Combinatorial Enumeration of Two-Dimensional Vesicles

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Abstract

I discuss two-dimensional lattice models of closed fluctuating membranes, or vesicles. The underlying mathematical model is that of self-avoiding polygons and their enumeration by perimeter and area. By adding the constraint of partial directedness, one gets solvable models in the sense that an explicit expression for the generating function can be given in terms of alternating q-series. An asymptotic analysis leads to an explicit calculation of the scaling behaviour around the critical point in terms of the Airy function.

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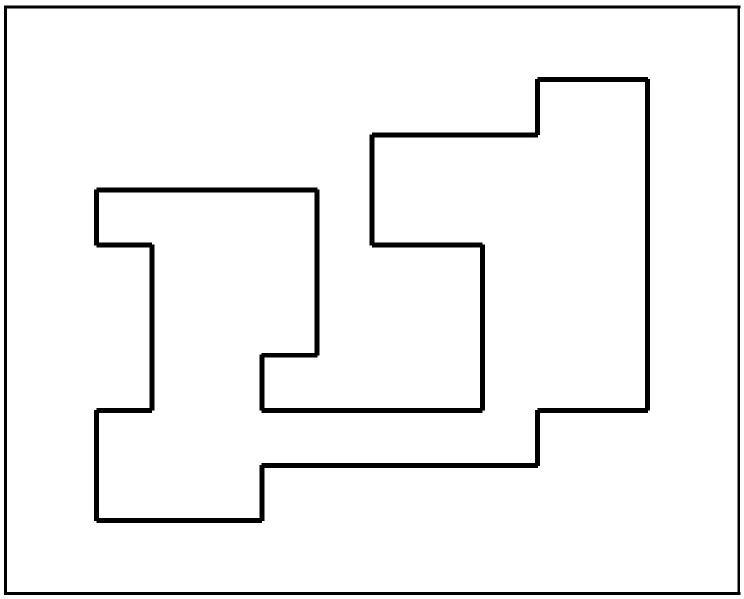
Combinatorial Enumeration of Vesicles

vesiculum (latin) = bubble

- physical motivation:
  - polygons as models of vesicles (= closed fluctuating membranes)
  - statistical mechanics of vesicles
  - phase transition in the thermodynamic limit
  - tricritical phase diagram
- partially directed vesicles – solvable models
  - non-linear functional equations
  - generating functions
- asymptotic analysis:
  - perturbation expansion → critical exponents
  - contour integral representation → coalescent saddle points → scaling function

Polygon Models of Vesicles

- 3-dimensional bubble (e.g. bi-lipid layer membrane), surface tension, osmotic pressure, ...
- 2-dimensional lattice model: polygons on the square lattice with area and perimeter fugacities



$c_{m,n}$ number of polygons with area m and perimeter $2n$

$$G(x,q) = \sum_{n,m} c_{m,n} x^n q^m \qquad \text{generating function}$$

wanted:

- an explicit formula for $G(x,q)$
- information on its singularity structure

Statistical Mechanics

- system size: area m, thermodynamic limit: $m \to \infty$
- energy of vesicle ϕ proportional to perimeter $2n$:

$$H(\phi) = -E\, n(\phi)$$

- finite-area partition function:

$$Z_m = \sum_{|\phi|=m} e^{-\beta H(\phi)} = \sum_n c_{m,n} e^{\beta E n}$$

- now write

$$Z_m(x) = \sum_n c_{m,n} x^n \qquad x = e^{\beta E}$$

  to identify with the generating function:

$$G(x,q) = \sum_m q^m Z_m(x)$$

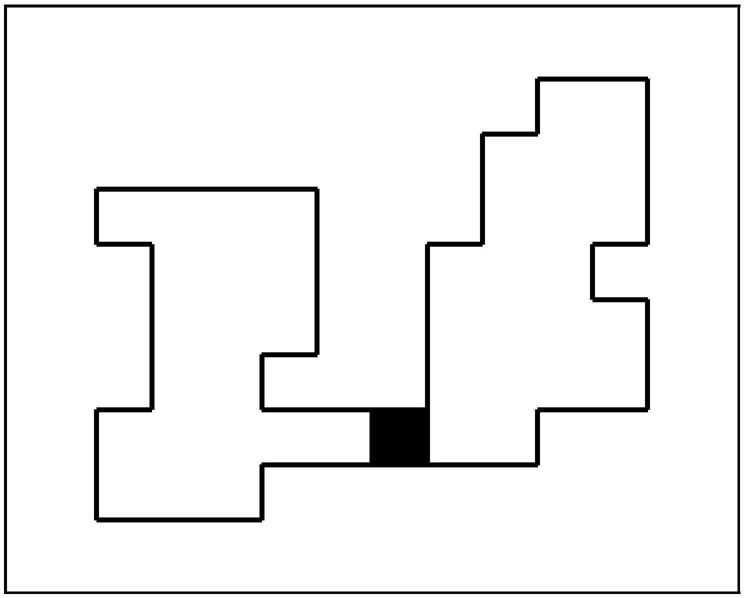
- thermodynamic limit: relation to radius of convergence

$$q_c(x) = \lim_{m\to\infty} (Z_m(x))^{-\frac{1}{m}}$$

- phase transition = non-analyticity in $q_c(x)$

Existence of Thermodynamic Limit and Phase Transition

Consider $Q_n(q) = \sum_m c_{m,n} q^m$



concatenation $\quad c_{m+1,n_1+n_2} \geq \sum_{m_1+m_2=m} c_{m_1,n_1} c_{m_2,n_2}$

implies $\quad Q_{n_1+n_2}(q) \geq q Q_{n_1}(q) Q_{n_2}(q)$

Subadditivity $\Rightarrow x_c(q) = \lim_{n\to\infty} (Q_n(q))^{-\frac{1}{n}}$ exists

$q = 1$: self-avoiding polygons

$$Q_n(1) \sim {\mu_{SAW}}^{2n} \quad \Rightarrow \quad x_c(1) = {\mu_{SAW}}^{-2}$$

$q > 1$: consider squares of size $n/2 \times n/2$

$$Q_n(q) \sim q^{n^2/4+O(n)} \quad \Rightarrow \quad x_c(q) = 0 \text{ for } q > 1$$

$\boxed{\text{Jump of } x_c(q) \text{ at } q = 1 \Rightarrow \text{Phase Transition!}}$

Further results for $q > 1$:

Theorem 1 (Prellberg, Owczarek (1999)) *Let $Q_n(q)$ be the finite-perimeter partition function of polygons on the square lattice. Then*

$$Q_n(q) \sim Q_n^{as}(q) = \frac{1}{(q^{-1};q^{-1})_\infty^4} \sum_{k=-\infty}^{\infty} q^{k(n-k)}$$

exponentially fast as $n \to \infty$.

