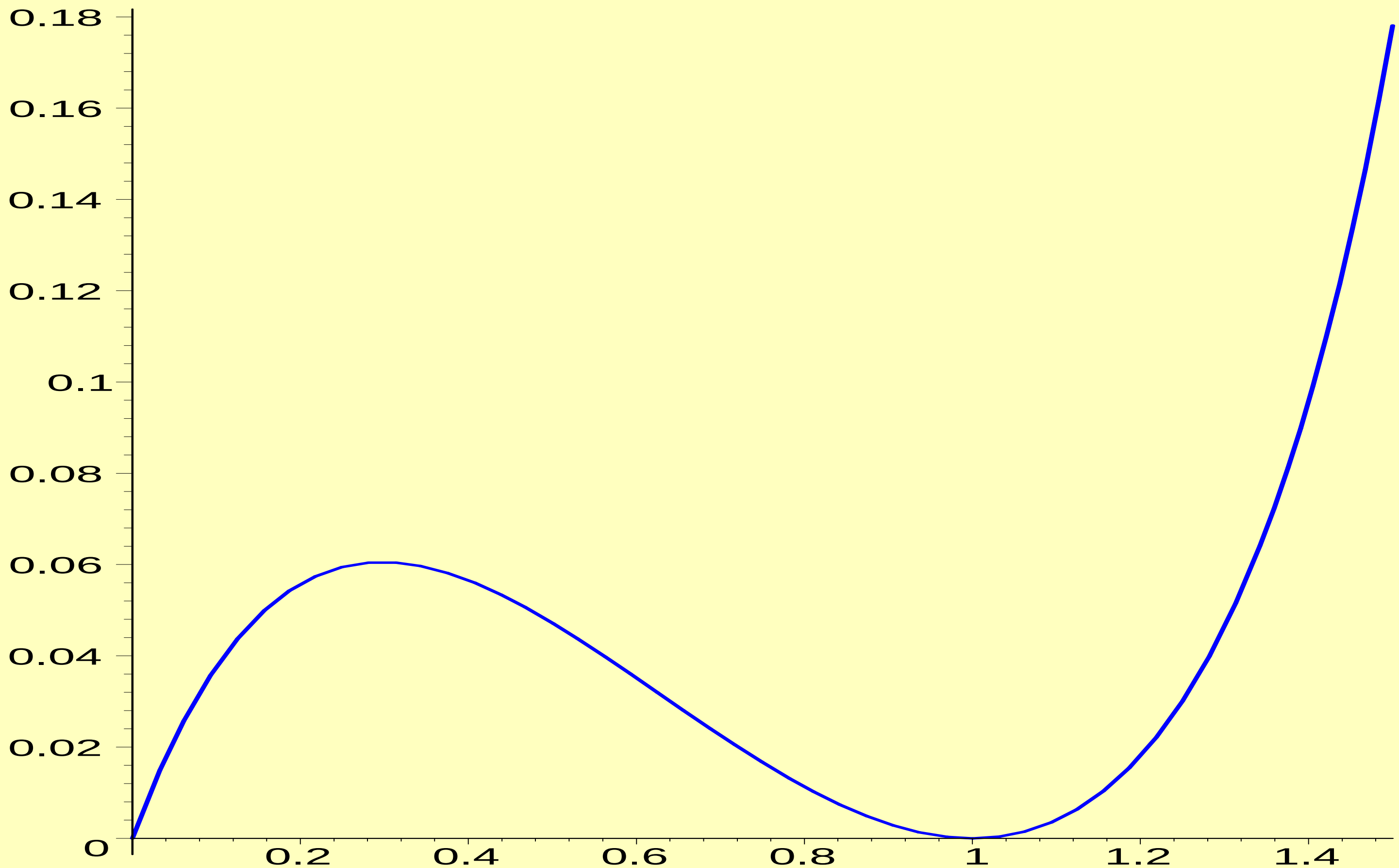
$$\mathbb{V}(X_{n,k}) \sim \phi(\alpha_{n,k})\mu_{n,k}^2,$$

where

$$\phi(x) := \frac{\Gamma(x+1)^2}{(1-\frac{x}{2})\Gamma(2x+1)} - 1.$$

A full asymptotic expansion can be derived.

