The Ising model
on planar maps

Gilles Schaeffer – CNRS, École Polytechnique

Joint work with Mireille Bousquet-Mélou
(CNRS – LaBRI, Bordeaux)

Summary of the talk.

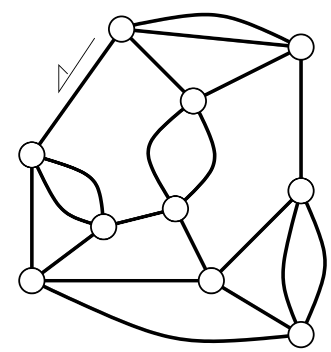
1. Planar maps, trees and a bijection.

2. Application to the Ising model on maps.

Planar maps, trees and bijection.

Planar maps. Definition.

Planar map : proper embedding of a connected graph in the plane, up to homeomorphisms of the plane.



= *graph structure* + *circular order around vertices*

Rooted maps : a root edge in the infinite face.

Tetravalent maps : all vertices of degree 4.

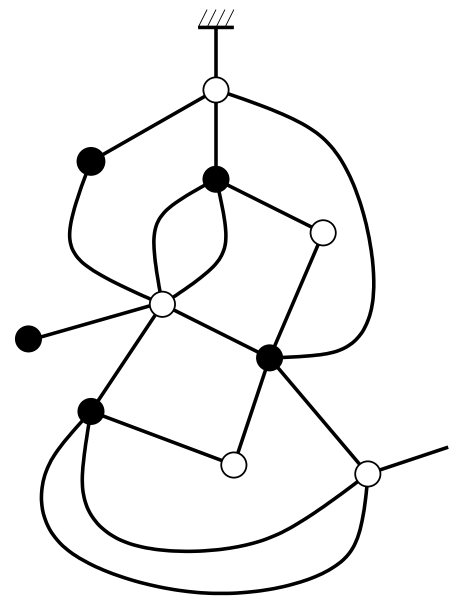
In the family of maps, I pick...

bipartite maps with k legs :

– Two colors of vertices: $\circ$ or $\bullet$.

– Edges $\circ - \bullet$.

– $k \geq 1$ legs $\circ$– in infinie face including a root.

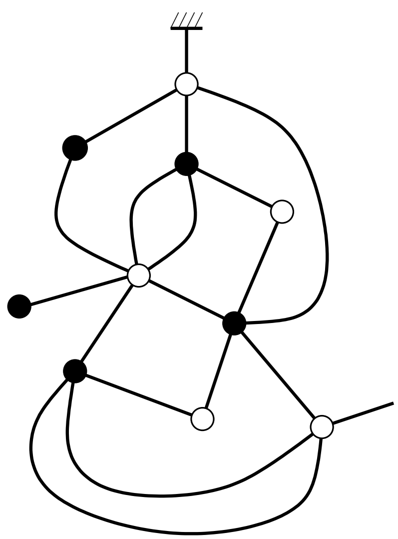


The generating function according to degrees.

The generating function of bipartite maps according to degrees:

$$F_k(\mathbf{x},\mathbf{y}) = \sum_B \mathbf{x}^{\circ(B)}\mathbf{y}^{\bullet(B)},$$

where $\mathbf{x}^{\circ(B)} = x_1^{d_1} \cdots x_k^{d_k}$ if B has d_i vertices $\circ$ of degree i
and $\mathbf{y}^{\bullet(B)} = y_1^{d'_1} \cdots y_k^{d'_k}$ if B has d'_i vertices $\bullet$ of degree i.



The contribution of this map is

$$x_2^2\, x_4^2\, x_6\; y_1\, y_2\, y_4^2\, y_5$$

The generating function according to degrees.

The generating function of bipartite maps according to degrees:

$$F_k(\mathbf{x},\mathbf{y}) = \sum_B \mathbf{x}^{\circ(B)}\mathbf{y}^{\bullet(B)},$$

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Hence, $F_k(\mathbf{x},\mathbf{y})\Big|_{\substack{x_i=t\\ y_i=t}}$ counts bipartite maps according to edges,

and $F_3(\mathbf{x},\mathbf{y})\Big|_{\substack{x_i=0,\ i\neq 3\\ y_i=0,\ i\neq 3}}$ counts bipartites cubic maps...

The latter two series have been computed by Tutte in 1962
(recurrence + generating fcn + quadratic method).

Generating functions according to the degree.

Generating functions according to the degree:

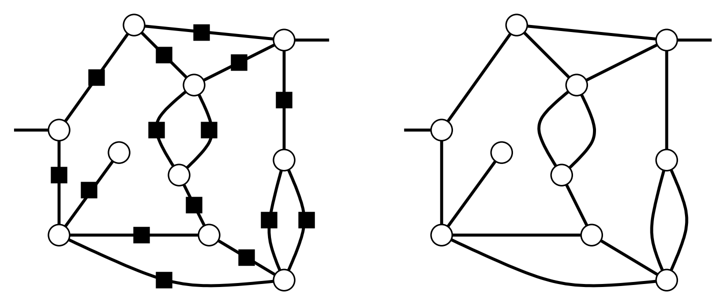
$$F_k(\mathbf{x}, \mathbf{y}) = \sum_B \mathbf{x}^{\circ(B)} \mathbf{y}^{\bullet(B)},$$

where $\mathbf{x}^{\circ(B)} = x_1^{d_1} \cdots x_k^{d_k}$ if B has d_i vertices $\circ$ of degree i
and $\mathbf{y}^{\bullet(B)} = y_1^{d'_1} \cdots y_k^{d'_k}$ if B has d'_i vertices $\bullet$ of degree i.

Trick to remember (taken in Robert Cori's PhD)

Bipartite maps extend arbitrary maps

The series $F_2(\mathbf{x}, \mathbf{y})|_{y_i=0,\ i\neq 2}$ count arbitrary maps according to degrees.