Ideas for proof:

- Partition function is dominated by convex polygons
- Convex polygons are obtained by cutting off corners from rectangles

Understanding $Q_n^{as}(q)$:

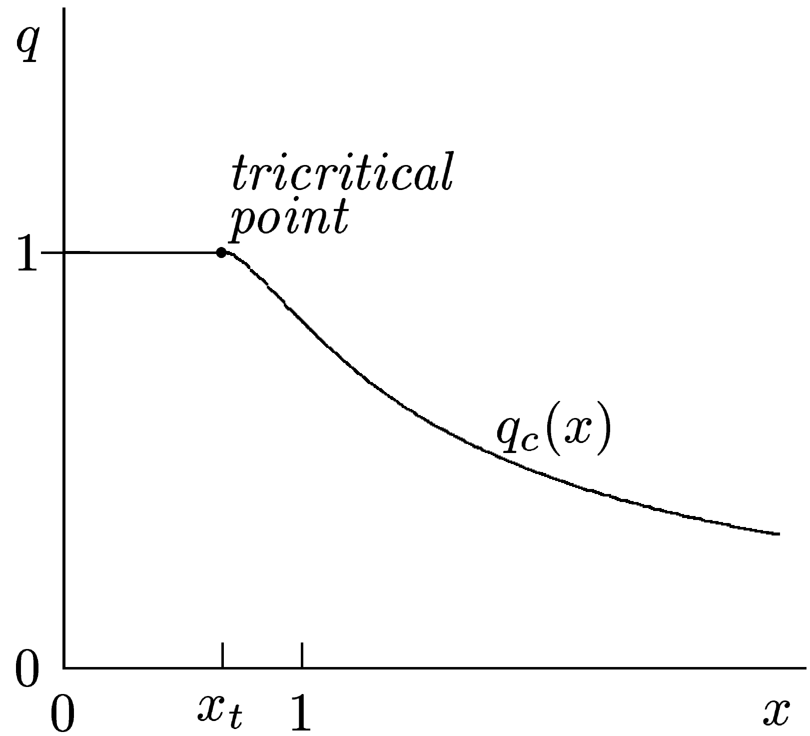
- Finite-perimeter partition function of rectangles:

$$\sum_{k=1}^{n-1} q^{k(n-k)}$$

- Area generating function of Ferrer diagrams (corners):

$$\frac{1}{(q;q)_\infty} = \frac{1}{1-q} \times \frac{1}{1-q^2} \times \frac{1}{1-q^3} \times \cdots$$

Tricritical Phase Diagram



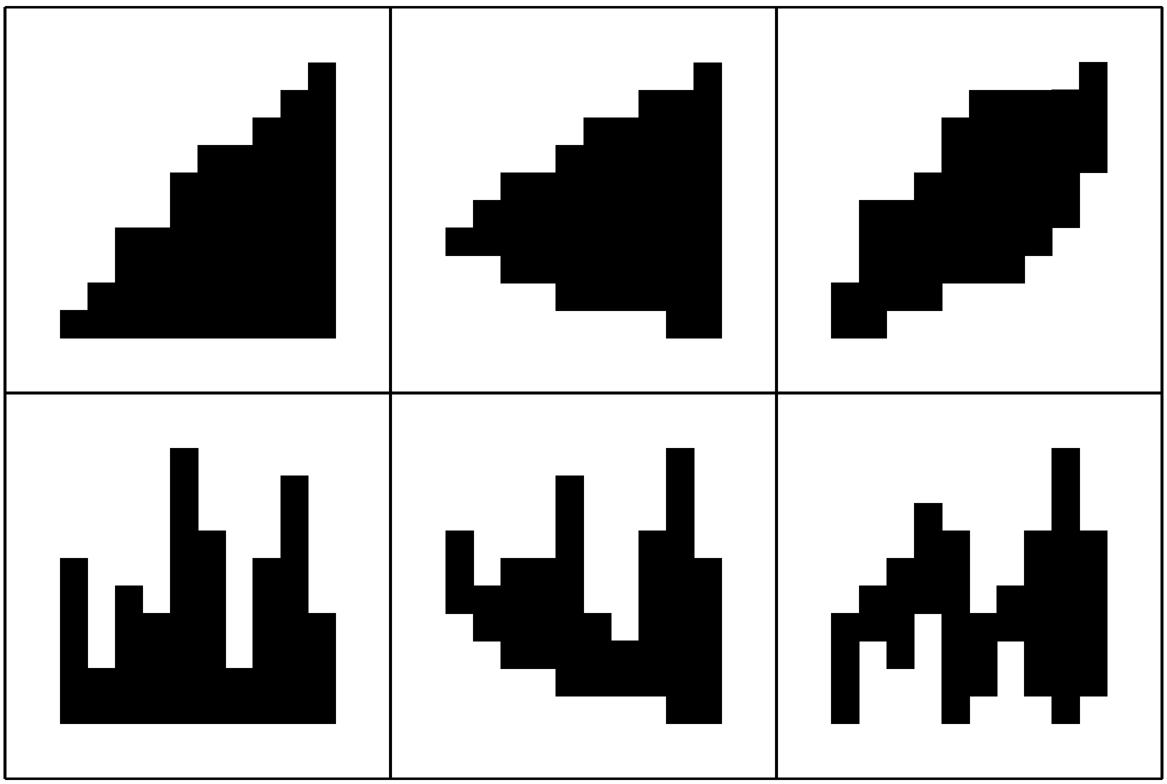
- $q_c(x)$ singular in $x = x_t \Rightarrow$ phase transition
- $G(x, q)$ diverges at $q_c(x)$ for $x > x_t$
- $G(x, q)$ is singular at $q_c(x) = 1$ for $x < x_t$
- $G(x, 1)$ is finite with singularity exponent γ_u as $x \to x_t$
- $G(x_t, q)$ has a singularity with exponent γ_t as $q \to 1$
- tricritical scaling function f with crossover exponent $\phi = \gamma_u/\gamma_t$:

$$G^{sing}(x, q) \sim (1 - q)^{-\gamma_t} f\left(\{1 - q\}^{-\phi}\{x_t - x\}\right)$$

where $f(z) \sim z^{-\gamma_u}$ as $z \to \infty$, $f(z) \sim 1$ as $z \to 0$

Partially Directed Vesicles

partial directedness leads to solvability:



Solution via e.g.