$$\phi(0) = \phi(1) = \phi'(1) = 0$$



MORE PRECISE ESTIMATES

$$\mathbb{V}(X_{n,k}) \sim \begin{cases} \dfrac{(\log n)^{2k-1}}{(2k-1)(k-1)!^2}, & \textbf{if } k = o(\log n), \\ p(t_n)\left(\dfrac{(\log n)^{k-1}}{k!}\right)^2, & \textbf{if } \begin{aligned} t_n &:= k - \log n \\ &= o(\log n), \end{aligned} \end{cases}$$

where

$$p(t_n) := \left(2 - \frac{\pi^2}{6}\right) t_n^2 - \left(2\zeta(3) + 4\gamma + \frac{\pi^2}{3}(1-\gamma) - 6\right) t_n + 2\gamma^2 - 6\gamma + 8 - 2\zeta(3)(1-\gamma) - \frac{\pi^2}{6}\left(\gamma^2 - 2\gamma + 3\right) - \frac{\pi^4}{360}.$$

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$$\frac{\phi''(1)}{2} = 2 - \frac{\pi^2}{6}$$

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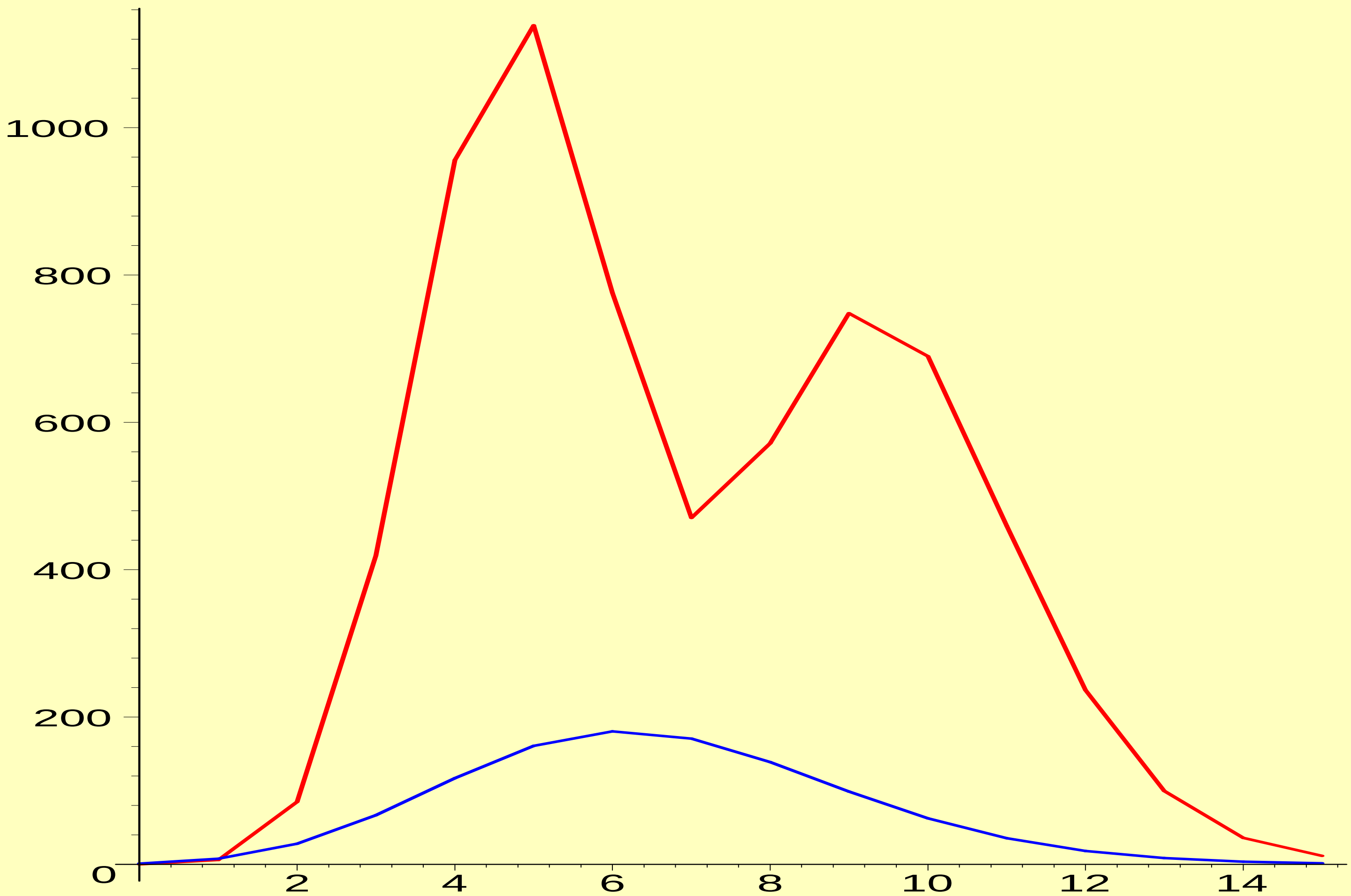
Note that $\mathbb{V}(X_{n,k}) \sim \frac{p(t_n)}{(\log n)^2} \cdot \mu_{n,k}^2$**.**