A bijection between maps and trees.

The number of tetravalent maps with n edges was computed by Tutte in 1962.

$$\frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}.$$

We recognise the number of complete binary trees.

$$\frac{1}{(n+1)} \binom{2n}{n}.$$

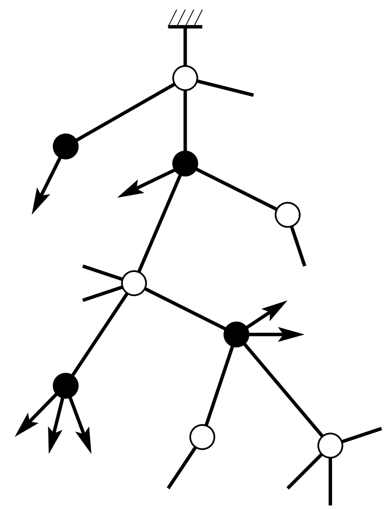
$\Rightarrow$ bijective proof by « closure of trees » (GS97)

Idem for the other two specialisations...

Unify and deal with the general case ?

Introduce the « right » trees...

Planted plane trees with:
black vertices (degree ≥ 1),
white vertices (degree ≥ 1),
and *half-edges* called:
- leaves if attached to a $\circ$,
- buds if attached to a $\bullet$.
The root is a half-edge.



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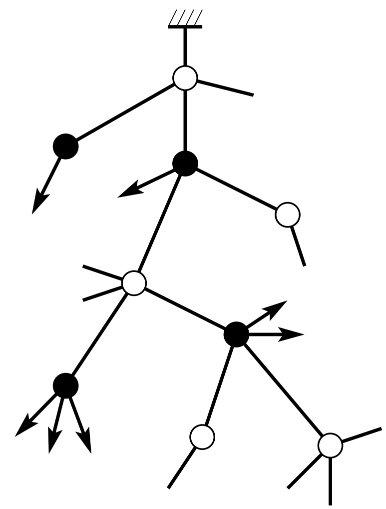
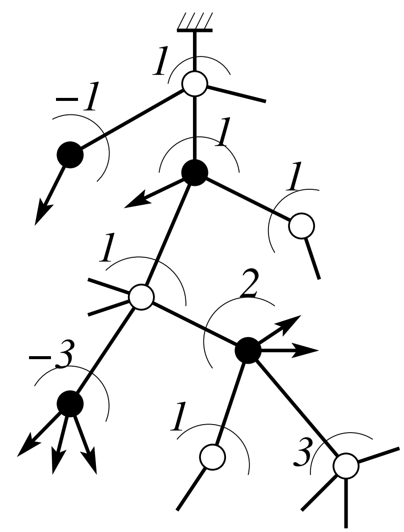


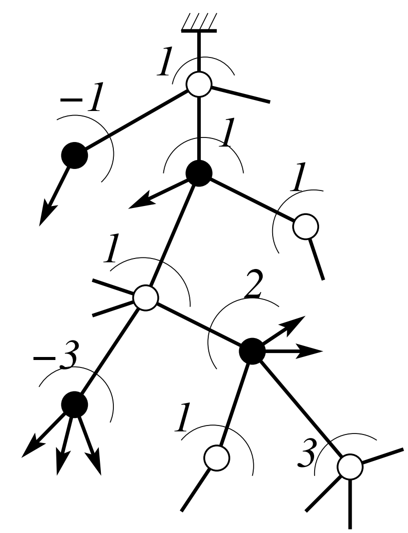
The *total charge* of a tree is the difference between the number of leaves and buds. Idem for the *charge* of a subtree.



Blossom trees. Definition

Black rule: the charge is ≤ 1.

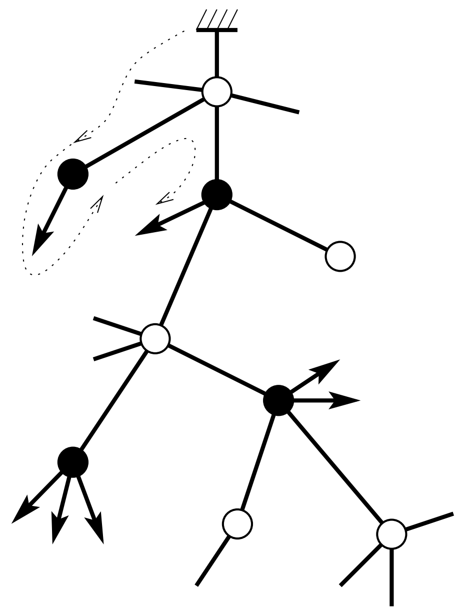
White rule: the charge is ≥ 0.



A tree is blossoming if all its strict subtrees satisfy the color rule at their top vertex.

The closure of a tree. Definition by example

On an example of tree with total charge 2.



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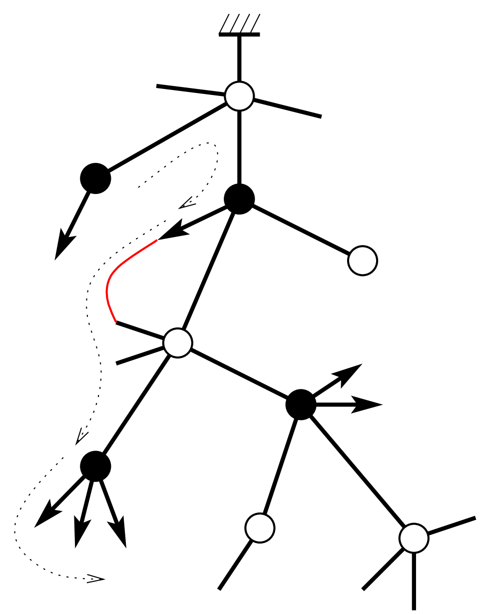