- recurrence relations (Temperley '52, Brak '90)
- q-extension of an algebraic language (Delest '84)
- functional equations (Bousquet-Melou '93)

single-variable generating function: algebraic or rational

two-variable generating function: quotient of q-series

Non-Linear Functional Equations

Method of Inflation:

Distinguish horizontal (x) and vertical (y) steps:

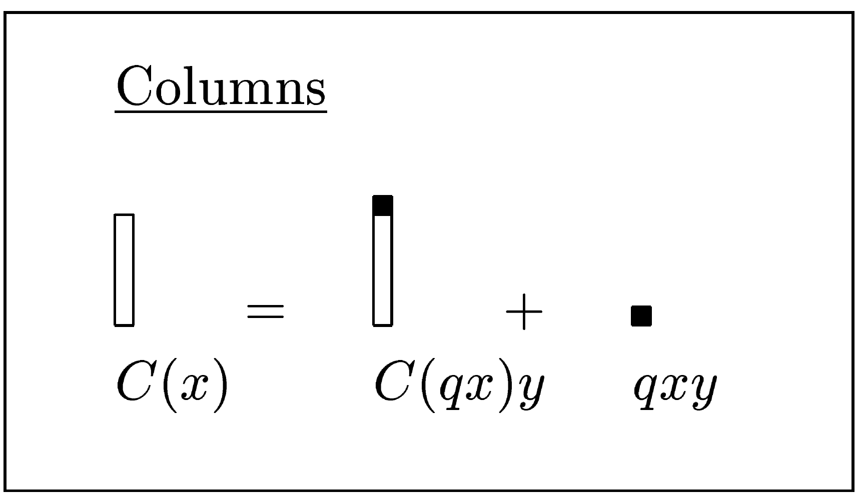
$$G(x, y, q) = \sum c_{m,n_x,n_y} x^{n_x} y^{n_y} q^m$$

Increase the height of each column by one:

$$G(x, y, q) \to G(qx, y, q)y$$

Construct set of polygons recursively.

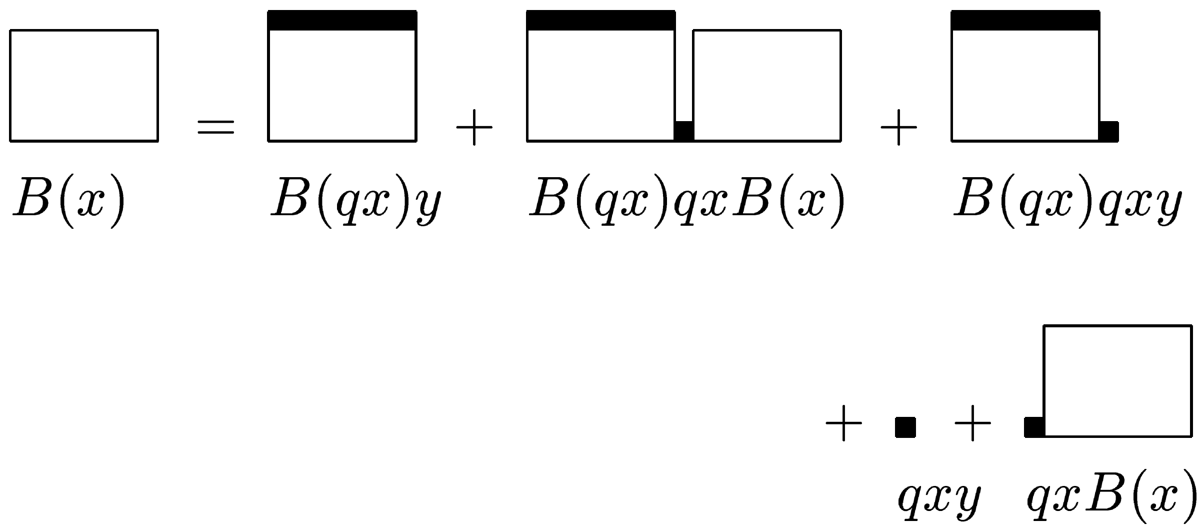
A simple example:



$$C(x, y, q) = C(qx, y, q)y + qxy \quad \to \quad C(x, y, q) = \frac{qxy}{1 - qy}$$

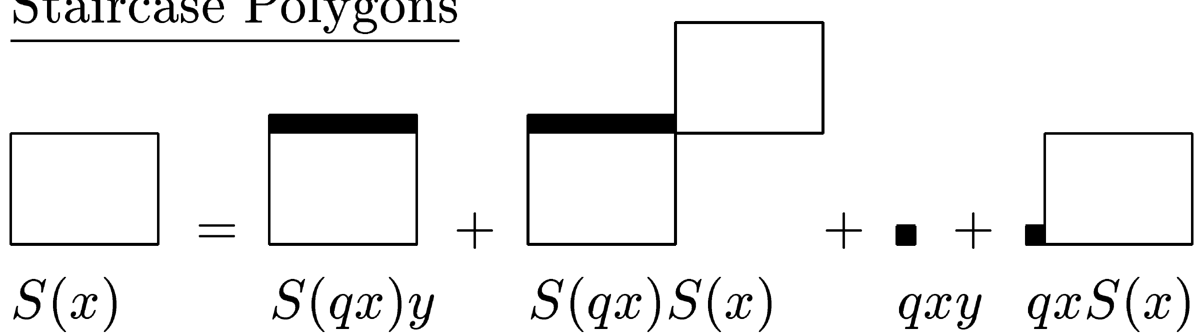
More complex models give non-linear equations:

Bar-graph Polygons



$$B(x, y, q) = B(qx, y, q)y + \{1 + B(qx, y, q)\}\, qx\, \{y + B(x, y, q)\}$$

Staircase Polygons



$$S(x, y, q) = \{S(qx, y, q) + qx\}\, \{y + S(x, y, q)\}$$

Directed Column-Convex Polygons:

Keep track of the height r of the rightmost column

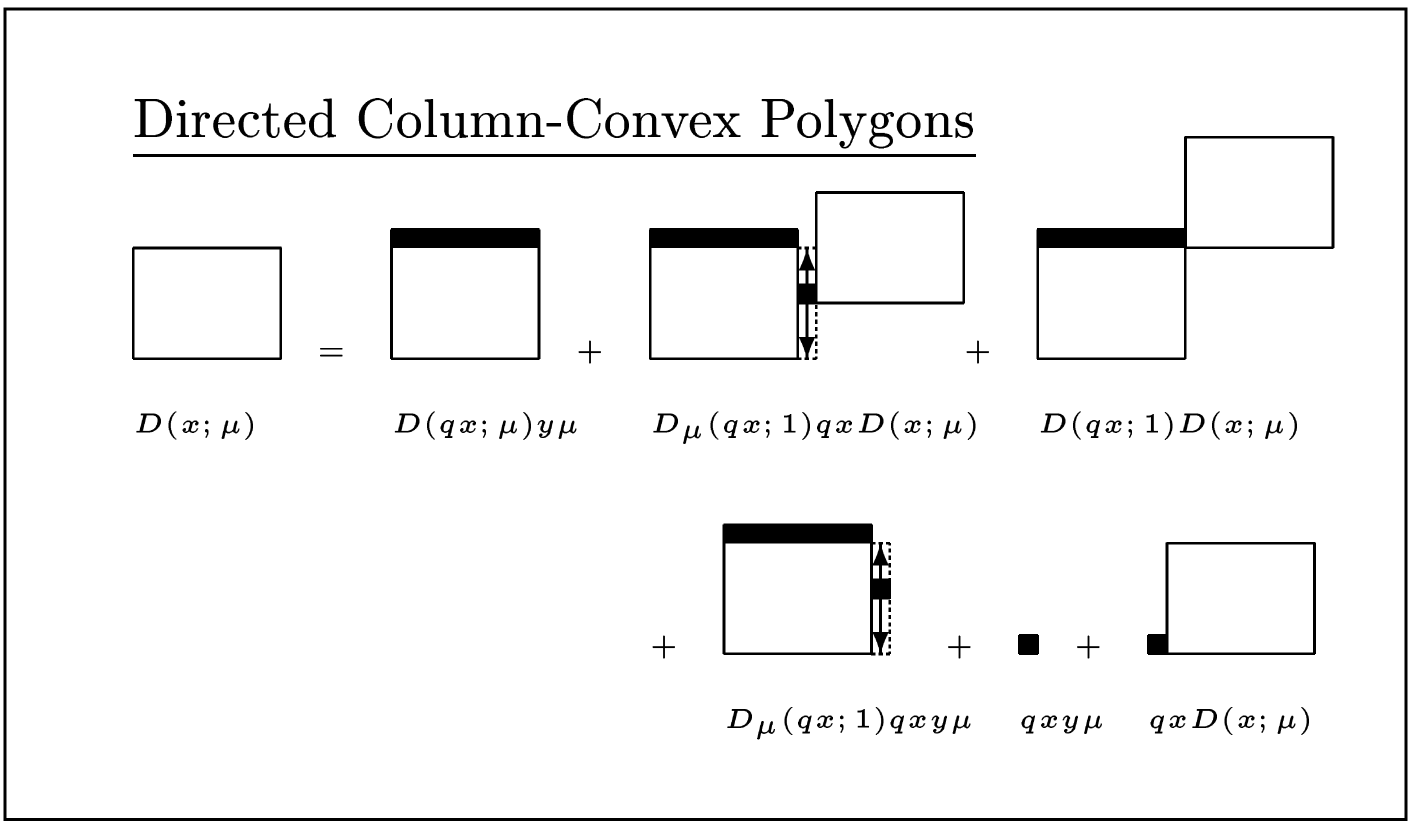
$$D(x,y,q;\mu) = \sum c_m^{n_x,n_y,r} x^{n_x} y^{n_y} q^m \mu^r$$

Inflation:

$$D(qx,y,q;\mu)\mu y$$

Multiplicity $(r-1)$ in sliding concatenation:

$$\frac{\partial}{\partial\mu} D(qx,y,q;\mu)y\Big|_{\mu=1} = D_\mu(qx,y,q;1)y$$



Functional equation:

$$\begin{aligned} D(x,y,q;\mu) &= \{1+D_\mu(qx,y,q;1)\}\,qx\,\{y\mu+D(x,y,q;\mu)\} \\ &+ D(qx,y,q;\mu)y\mu+D(qx,y,q;1)D(x,y,q;\mu) \end{aligned}$$

Differentiate with respect to μ and set $\mu=1$:

$$\begin{aligned} d &= \{1+D_\mu\}\,qx\,\{y+d\}+D\,\{y+d\} \\ d_\mu &= \{1+D_\mu\}\,qx\,\{y+d_\mu\}+D\,\{y+d_\mu\}+D_\mu y \end{aligned}$$

where now

$$d(x,y,q)=D(x,y,q;1) \qquad d_\mu(x,y,q)=D_\mu(x,y,q;1)$$

and

$$D(x,y,q)=d(qx,y,q) \qquad D_\mu(x,y,q)=d_\mu(qx,y,q)$$

Simplify this system further to get one equation in $D(x)=D(x,y,q;1)$:

$$\begin{aligned} 0 &= D(q^2x)D(qx)D(x) \\ &+ yD(q^2x)D(qx)+yD(q^2x)D(x)-(1+q)D(qx)D(x) \\ &+ y^2D(q^2x)-y(1+q)D(qx)+q(1+qx(y-1))D(x) \\ &+ yq^2x(y-1) \end{aligned}$$

It turns out that this equation can be solved explicitely...

Solving the Functional Equations

Functional equations of the form

$$G(x)G(qx) + a(x)G(x) + b(x)G(qx) + c(x) = 0$$

can be linearised with the transformation

$$G(x) = \alpha \frac{H(qx)}{H(x)} - b(x)$$

which yields

$$\alpha^2 H(q^2 x) + \alpha[a(x) - b(qx)]H(qx) + [c(x) - a(x)b(x)]H(x) = 0$$

We will apply the following lemma:

Lemma 1 *Consider a linear functional equation of the form*

$$0 = xH(qx) + \sum_{k=0}^{N} \alpha_k H(q^k x) \qquad \textit{with} \qquad \sum_{k=0}^{N} \alpha_k = 0$$

with α_k independent of x. The solution which is regular at $x = 0$ is given by

$$H(x) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{\prod_{m=1}^{n} \Lambda(q^m)} \quad \textit{with} \quad \Lambda(t) = \sum_{k=0}^{N} \alpha_k t^k$$

Note: the condition $\Lambda(1) = 0$ is crucial.