More precise estimates can be derived.

$\mathbb{E}(X_{500,k})$ AND $\mathbb{V}(X_{500,k})$



A GLOBAL DESCRIPTION OF $\mathbb{V}(X_{n,k})$

$$\frac{\log \mathbb{V}(X_{n,k})}{\log n} \to \begin{cases} 2\alpha(1-\log\alpha), & \textbf{if } 0 \le \alpha \le 2; \\ 4-\alpha\log 4, & \textbf{if } 2 \le \alpha \le 4; \\ \alpha(1-\log\alpha), & \textbf{if } 4 \le \alpha \le K. \end{cases}$$

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Thus $\mathbb{V}(X_{n,k}) \begin{cases} \asymp \mu_{n,k}^2, & \textbf{if } 0 \le \alpha < 2; \\ \gg \mu_{n,k}^2, \mu_{n,k}, & \textbf{if } 2 \le \alpha \le 4; \\ \asymp \mu_{n,k}, & \textbf{if } 4 < \alpha \le K. \end{cases}$

LIMIT DISTRIBUTION: GENESIS

Let $\bar{X}_{n,k} := X_{n,k}/\mu_{n,k}$ **and** $I_n :=$ uniform$[1, n-1]$.

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Then from

$$X_{n,k} \stackrel{d}{=} X_{I_n,k-1} + X^*_{n-I_n,k},$$

it follows that

$$\bar{X}_{n,k} \stackrel{d}{=} \frac{\mu_{I_n,k-1}}{\mu_{n,k}} \bar{X}_{I_n,k-1} + \frac{\mu_{n-I_n,k}}{\mu_{n,k}} \bar{X}^*_{n-I_n,k}.$$

LIMIT DISTRIBUTION: GENESIS

Since $\mu_{n,k} \approx (\log n)^k/k!$ **and** $I_n = \lceil (n-1)U \rceil$**, we expect that**

$$\frac{\mu_{I_n,k-1}}{\mu_{n,k}} \approx \frac{k}{\log n}\left(\frac{\log n + \log U}{\log n}\right)^{k-1} \xrightarrow{d} \alpha U^{\alpha},$$

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and similarly $\dfrac{\mu_{n-I_n,k}}{\mu_{n,k}} \xrightarrow{d} (1-U)^\alpha$**. Thus if**

$\bar{X}_{n,k} \xrightarrow{d} X_\alpha$**, then** $X_\alpha \stackrel{d}{=} \alpha U^\alpha X_\alpha + (1-U)^\alpha X_\alpha^*$**.**

LIMIT DISTRIBUTIONS: NEW RESULTS

For $0 \leq \alpha < e$

$$\frac{X_{n,k}}{\mu_{n,k}} \xrightarrow{d} X_\alpha,$$

with convergence of the first m **moments for** $0 \leq \alpha < m^{1/(m-1)}$**.**