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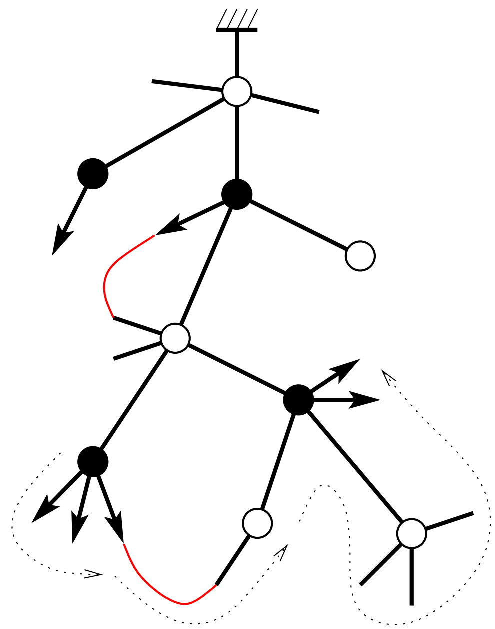

The closure of a tree. Definition by example

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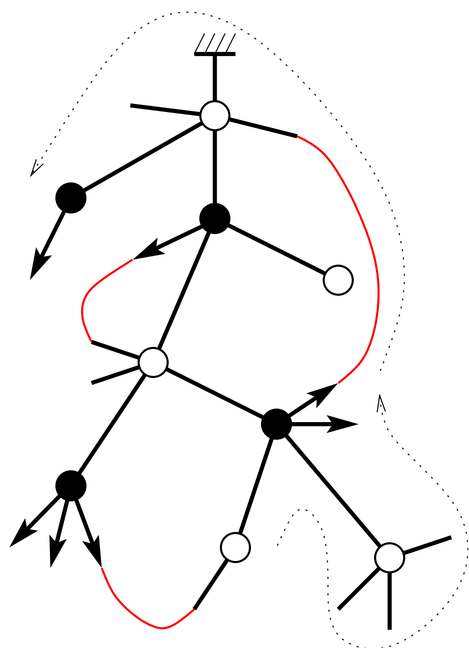

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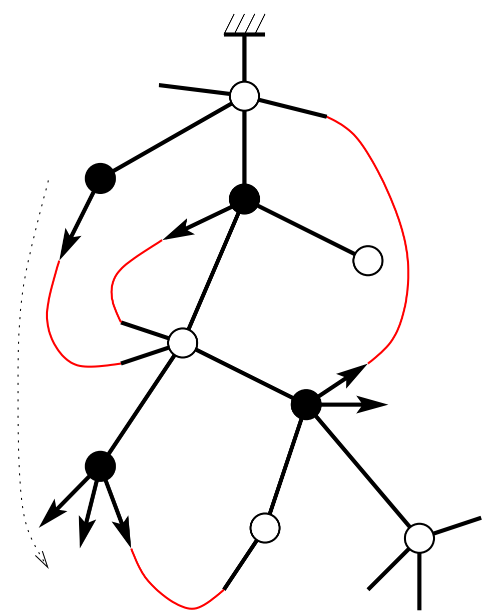

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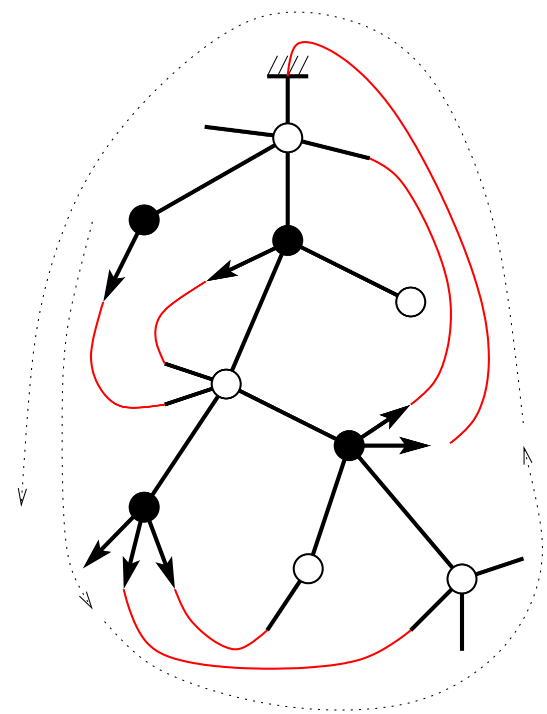

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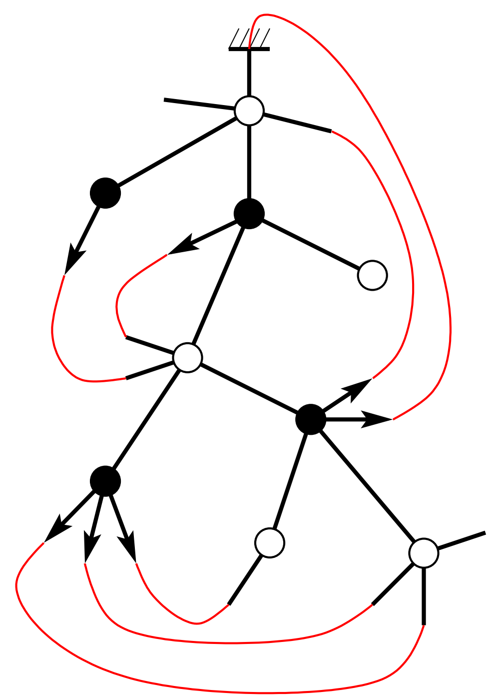