Staircase polygons:

$$S(x,y,q) = \{S(qx,y,q) + qx\}\{y + S(x,y,q)\}$$

choose $\alpha = y$ and write

$$S(x) = y\left(\frac{T(qx)}{T(x)} - 1\right)$$

this implies

$$0 = yT(q^2x) + (qx - 1 - y)T(qx) + T(x)$$

whence

$$\Lambda(t) = \frac{1}{q} - \frac{1+y}{q}t + \frac{y}{q}t^2$$

now $\Lambda(1) = 0$, and applying Lemma 1 gives

$$T(x) = \sum_{n=0}^{\infty} \frac{(-qx)^n q^{\binom{n}{2}}}{(q, qy; q)_n}$$

with the q-product notation

$$(x_1, x_2, \ldots, x_k; q)_n = \prod_{m=0}^{n-1} (1 - x_1 q^m)(1 - x_2 q^m) \ldots (1 - x_k q^m)$$

thus

$$S(x,y,q) = y\left(\frac{\sum_{n=0}^{\infty} \frac{(-q^2x)^n q^{\binom{n}{2}}}{(q,qy;q)_n}}{\sum_{n=0}^{\infty} \frac{(-qx)^n q^{\binom{n}{2}}}{(q,qy;q)_n}} - 1\right)$$

Directed column-convex polygons:

"Cubic" functional equation

$$\begin{aligned}
0 &= D(q^2x)D(qx)D(x) \\
&+ yD(q^2x)D(qx) + yD(q^2x)D(x) - (1+q)D(qx)D(x) \\
&+ y^2D(q^2x) - y(1+q)D(qx) + q(1+qx(y-1))D(x) \\
&+ yq^2x(y-1)
\end{aligned}$$

Surprisingly, the transformation

$$D(x) = y\left(\frac{E(qx)}{E(x)} - 1\right)$$

leads again to a linearisation

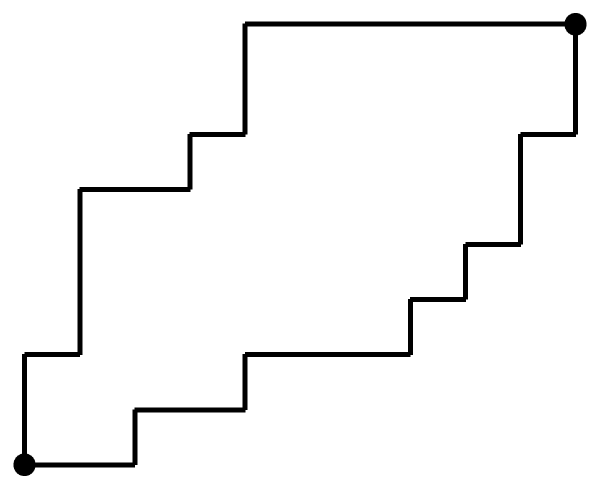
$$\begin{aligned}
0 &= y^2E(q^3x) - y[q+y+1]E(q^2x) \\
&+ [y+q+qy+q^2x(y-1)]E(qx) - qE(x)
\end{aligned}$$

The conditions of Lemma 1 are satisfied, and one gets

$$E(x) = \sum_{n=0}^{\infty} \frac{((y-1)qx)^n q^{\binom{n}{2}}}{(q, qy, y; q)_n}$$

Focus on Staircase Polygons

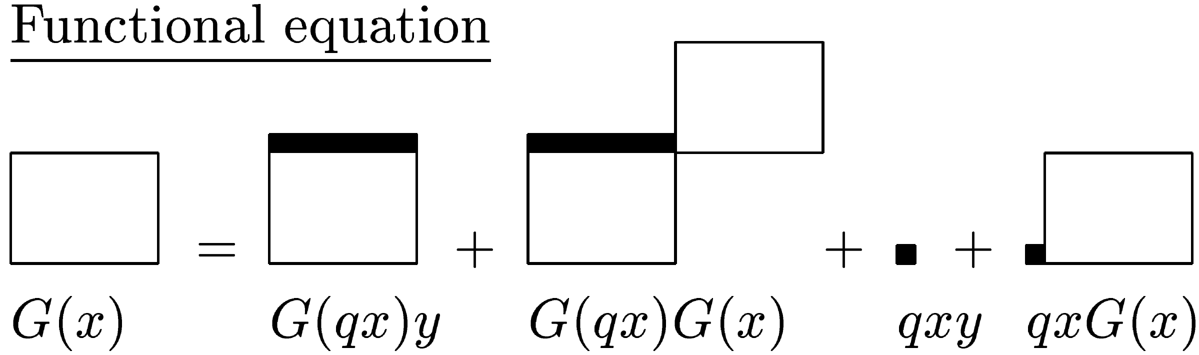
Notation: replace $S(x, y, q)$ by "generic" $G(x, y, q)$



Distinguish horizontal (x) and vertical (y) steps:

$$G(x, y, q) = \sum c_{m,n_x,n_y} x^{n_x} y^{n_y} q^m$$

Functional equation



$$G(x, y, q) = \{G(qx, y, q) + qx\} \{y + G(x, y, q)\}$$

Solution of the Functional Equation

$$G(x,y,q) = \{G(qx,y,q) + qx\}\{y + G(x,y,q)\}$$

is solved by

$$G(x,y,q) = y\left(\frac{H(q^2x,qy,q)}{H(qx,qy,q)} - 1\right)$$

where

$$H(x,y,q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (y;q)_n} = {}_1\phi_1(0;y;q,x)$$

with the q-product notation

$$(t;q)_n = \prod_{m=0}^{n-1} (1 - tq^m) \; .$$

The perimeter-generating function is algebraic:

$$G(x,y,1) = \frac{1-x-y}{2} - \sqrt{\left(\frac{1-x-y}{2}\right)^2 - xy}$$

$\boxed{\text{How can one understand the limit } q \to 1?}$

Main Result

We get an asymptotic expression for $G(x,y,q)$ as $\varepsilon = 1-q \to 0$:

$$G(x,q) \sim \frac{1}{2} - x + 4^{-2/3}\varepsilon^{1/3}\frac{\mathrm{Ai}'\left(4^{4/3}\varepsilon^{-2/3}\{1/4-x\}\right)}{\mathrm{Ai}\left(4^{4/3}\varepsilon^{-2/3}\{1/4-x\}\right)}$$

The limit $\varepsilon \to 0$ gives

$$G(x,1) = \frac{1}{2} - x - \sqrt{\frac{1}{4} - x}$$

Important: our result is valid *uniformly* in a whole neighbourhood of $x = 1/4$ as $\varepsilon \to 0$, and not just in the scaling limit which involves simultaneously $\varepsilon \to 0$ and $x \to 1/4$.

Comparing to

$$G^{sing}(x,q) \sim (1-q)^{-\gamma_t} f\left(\{1-q\}^{-\phi}\{x_t - x\}\right)$$

gives

$$f(z) = -\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}$$

and

$$\gamma_u = -\frac{1}{2}\,, \qquad \phi = \frac{2}{3}\,, \qquad \gamma_t = -\frac{1}{3}$$

Ingredients of the Proof

- Finding a suitable contour integral representation
- q-product asymptotics
- Uniform saddle point asymptotics

Finding a Contour Integral

We need to evaluate

$$H(x, y, q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q; q)_n (y; q)_n}$$

<u>Standard Trick:</u> write an alternating series as a contour integral.

$$\sum_{n=0}^{\infty} (-x)^n c_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} x^s c(s) \frac{\pi}{\sin(\pi s)} ds$$

$\mathcal{C}$ runs counterclockwise around the zeros of $\sin(\pi s)$.