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In particular, convergence of all moments holds only for $\alpha \in [0, 1]$.

$$X_0 = X_1 \equiv 1$$

RECURRENCE OF MOMENTS

Let $\nu_m := \mathbb{E}(X_\alpha^m)$**. Then** $\nu_0 = \nu_1 = 1$ **and**

$$\nu_m = \frac{1}{m - \alpha^{m-1}} \sum_{1 \le j < m} \binom{m}{j} \nu_j \nu_{m-j} \alpha^{j-1} \times \frac{\Gamma(j\alpha + 1)\Gamma((m-j)\alpha + 1)}{\Gamma(m\alpha + 1)} \qquad (m \ge 2),$$

for $0 \le \alpha < m^{1/(m-1)}$**.**

ASYMPTOTIC NORMALITY WHEN $\alpha = 0$

If $1 \le k = o(\log n)$, then

$$\sup_x \left| \mathbb{P}\left(\frac{X_{n,k} - \frac{(\log n)^k}{k!}}{\sqrt{\frac{(\log n)^{2k-1}}{(k-1)!^2(2k-1)}}} < x \right) - \Phi(x) \right| = O\left(\sqrt{\frac{k}{\log n}} \right),$$

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where $\Phi(x)$ denotes the standard normal distribution.

In particular, $\begin{cases} X_{n,1} \sim N(\log n, \log n) \\ X_{n,2} \sim N(\frac{1}{2}(\log n)^2, \frac{1}{3}(\log n)^3) \end{cases}$ **...**

A QUICKSORT-TYPE LIMIT LAW WHEN $\alpha = 1$

If $t_n := k - \log n = o(\log n)$ **and** $t_n \to \infty$**, then**

$$\frac{X_{n,k} - \mu_{n,k}}{t_n \frac{(\log n)^{k-1}}{k!}} \xrightarrow{m} X_1',$$

where $X_1' = (\mathbf{d}X_\alpha / \mathbf{d}\alpha)|_{\alpha=1}$ **or**

$$X_1' \stackrel{d}{=} UX_1' + (1-U)X_1'^{*} + U + U\log U + (1-U)\log(1-U).$$

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Same law as total path length (or left path length in BST; Dobrow, Fill, 1999) and cost of an in-situ permutation algorithm.

NONEXISTENCE OF LIMIT LAW WHEN $k = \log n + O(1)$

If $k = \log n + O(1)$, then the limit distribution of $(X_{n,k} - \mu_{n,k})/\sqrt{\mathbb{V}(X_{n,k})}$ does not exist.

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Main step of the proof:

$$\mathbb{E}(X_{n,k} - \mu_{n,k})^m \sim \underbrace{\text{Polynomial}}_{\text{degree}=m}(t_n) \left(\frac{(\log n)^{k-1}}{k!} \right)^m ;$$

the remaining proof is similar to that used for the space requirement of random m-ary search trees when $m \geq 27$.

APPROACHES USED

Convergence in distribution: contraction method

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Common to both approaches is the resolution of the double-indexed recurrence

$$a_{n,k} = b_{n,k} + \frac{1}{n-1} \sum_{1 \le j < n} (a_{j,k-1} + a_{j,k}).$$

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(Martingale arguments also apply to recursive trees.)

SOLUTION OF THE RECURRENCE

If

$$a_{n,k} = \frac{1}{n-1} \sum_{1 \le j < n} \left(a_{j,k} + a_{j,k-1} \right) + b_{n,k}, \quad (n \ge 2; k \ge 1),$$

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$$a_{n,k} = b_{n,k} + \sum_{2 \le j < n} j^{-1} \sum_{0 \le r \le k} b_{j,k-r} [w^r](w+1) \prod_{j < \ell < n} \left(1 + \frac{w}{\ell}\right).$$

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Proof by GF

WHICHEVER APPROACH REQUIRES MEAN VALUE

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$$= \frac{\text{Stirling1}(n, k+1)}{(n-1)!}$$

$$= \frac{(\log n)^k}{\Gamma(1 + \frac{k}{\log n}) k!} \left(1 + O\left(\frac{1}{\log n}\right)\right).$$