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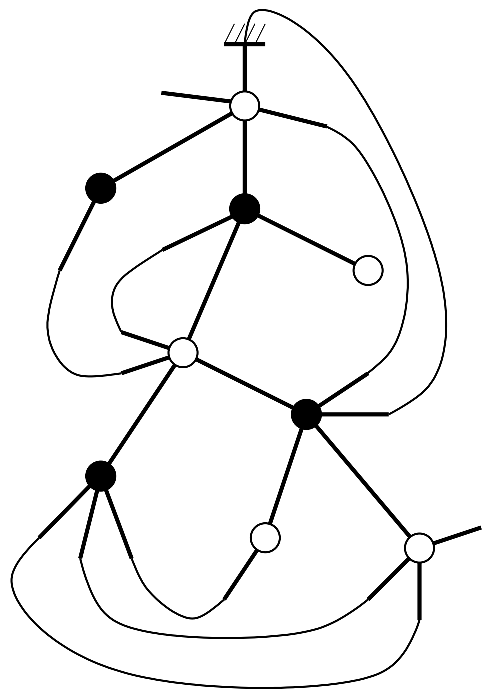
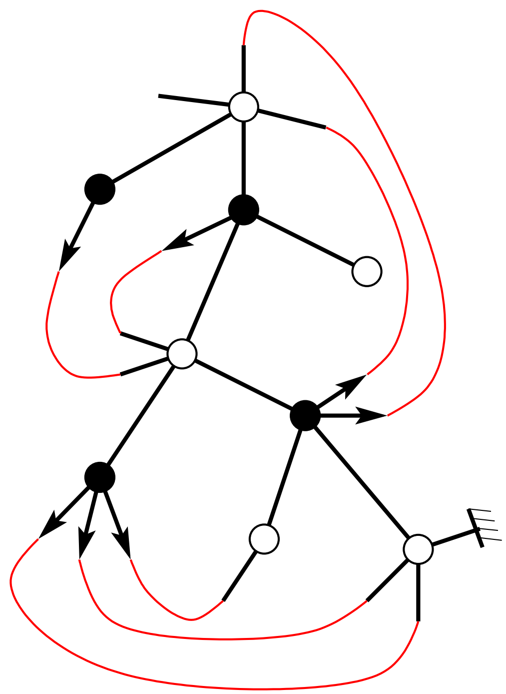


Balanced trees

A tree is *balanced* if its root remains free after closure.



Closure is one-to-one

Theorem (MBM-GS 2002)

For $k = 1, 2$, closure is one-to-one between:
– blossom trees with total charge k
– and bipartite maps with k legs.

Moreover it preserves black and white degree distributions.

This theorem unifies and extends the following bijections:
tetravalent maps (GS97), bipartite cubic and
constellations (MBM+GS00), and arbitrary maps (BdFG02).

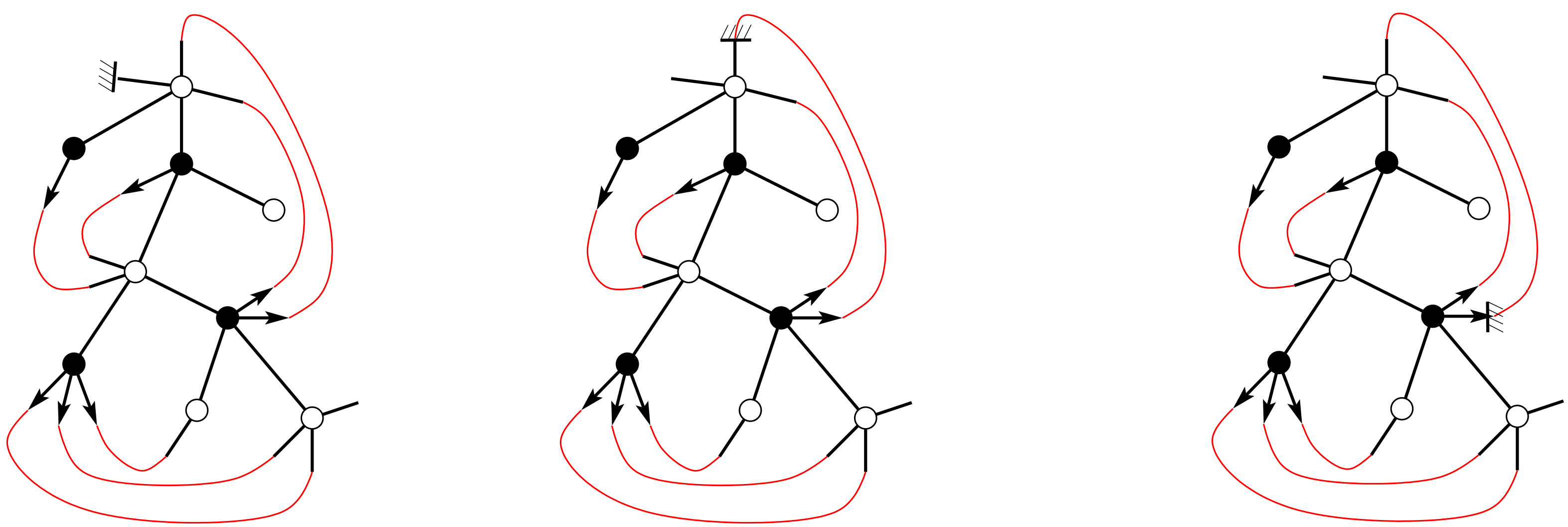
i.e. all tree conjugations appart from those with D. Poulalhon.

Application to enumeration.

The series $F(\mathbf{x}, \mathbf{y})$ is thus the series of balanced blossom trees.