Simple Example A

$$\exp(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \quad = \quad \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{x^s}{\Gamma(s+1)} \frac{\pi}{\sin(\pi s)}$$

$$= \quad \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} x^s \Gamma(-s) ds$$

making use of the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \; .$$

Simple Example B

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n} = -\frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} \frac{x^s q^{\binom{s}{2}}}{(q;q)_s} \frac{\pi}{\sin(\pi s)} ds$$

where the q-product $(x;q)_n$ is extended to complex values of n via

$$(x;q)_s = \prod_{n=0}^{\infty} \frac{1-xq^n}{1-xq^{s+n}} = \frac{(x;q)_\infty}{(xq^s;q)_\infty}$$

No reflection formula: bad for the derivation of asymptotic expansions.

q-analogue of reflection formula

Consider that

$$\Gamma_q(s) = (1-q)^{1-s}(q;q)_{s-1} \to \Gamma(s) \quad \text{as } q \to 1$$

In analogy to $\Gamma(s)\Gamma(1-s)$ consider

$$\Lambda_q(s) = (q;q)_{s-1}(q;q)_{-s} \ .$$

- We have

$$\begin{aligned}
\Lambda_q(s) &= \Lambda_q(1-s) \\
\Lambda_q(s) &= \Lambda_q(s + \frac{2\pi i}{-\log q}) \\
\Lambda_q(s) &= -q^{-s}\Lambda_q(s+1)
\end{aligned}$$

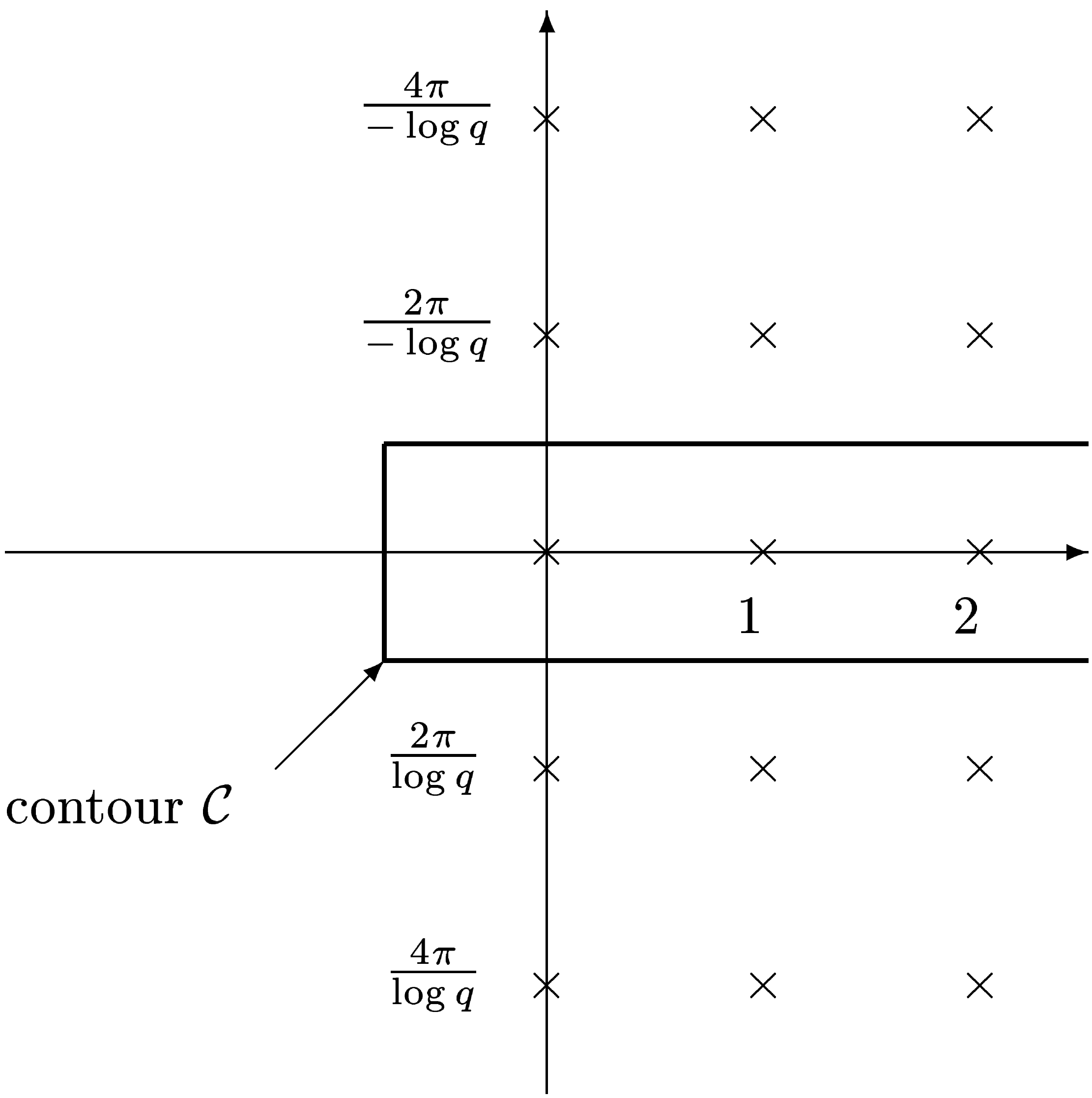
- $\Lambda_q(s)$ has simple poles at $s = n + m\frac{2\pi i}{-\log q}$ for integer m and n with residues

$$\text{Res}\left[\Lambda_q(s); s = n + m\frac{2\pi i}{-\log q}\right] = \frac{(-1)^n q^{\binom{n}{2}}}{-\log q}$$

- $\Lambda_q(s)$ has no zeros.