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Sufficient for all ranges except $k = \log n + O(1)$**.**

ASYMPTOTICS OF HIGHER MOMENTS

For method of moments: all moments (centered or not) satisfy the same type of recurrences

$$a_{n,k} = \frac{1}{n-1} \sum_{1 \le j < n} (a_{j,k} + a_{j,k-1}) + b_{n,k},$$

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with different $b_{n,k}$, and we need the $\sim$-transfer:

if $b_{n,k} \sim c\left(\frac{(\log n)^k}{\Gamma(1+\alpha)k!}\right)^m$**, then** $a_{n,k} \sim c\frac{m\alpha+1}{m\alpha-\alpha^m}\left(\frac{(\log n)^k}{\Gamma(1+\alpha)k!}\right)^m$**.**

$$c = c(\alpha)$$

CONTRACTION METHOD: CONVERGENCE IN DISTRIBUTION

For contraction method: (i) **choose** $s \in (1, 2]$ **such that**

$$\mathbb{E}\left(\alpha^s U^{s\alpha} + (1-U)^{\alpha s}\right) = \frac{\alpha^s + 1}{\alpha s + 1} < 1,$$

or, equivalently, $s - \alpha^{s-1} > 0$**.**

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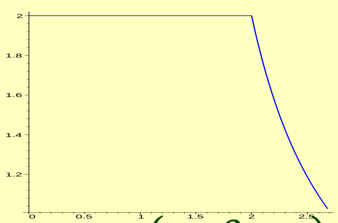

(ii) **need the O-transfer: If** $b_{n,k} = o(\mu_{n,k}^s)$**, where** $s > 1$ **satisfies** $s - \alpha^{s-1} > 0$**, then** $a_{n,k} = o(\mu_{n,k}^s)$**.**

CONTRACTION METHOD: CONVERGENCE RATE

For contraction method, a convergence rate in Zolotarev distance can be derived when $0 \le \alpha < 2$ by proving the stronger O-transfer:

$$\text{if } b_{n,k} = O\left(\frac{|k - \alpha \log n| \vee 1}{\log n} \mu_{n,k}^s\right),$$

$$\text{then } a_{n,k} = O\left(\frac{|k - \alpha \log n| \vee 1}{\log n} \mu_{n,k}^s\right),$$

where $s > 1$ satisfies $s - \alpha^{s-1} > 0$.

THE CENTRAL RANGE $\alpha = 1$

When $k = \log n + o(\log n)$**, same moments approach but starting with** $\bar{P}_{n,k}(y) := \mathbb{E}(e^{(X_{n,k}-\mu_{n,k})y})$**, which satisfies**

$$\bar{P}_{n,k}(y) = \frac{1}{n-1} \sum_{1 \le j < n} \bar{P}_{j,k-1}(y) \bar{P}_{n-j,k}(y) e^{-\Delta_{n,k}(j)y},$$

for $n \ge 2, k \ge 1$**, where** $\Delta_{n,k}(j) := \mu_{j,k-1} + \mu_{n-j,k} - \mu_{n,k}$**.**

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for $n \ge 2, k \ge 1$, where $\Delta_{n,k}(j) := \mu_{j,k-1} + \mu_{n-j,k} - \mu_{n,k}$.

A more uniform estimate for $\Delta_{n,k}(j)$ is needed.

THE NORMAL RANGE: $\alpha = 0$

Let $\sigma_{n,k} := \sqrt{\frac{(\log n)^{2k-1}}{(2k-1)(k-1)!^2}}$ **and** $\Lambda := \frac{\log n}{k}$**.**

Two estimates are needed: (i) **uniformly for** $|\theta| \leq \varepsilon \Lambda^{1/6}$

$$\left| \mathbb{E}\left(e^{\frac{X_{n,k} - (\log n)^k / k!}{\sigma_{n,k}} i\theta} \right) - e^{-\theta^2/2} \right| = O\left(\frac{|\theta| + |\theta|^3}{\sqrt{\Lambda}} e^{-\theta^2/2} + n^{-\varepsilon} \right);$$