It can be expressed in terms of generating functions of blossom trees:

$$F = A^{\circ\!-} - A^{\bullet\!-} \qquad \text{(for all degrees even)}$$



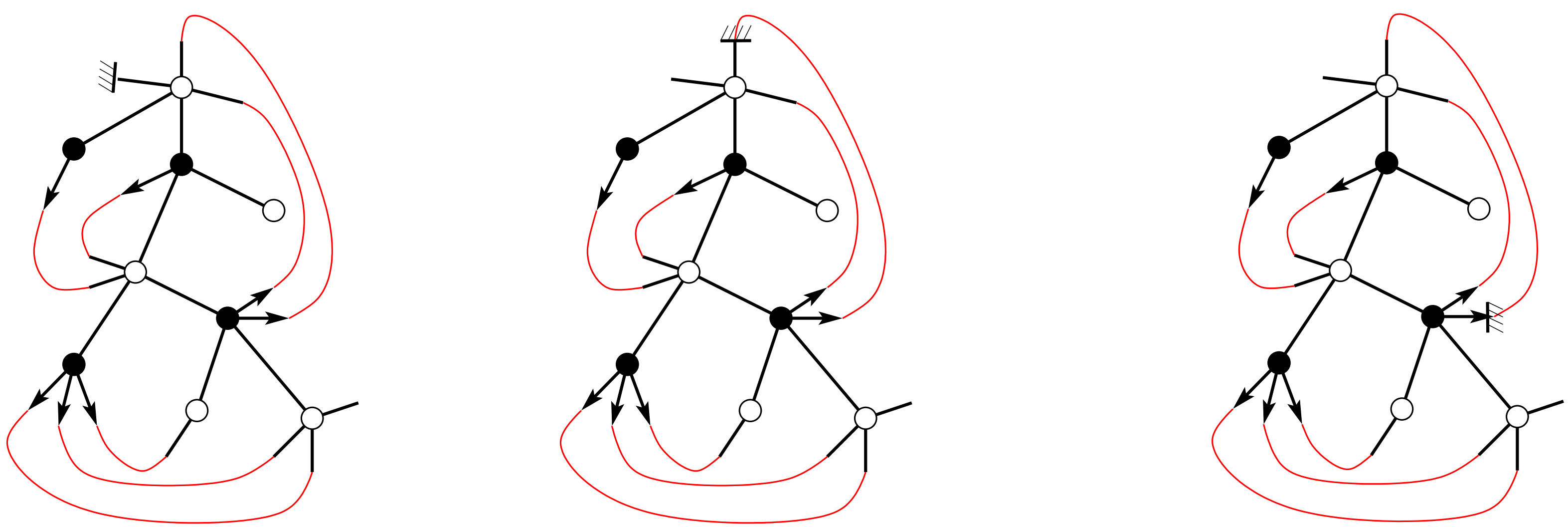
(Conjugaison as seen by BdFG02)

Application to enumeration.

The series $F(\mathbf{x}, \mathbf{y})$ is thus the series of balanced blossom trees.

It can be expressed in terms of generating functions of blossom trees:

$$F = A_1^{\circ\!-} - A_3^{\bullet\!-} \qquad \text{(charge at the root)}$$



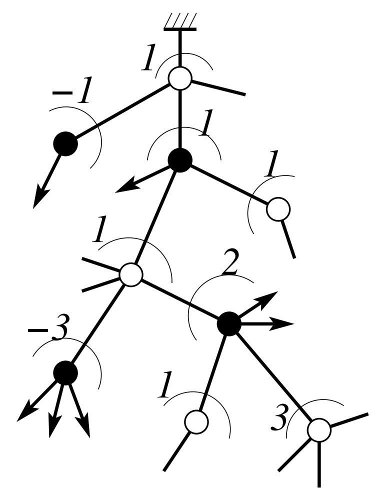
(Conjugaison as seen by BdFG02)

Counting trees, this is easier...

We therefore want the series $A^{\circ\!-}$ and $A^{\bullet\!-}$ of blossom trees.

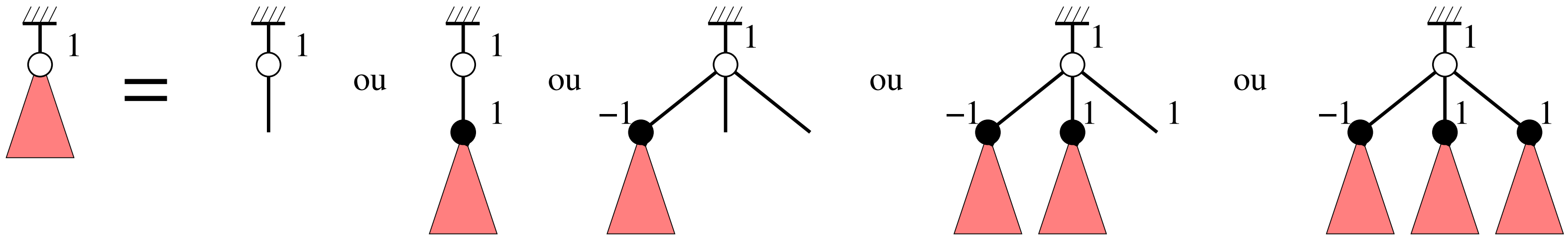
Recall... A tree is blossom if all its strict subtrees satisfy the colored rule at their top vertex.

Black rule: charge is ≤ 1.
White rule: charge is ≥ 0.



$\Rightarrow$ equations on series $A_k^{\circ\!-}$ and $A_k^{\bullet\!-}$ that count blossom trees with charge k at the root.

Example: degrees 2 and 4.



$$A_1^{\circ-} = x_2(1 + A_1^{\bullet-}) + 3x_4 A_{-1}^{\bullet-}(1 + A_1^{\bullet-})^2, \qquad A_3^{\circ-} = x_4(1 + A_1^{\bullet-})^3,$$

$$A_1^{\bullet-} = y_2 A_1^{\circ-} + 3y_4(A_3^{\circ-} + A_1^{\circ-\,2}), \qquad A_{-1}^{\bullet-} = y_2 + 3y_4 A_1^{\circ-}.$$

All these series can be expressed in terms of only one $P = 1 + A_1^{\bullet-}$:

$$P = 1 + 3x_4y_4P^3 + \frac{P(x_2 + 3x_4y_2P)(y_2 + 3y_4x_2P)}{(1 - 9x_4y_4P^2)^2}.$$

and $F(\mathbf{x}, \mathbf{y}) = A_1^{\circ-} - A_3^{\bullet-}$. (Tutte is recovered for $x_2 = y_2 = 0$).