Simple Example B (revisited)

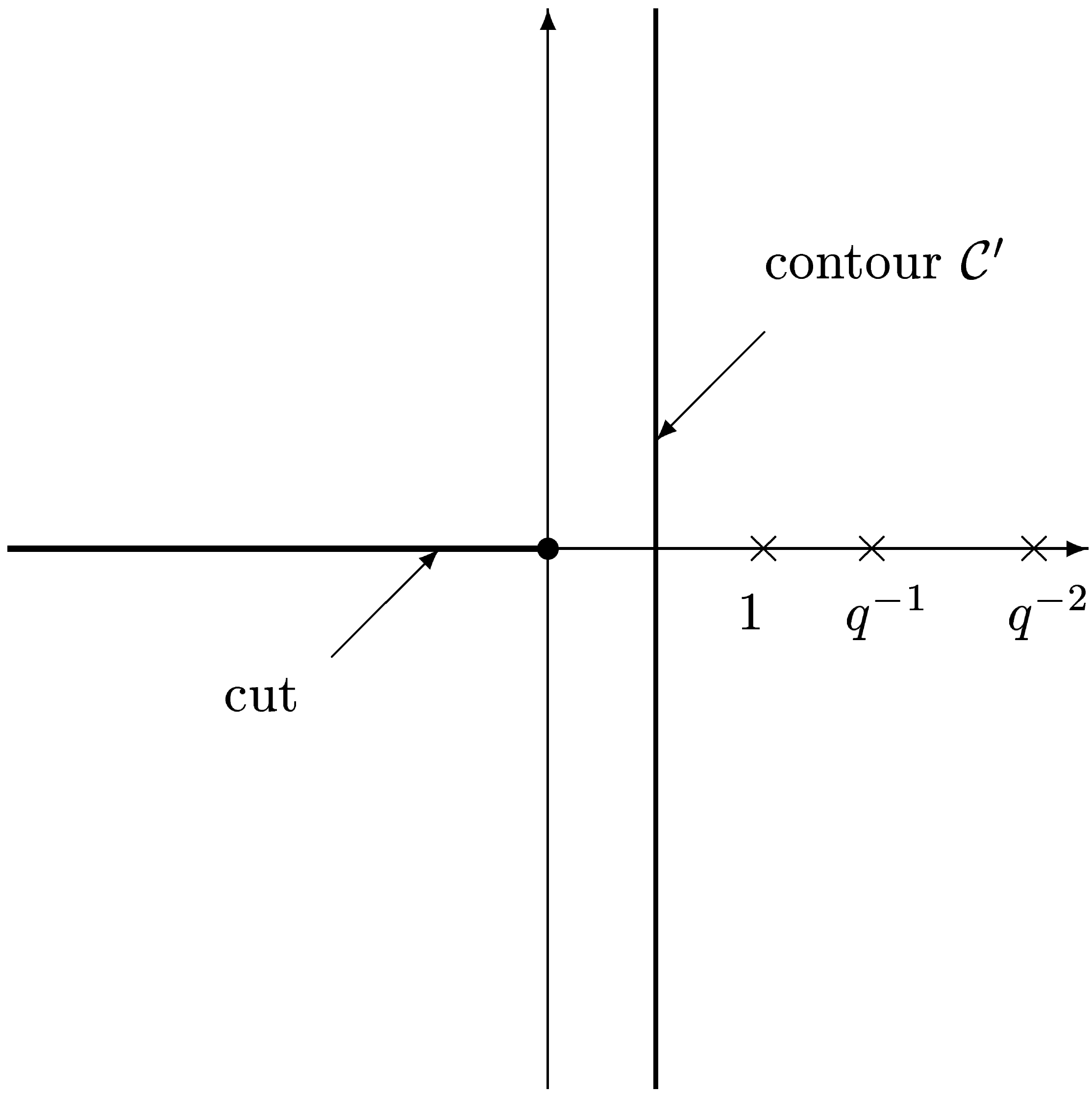
$$
\begin{aligned}
(x;q)_\infty &= \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n} \\
&= \frac{-\log q}{2\pi i} \oint_{\mathcal{C}} \frac{x^s}{(q;q)_s} \Lambda_q(s) ds \\
&= \frac{\log q}{2\pi i} \oint_{\mathcal{C}} \left(\frac{x}{q}\right)^s (q;q)_{-s-1} ds
\end{aligned}
$$



$$
(x;q)_\infty = \frac{\log q}{2\pi i} \oint_{\mathcal{C}} \left(\frac{x}{q}\right)^s (q;q)_{-s-1} ds
$$

$$(x;q)_\infty = \frac{\log q}{2\pi i} \oint_{\mathcal{C}} \left(\frac{x}{q}\right)^s (q;q)_{-s-1} ds$$

Change the integration variable to $z = q^{-s}$



$$(x;q)_\infty = -\frac{(q;q)_\infty}{2\pi i} \oint_{\mathcal{C}'} \frac{z^{-\frac{\log x}{\log q}}}{(z;q)_\infty} dz \ .$$

Formally, we summarize this in

Lemma 2 *For complex x with $|\arg(x)| < \pi$ and $0 < q < 1$ we have for $0 < \rho < 1$*

$$\frac{(x;q)_\infty}{(q;q)_\infty} = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} \frac{z^{-\frac{\log x}{\log q}}}{(z;q)_\infty} dz$$

Analogously, we obtain

Lemma 3 *For complex x with $|\arg(x)| < \pi$, complex y with $y \neq q^{-n}$ for non-negative integer n, and $0 < q < 1$ we have for $0 < \rho < 1$*

$$H(x,y,q) = \frac{1}{2\pi i} \frac{(q;q)_\infty}{(y;q)_\infty} \int_{\rho - i\infty}^{\rho + i\infty} \frac{(y/z;q)_\infty}{(z;q)_\infty} z^{-\frac{\log x}{\log q}} dz$$

Proof: Calculus of residues ■

In hindsight, simply observe that

$$\text{Res}\left[(z;q)_\infty^{-1}; z = q^{-n}\right] = -\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n (q;q)_\infty} \qquad n = 0, 1, 2, \ldots$$

contains much of the structure of

$$H(x,y,q) = \sum_{n=0}^{\infty} \frac{(-x)^n q^{\binom{n}{2}}}{(q;q)_n (y;q)_n}$$

Asymptotics of q-Products

We expand

$$\begin{aligned} \log(t;q)_\infty &= \sum_{n=0}^{\infty} \log(1-tq^n) \\ &= -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{(tq^n)^m}{m} \\ &= -\sum_{m=1}^{\infty} \frac{1}{m}\frac{t^m}{1-q^m} \end{aligned}$$

Heuristically, applying

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

and blatantly exchanging the order of summation, we get

$$\log(t;q)_\infty \sim \frac{1}{\log q}\mathrm{Li}_2(t) + \frac{1}{2}\log(1-t) + R(t,q)$$

with

$$R(t,q) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(\log q)^{2n-1}\left(t\frac{d}{dt}\right)^{2n-2}\frac{t}{1-t}$$

and the Euler dilogarithm

$$\mathrm{Li}_2(t) = \sum_{m=1}^{\infty} \frac{t^m}{m^2}$$

Asymptotics of q-Products (ctd.)