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and (ii) **uniformly for** $\varepsilon\Lambda^{1/6} \le |\theta| \le \varepsilon\sqrt{\Lambda}$

$$\mathbb{E}\left(e^{\frac{X_{n,k}-(\log n)^k/k!}{\sigma_{n,k}} i\theta}\right) = O(e^{-\theta^2/4} + n^{-\varepsilon}).$$

IDEA OF PROOF

Let $Q_k(z,y) := \sum_{n\ge 1} \mathbb{E}(e^{X_{n,k}y}) z^{n-1}$**. Then**

$$\begin{cases} Q_0(z,s) = \dfrac{e^s}{1-z}, \\ Q_k(z,s) = \exp\left(\displaystyle\int_0^z Q_{k-1}(t,s)\,\mathbf{d}t\right), \qquad (k \ge 1). \end{cases}$$

IDEA OF PROOF

Decompose Q_k in two ways as follows.

$$Q_k(z,s) := \exp\left(\sum_{m\geq 0} \frac{V_{k,m}(-\log(1-z))}{m!} s^m\right)$$

$$:= \frac{1}{1-z} \sum_{m\geq 0} \frac{W_{k,m}(-\log(1-z))}{m!} s^m.$$

Then manipulate inductively the recurrences of V and W.

PROFILE OF BST

Our approach based on contraction method and method of moments is applicable to BSTs.

$$\begin{cases} Y_{n,k} & := \textbf{the number of external nodes at level } k \\ Z_{n,k} & := \textbf{the number of internal nodes at level } k \end{cases}$$

which satisfies the same type of recurrences

$$C_{n,k} \stackrel{d}{=} C_{\text{uniform}[0,n-1],k-1} + C^*_{n-1-\text{uniform}[0,n-1],k-1},$$

with different initial conditions.

MEAN VALUES

Mean values known since 1960's: **Lynch (1965), Knuth (1998), Brown, Shubert (1984), Mahmoud, Pittel (1984), Pittel (1984), Louchard (1987), Devroye (1988).**

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$$\begin{aligned}\mathbb{E}(Y_{n,k}) &= \frac{2^k}{n!}\,\text{Stirling1}(n,k)\\ &= \frac{(2\log n)^k}{\Gamma(\alpha_{n,k})k!n}\left(1+O\left(\frac{1}{\log n}\right)\right),\end{aligned}$$

uniformly for $0 \le k \le K\log n$**.**

THE BINARY SEARCH TREE CONSTANTS

$$\frac{\log \mathbb{E}(Y_{n,k})}{\log n} \to \lambda(\alpha) := \alpha - 1 - \alpha \log(\alpha/2).$$

Thus $\mathbb{E}(Y_{n,k}) \to \infty$ **when** $\alpha_- < \alpha < \alpha_+$**, where** $0 < \alpha_- < 1 < \alpha_+$ **are the two real zeros of the equation** $z - 1 - z\log(z/2)$ **or** $e^{(z-1)/z} = z/2$**.**

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Also the estimate for $\mu_{n,k}$ **implies that the expected height is bounded above by**

$$\mathbb{E}(H_n) \le \alpha_+ \log n - \frac{\alpha_+}{2(\alpha_+ - 1)} \log\log n + O(1).$$

LIMIT DISTRIBUTIONS

Chauvin et al. (2001): almost sure convergence of

$$\frac{Y_{n,k}}{\mathbb{E}(Y_{n,k})}, \frac{Z_{n,k}}{\mathbb{E}(Z_{n,k})} \xrightarrow{d} Y_\alpha,$$

for $1.2 \le \alpha \le 2.8$**, where**

$$Y_\alpha \stackrel{d}{=} \frac{\alpha}{2} U^{\alpha-1} Y_\alpha + \frac{\alpha}{2}(1-U)^{\alpha-1} Y_\alpha^*.$$

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$$Y_\alpha \stackrel{d}{=} \frac{\alpha}{2} U^{\alpha-1} Y_\alpha + \frac{\alpha}{2}(1-U)^{\alpha-1} Y_\alpha^*.$$