Enumerative Result

Corollary.

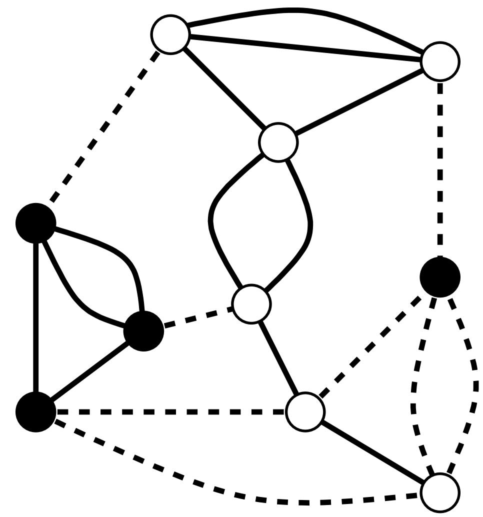
For any finite set of degrees, the generating function of bipartite maps with 2 legs is algebraic.

This results unifies and extends most of the known results for families of planar maps counted according to vertex degrees *without connectivity conditions*.

Application to the Ising model on maps

The Ising model on a graph.

$\mathcal{G} = (V, E)$ a graph.
An Ising configuration on $\mathcal{G}$:
$\sigma : V \to \{\circ, \bullet\}$
(*i.e.* colors on vertices)



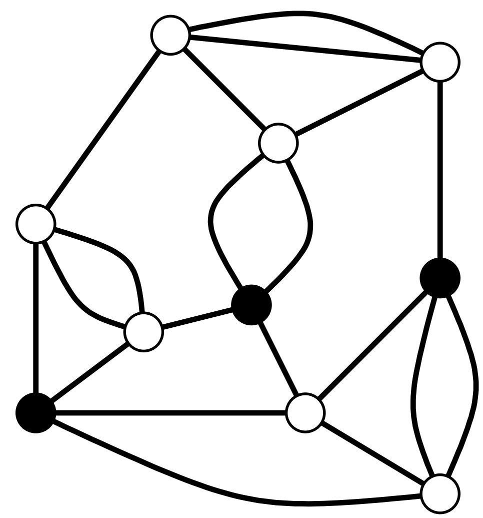
An edge is *frustrated* if it is of the form $\circ\!\!-\!\!\bullet$.
The distribution is studied (for x, y, u positive reals)

$$Z_{\mathcal{G}}(x,y,u) = \sum_{\sigma} x^{\circ(\sigma)} y^{\bullet(\sigma)} u^{\circ\!-\!\bullet(\sigma)}, \qquad \Pr(\sigma) = \frac{x^{\circ\sigma} y^{\bullet\sigma} u^{\circ\!-\!\bullet\sigma}}{Z_{\mathcal{G}}(x,y,u)}$$

Hard particle model on a graph

(random independent sets).

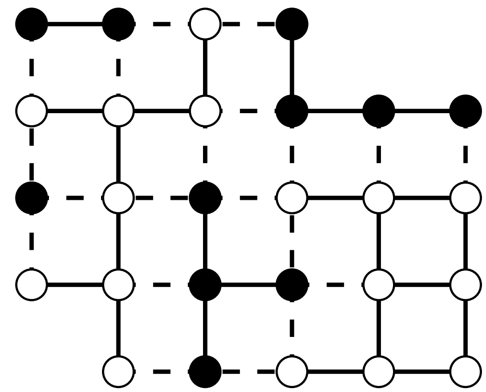
$\mathcal{G} = (V, E)$ a graph.
A configuration on $\mathcal{G}$:
$\sigma : V \to \{\circ, \bullet\}$
($i.e.$ colors on vertices)



Edges $\bullet\!\!-\!\!\bullet$ are forbidden (less possible configurations).
We study the distribution (for x, α, u positive reals)

$$H_{\mathcal{G}}(x, \alpha, u) = \sum_{\sigma} x^{\odot(\sigma)} \alpha^{\bullet(\sigma)} u^{\odot-\odot(\sigma)}, \qquad \Pr(\sigma) = \frac{x^{\odot\sigma} \alpha^{\bullet\sigma} u^{\odot-\odot\sigma}}{H_{\mathcal{G}}(x, \alpha, u)}$$

The Ising model on the square lattice.



In "classical" statistical physics:
on a portion of the square lattice $\mathbb{Z}^2$,
or more generally on $\mathbb{Z}^d$.
cf. Baxter's book.

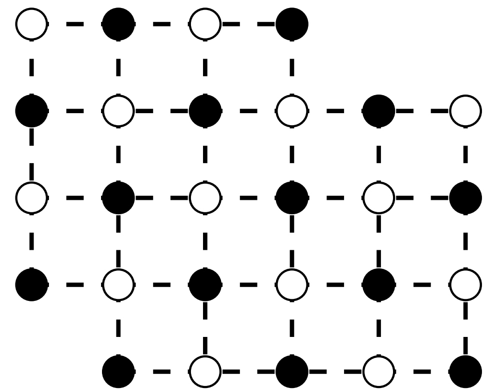
Here it is convenient to set $1/u = e^{-k/T}$:

- $T \to \infty$, $u = 1$: independant colors on vertices.

∘ $T \to 0$, $u \to \infty$: maximisation of nb of frustrated edges
$\Rightarrow$ the 2 "frozen" bipartite configurations.