Chauvin et al. (2003+): convergence in probability for $Y_{n,k}/\mathbb{E}(Y_{n,k})$ **when** $k = \alpha \log n + O(\sqrt{\log n})$**,** $\alpha_- < \alpha < \alpha_+$**.**

NEW RESULTS

Define $\bar{Z}_{n,k} := \begin{cases} 2^k - Z_{n,k}, & \textbf{if } \alpha_- \le \alpha < 1; \\ Z_{n,k}, & \textbf{if } 1 \le \alpha < \alpha_+. \end{cases}$

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If $k = \alpha \log n + o(\log n)$**, where** $\alpha_- < \alpha < \alpha_+$**, then**

$$\frac{Y_{n,k}}{\mathbb{E}(Y_{n,k})}, \frac{\bar{Z}_{n,k}}{\mathbb{E}(\bar{Z}_{n,k})} \xrightarrow{d} Y_\alpha,$$

with convergence of all moments for $\alpha \in [1, 2]$ **but not for** α **outside** $[1, 2]$**.**

MOMENTS OF THE LIMIT LAW

$\eta_0 = \eta_1 = 1$ **and for** $m \geq 2$

$$\eta_m = \frac{(\alpha/2)^m}{m(\alpha-1)+1-2(\alpha/2)^m} \times \sum_{1\leq j<m} \binom{m}{j} \eta_j \eta_{m-j} \frac{\Gamma(j(\alpha-1)+1)\Gamma((m-j)(\alpha-1)+1)}{\Gamma(m(\alpha-1)+1)}.$$

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The polynomial $m(z-1)+1-2(z/2)^m$ **has two positive zeros** z_m^- **and** z_m^+**, where** $z_m^- \uparrow 1$**, and** $z_m^+ \downarrow 2$**.**

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If $k = 2\log n + t_n$, where $t_n = o(\log n)$ and $t_n \to \infty$, then

$$\frac{Y_{n,k} - \mathbb{E}(Y_{n,k})}{t_n n^{\lambda(\alpha_{n,k})}/\sqrt{4\pi(\log n)^3}}, \frac{Z_{n,k} - \mathbb{E}(Z_{n,k})}{t_n n^{\lambda(\alpha_{n,k})}/\sqrt{4\pi(\log n)^3}} \xrightarrow{m} Y_2',$$

where $Y_2' := (dY_\alpha/d\alpha)|_{\alpha=2}$.

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where $Y_2' := (dY_\alpha/d\alpha)|_{\alpha=2}$.

The limit law Y_2' is essentially the quicksort limit law

$$Y_2' \stackrel{d}{=} UY_2' + (1-U)Y_2'^{*} + \frac{1}{2} + U\log U + (1-U)\log(1-U).$$

NONEXISTENCE OF LIMIT DISTRIBUTION

If $k = 2\log n + O(1)$**, then neither of the limit distribution of**

$$\left\{ \frac{Y_{n,k} - \mathbb{E}(Y_{n,k})}{\sqrt{\mathbb{V}(Y_{n,k})}}, \frac{Z_{n,k} - \mathbb{E}(Z_{n,k})}{\sqrt{\mathbb{V}(Z_{n,k})}} \right\}$$

exists.

THE RANGE $\alpha = 1$

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$$Y_1' \stackrel{d}{=} \frac{1}{2}Y_1' + \frac{1}{2}Y_1'^{*} + 1 + \frac{1}{2}\log U + \frac{1}{2}\log(1-U).$$

Any naturally defined RVs on trees leading to Y_1'?

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No normal range for BST-profile

GENERALITY (METHODOLOGY)

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No martingale in general

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- quadtrees: extend Flajolet et al. (1993)'s direct DE and Flajolet et al. (1995)'s Euler-transform approaches for asymptotics of moments.

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5. **nonexistence of limit law when** $k = \alpha_2 \log n + O(1)$

ARE THERE LOG-TREES WITH DIFFERENT BEHAVIOR FOR THEIR PROFILE?