Study complicated parameters (cluster's size, their shape).

The Ising model on the square lattice.



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Study complicated parameters (cluster’s size, their shape).

The Ising model in Physics.

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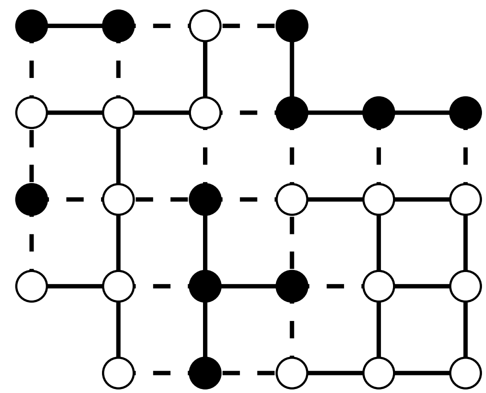
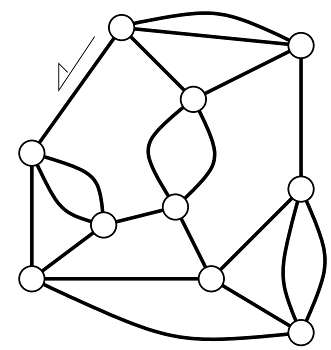


Grid = natural discretisation of a fixed metric (the plane).

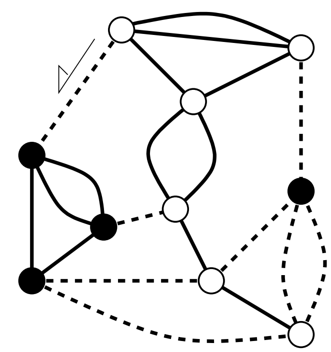
In *quantum gravity*, the space is endowed with a random metric
$\Rightarrow$ (since the 80's): random combinatorial maps are the right discretisation of the 2d euclidean quantum gravity metric.

The Ising model on random maps.

Planar map : proper embedding of a connected graph in the plane, up to deformation of the plane.



The Ising model on maps : uniform distribution on the set of all maps with n edges with an Ising configuration.



The Ising model on random maps.

"Canonical ensemble": for x, y, u positive reals,

$$Z_n(x,y,u) = \sum_{|C|=n} Z_C(x,y,u), \qquad \Pr((C,\sigma)) = \frac{x^{\circ} y^{\bullet} u^{\circ\!-\!\bullet}}{Z_n(x,y,u)}$$

"Grand canonical ensemble": for t, x, y, u positive reals, $|t| < \rho$

$$Z(t;x,y,u) = \sum_{n\geq 0} Z_n(x,y,u)\, t^n. \qquad \Pr((C,\sigma)) = \frac{t^n x^{\circ} y^{\bullet} u^{\circ\!-\!\bullet}}{Z(t;x,y,u)}$$

Warning: in general this is not uniform on underlying maps.

The same definitions apply to the hard particle model.

Enumerative results. An example.

Matrix integral methods allow to predict the generating functions H and Z for various families of maps (tetravalent, trivalent, etc.)

Theorem (Bouttier, Di Francesco, Guitter 2002). *The generating function of the hard particle model on tetravelent maps can be expressed rationaly in* x, α *and in the series* $P \equiv P(x, \alpha)$, *solution of*

$$P = 1 + 3x\alpha P^3 + \frac{3xP^2}{(1 - 9x\alpha P^2)^2}.$$

This series looks very much like a tree series!   via bipartite ?

Hard particles as a specialisation of bipartite maps.

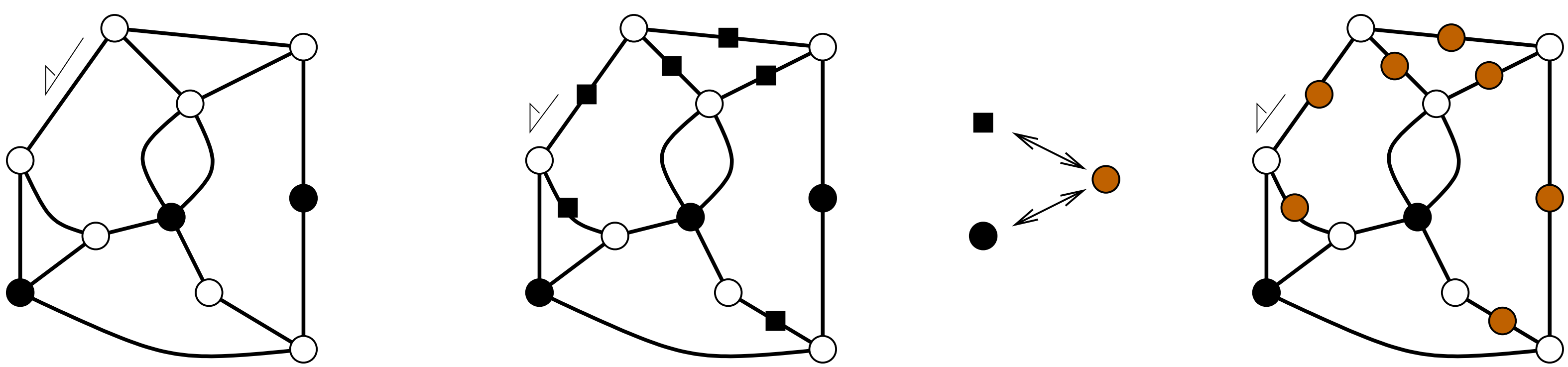
Bipartite : edges $\circ\!-\!\bullet$ only; $\circ\!-\!\circ$ and $\bullet - \bullet$ forbidden.
Hard particle: edges $\circ\!-\!\bullet$ or $\circ\!-\!\circ$; $\bullet - \bullet$ forbidden.