REPRESENTATIVE CLASSES OF INCREASING TREES

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- ☛ **ordered recursive (or heap-ordered, plane-oriented) trees:** $y' = \dfrac{1}{1-y}$

HEAP-ORDERED (ORDERED RECURSIVE) TREES

The profile $V_{n,k}$ satisfies the recurrence

$$V_{n,k} \stackrel{d}{=} V_{J_n,k-1} + V^*_{n-J_n,k},$$

where ($C_n := \binom{2n-2}{n-1}/n$)

$$\mathbb{P}(J_n = j) = \frac{2(n-j)C_j C_{n-j}}{nC_n} \qquad (1 \le j < n).$$

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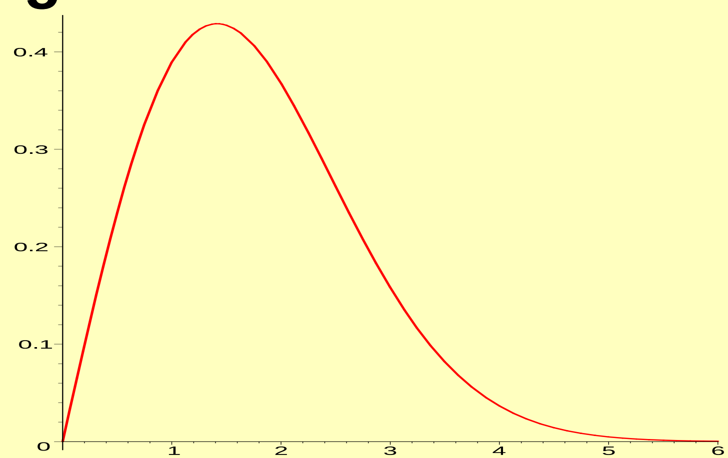
Note that $J_n \xrightarrow{d} J$, where $\mathbb{P}(J = h) = 2C_h/4^h$, but no convergence of any moment of order ≥ 1.

METHOD OF MOMENTS: $k = o(\log n)$

The method of moments applies: for $1 \leq k = o(\log n)$

$$\frac{V_{n,k}}{\sqrt{n}(\frac{1}{2}\log n)^{k-1}/(k-1)!} \longrightarrow \mathcal{R},$$

where $\mathcal{R}$ **is Rayleigh distributed with the density** $te^{-t^2/4}/2$**.**



METHOD OF MOMENTS: $0 \le \alpha \le 1/2$

If $0 \le \alpha \le 1/2$ then $\dfrac{V_{n,k}}{\mathbb{E}(V_{n,k})} \xrightarrow{m} V_\alpha$, for some V_α whose moments can be recursively computed.

Same bimodality for variance, as well as other log-profile phenomena as above for $\alpha = 1/2$.

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But so far no contraction proof for $1/2 < \alpha < c$, where $c \approx 1.79556$ solves the equation $\frac{1}{2} + z - z\log(2z) = 0$, because it's not obvious how to write a fixed-point equation from the moment sequence of the limit law.

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New method is needed for proving the convergence in distribution (anticipated) of $\dfrac{V_{n,k}}{\mathbb{E}(V_{n,k})}$ in the range $(1/2, c)$.

PROFILES OF RANDOM TREES WITH $\sqrt{n}$ HEIGHT

Much has been known for such trees (random binary trees being typical): Stepanov (1969), Takács (1991), Aldous (1993), Drmota, Gittenberger (1997), Pitman (1998), Gittenberger (1998), Kersting (1998), . . .

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For example, for random binary trees,

$$\begin{cases} \dfrac{X_{n,k}}{k/4} \xrightarrow{d} \text{Gamma}(1), & \textbf{if } k \to \infty, k = o(\sqrt{n}), \\ 2\sqrt{2}\dfrac{X_{n,k}}{\sqrt{n}} \xrightarrow{d} \text{Stepanov}_\alpha, & \textbf{if } \dfrac{k}{\sqrt{n}} \to 2\sqrt{2}\alpha. \end{cases}$$

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- **More asymptotic transfer for the double-indexed recurrence?**
- **How to plot or simulate the limit law?**

PROFILE OF RANDOM TREES: A RICH SOURCE OF INTRIGUING PHENOMENA