To reduce hard particles to bipartite maps we use again the bipartite trick: $\circ\!-\!\circ \Leftrightarrow \circ\!-\!\blacksquare\!-\!\circ$.

Hence

$$H(\mathbf{x},\alpha) = \sum_{D} \mathbf{x}^{\odot}\,\alpha^{\bullet} \;=\; F(\mathbf{x},\mathbf{y})\Big|_{\substack{y_2=1+\alpha x_2,\\ y_i=\alpha x_i, i\neq 2}}.$$

Hard particles as a specialisation of bipartite maps.



Hence

$$H(\mathbf{x}, \alpha) = \sum_{D} \mathbf{x}^{\odot} \alpha^{\bullet} \; = \; F(\mathbf{x}, \mathbf{y})\Big|_{\substack{y_2 = 1 + \alpha x_2, \\ y_i = \alpha x_i, i \neq 2}}.$$

Hard particle as a specialisation of bipartite maps.

We want

$$H(\mathbf{x},\alpha) = \sum_{D} \mathbf{x}^{\odot}\,\alpha^{\bullet} \;=\; F(\mathbf{x},\mathbf{y})\Big|_{\substack{y_2=1+\alpha x_2,\\ y_i=\alpha x_i, i\neq 2}}.$$

We had expressed $F(\mathbf{x},\mathbf{y})$ in terms of P:

$$P = 1 + 3x_4y_4P^3 + \frac{P(x_2+3x_4y_2P)(y_2+3y_4x_2P)}{(1-9x_4y_4P^2)^2}.$$

The specialisation of variables yield back BdFG02's parametrization.

$\Rightarrow$ *a tree interpretation*
and a matrix integral free proof.

More generally the Ising model.

Add vertices —■— and —□— to apply the trick to ○—○ and ●—●.

But this creates chains —■—□— ⋯ —■—, that must be eliminated.

This is done with a hardly more subtle substitution.

$$Z(\mathbf{X},\mathbf{Y},u) = \sum_{(C,\sigma)} \mathbf{X}^{\circ}\mathbf{Y}^{\bullet}\, u^{\circ\!-\!\bullet} \;=\; F(\mathbf{x},\mathbf{y})\Big|_{\substack{x_2=\bar{u}+X_2(u-\bar{u})^2,\; x_i=X_i(u-\bar{u})^2 \\ y_2=\bar{u}+Y_2(u-\bar{u})^2,\; y_i=Y_i(u-\bar{u})^2}}.$$

(here you must take my word)

A bit of singularity analysis to close...

Asymptotic and normality: hard particles

The generating series of the hard particle models is a rational function in x, α and the parametrization

$$P_\alpha(x) = x\phi(P_\alpha(x)), \quad \text{avec} \quad \phi(y) = \frac{1}{1 - 3y(\alpha y + \frac{1}{(1-9\alpha y^2)^2})}.$$

$\Rightarrow$ study the dominant singularities of P_α.

(no new dominant singularities in H w.r.t. $P_\alpha(x)$)

Asymptotic and normality: hard particles

$$P_\alpha(x) = x\phi(P_\alpha(x)), \quad \text{avec} \quad \phi(y) = \frac{1}{1 - 3y(\alpha y + \frac{1}{(1-9\alpha y^2)^2})}.$$

For $\alpha \geq 0$, the usual theory of simple trees:
the series $\phi(y)$ has positive coeff, hence P_α has a real dominant singularity in $x_\alpha = \frac{\tau}{\phi(\tau)}$, where $(\frac{y}{\phi(y)})'_{y=\tau} = \frac{\phi(\tau)-\tau\phi'(\tau)}{\phi(y)^2} = 0.$

$$x_\alpha - x = \frac{\tau}{\phi(\tau)} - \frac{y}{\phi(y)} = -\frac{\phi''(\tau)}{2\phi(\tau)} \cdot (\tau - y)^2 + O((\tau - y)^3)$$

$\Rightarrow$ square root dominant singularity.

The asymptotic "number" of configurations is *normal*:

$$\sum_{|C|=n} \alpha^{\bullet(C)} \underset{n\infty}{\sim} c(\alpha) \cdot (1/x_\alpha)^n \, n^{\gamma-2}, \qquad \gamma = -1/2.$$

(See also V. Malyshev's cluster expansion method.)

Asymptotic and criticality: hard particles

But for $\alpha < 0$?

Starting from $\alpha = 0$, as long as $\phi''(\tau_\alpha) > 0$ nothing changes

$x_\alpha - x = -\frac{\phi''(\tau)}{2\phi(\tau)} \cdot (\tau - y)^2 + O((\tau - y)^3)$ $\quad \Rightarrow$ square root singularity.

The model becomes *critical* for $\alpha_c = -\frac{25}{8192}(11\sqrt{5} + 25)$:

$x_\alpha - x = c \cdot (\tau - y)^3 + O((\tau - y)^4)$ $\quad \Rightarrow$ cubic root singularity.

$\Rightarrow$ asymptotic of the “number” of configurations with $\gamma = -1/3$.

But $\alpha < 0 \Leftrightarrow$ negative weights... probabilistic interpretation ??

Lee-Yang “non-physical” singularity.

Hard particles on bipartite cubic maps

Bouttier, Di Francesco, Guitter, 2002.

Bipartition $\Rightarrow$ two maximale canonical configurations.

Series remain algebraic, via the parametrization

$$P = \frac{x(1+2\alpha P)^2}{1-2P(1+2\alpha P)^2(1-2\alpha^2 P^2)}.$$

This parametrization degenerate at $\alpha = 32$, with $\phi''(\tau) = 0$
$\Rightarrow$ cubic root singularity, $\gamma = -1/3$.

No combinatorial interpretation to this series with positive terms.
What kind of trees can we expect ? It cannot be standard simple trees..