Lemma 4 *For t in any compact domain such that $|\arg(1-t)| < \pi$ and real $0 < q < 1$*

$$\log(t;q)_\infty = \frac{1}{\log q}\mathrm{Li}_2(t) + \frac{1}{2}\log(1-t) + O(\log q)$$

as $q \to 1$.

Proof: Euler-Maclaurin formula ■

For $t = q$ we use the conjugate modulus transformation

$$(q;q)_\infty = (r/q)^{1/24}\sqrt{\frac{2\pi}{-\log q}}\frac{1}{(r;r)_\infty}$$

where $r = \exp\left(4\pi^2/\log q\right)$ and $0 < q < 1$. Thus

$$\log(q;q)_\infty = \frac{\pi^2}{6\log q} + \frac{1}{2}\log\frac{2\pi}{-\log q} + O(\log q)$$

Needed later:

$$\mathrm{Li}_2(t) = \sum_{m=1}^{\infty}\frac{t^m}{m^2} = -\int_0^t \frac{\log(1-u)}{u}du$$

obeys a functional equation for $0 \leq x \leq 1$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x)$$

Uniform q-Bessel Asymptotics

Restrict to $0 < x, y, q < 1$ and write $\varepsilon = -\log q$.

Lemma 5

$$H(x, y, q) =$$

$$\frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} e^{\frac{1}{\varepsilon}\{\log(z)\log(x)+\mathrm{Li}_2(z)-\mathrm{Li}_2(y/z)\}} \sqrt{\frac{1-y/z}{1-z}} dz$$

$$\times e^{\frac{1}{\varepsilon}\left\{\mathrm{Li}_2(y)-\frac{\pi^2}{6}\right\}} \sqrt{\frac{2\pi}{\varepsilon(1-y)}} \{1 + O(\varepsilon)\}$$

where $y < \rho < 1$.

Proof: Carefully insert the q-product asymptotics into

$$H(x, y, q) = \frac{1}{2\pi i} \frac{(q;q)_\infty}{(y;q)_\infty} \int_{\rho-i\infty}^{\rho+i\infty} \frac{(y/z;q)_\infty}{(z;q)_\infty} z^{-\frac{\log x}{\log q}} dz$$

■

We now have an asymptotic representation of $H(x, y, q)$ as a genuine Laplace-type integral

$$\int_{\mathcal{C}} e^{\frac{1}{\varepsilon} g(z)} f(z) dz$$

Saddle point analysis

To analyse the saddle point structure of

$$\int_{\mathcal{C}} e^{\frac{1}{\varepsilon}g(z)} f(z)dz$$

consider

$$g(z) = \log(z)\log(x) + \mathrm{Li}_2(z) - \mathrm{Li}_2(y/z)$$

There are two saddle points are the zeros $z_{1,2} = z_m \pm \sqrt{d}$ of

$$(z-1)(z-y) + zx = 0$$

and we find coalescent saddles as d changes sign.

Reparametrizing locally by a cubic

$$g(z) = \frac{1}{3}u^3 - \alpha u + \beta \text{ with } u = \pm\alpha^{1/2} \longleftrightarrow z = z_{1,2}$$

determines α and β

$$g(z_1) = -\frac{2}{3}\alpha^{3/2} + \beta \qquad g(z_2) = \frac{2}{3}\alpha^{3/2} + \beta$$

The transformation is one-to-one and analytic in a neighbourhood of $d = 0$.

We proceed with the analysis of

$$\int_{\mathcal{C}} e^{\frac{1}{\varepsilon}g(z)} f(z) dz$$

by expanding

$$f(z)\frac{dz}{du} = \sum_{n=0}^{\infty} (p_m + q_m u)(u^2 - d)^m \ .$$

Denoting the image of $\mathcal{C}$ as $\mathcal{C}'$ and writing

$$V(\lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}'} e^{u^3/3 - \lambda u} du$$

we get the asymptotic expansion

$$\begin{aligned} e^{-\beta/\varepsilon} I(\varepsilon; d) \quad &\sim \quad \varepsilon^{1/3} V(\alpha \varepsilon^{-2/3}) \sum_{m=0}^{\infty} a_m \varepsilon^m \\ &+ \quad \varepsilon^{2/3} V'(\alpha \varepsilon^{-2/3}) \sum_{m=0}^{\infty} b_m \varepsilon^m \end{aligned}$$

$V(\lambda)$ is expressible using $\mathrm{Ai}(\lambda)$, depending on the contour $\mathcal{C}'$.

Explicit formulas for the coefficients $a_0 = p_0$ and $b_0 = q_0$ of the leading order terms are

$$p_0 \pm q_0 \alpha^{1/2} = f(z_{1,2}) \sqrt{\frac{2\alpha^{1/2}}{\pm g''(z_{1,2})}}$$

Applying this method leads to our main technical result.

Lemma 6 *Let* $0 < x, y < 1$ *and* $q = e^{-\varepsilon}$ *for* $\varepsilon > 0$*. Then*

$$H(x,y,q) = \left\{p_0 \varepsilon^{1/3} \mathrm{Ai}(\alpha\varepsilon^{-2/3}) + q_0 \varepsilon^{2/3} \mathrm{Ai}'(\alpha\varepsilon^{-2/3})\right\}$$
$$\times e^{\frac{1}{\varepsilon}\left\{\mathrm{Li}_2(y) - \frac{\pi^2}{6} + \log(x)\log(y)/2\right\}} \sqrt{\frac{2\pi}{\varepsilon(1-y)}}\,\{1 + O(\varepsilon)\}$$

where

$$\frac{4}{3}\alpha^{3/2} = \log(x)\log\frac{z_m - \sqrt{d}}{z_m + \sqrt{d}} + 2\mathrm{Li}_2(z_m - \sqrt{d}) - 2\mathrm{Li}_2(z_m + \sqrt{d})$$

with

$$z_{1,2} = z_m \pm \sqrt{d} \qquad z_m = \frac{1+y-x}{2} \quad \textit{and} \quad d = z_m^2 - y$$

and

$$p_0 = \left(\frac{\alpha}{d}\right)^{1/4}(1-x-y)\,, \qquad q_0 = \left(\frac{d}{\alpha}\right)^{1/4}\,.$$

Proof: coalescing saddle point asymptotics ■

Asymptotics for Staircase Polygons

Theorem 2 (Prellberg(1995)) *Let* $0 < x, y < 1$ *and* $q = e^{-\varepsilon}$ *for* $\varepsilon > 0$. *Then*

$$G(x,y,q) = \frac{1-x-y}{2} + \sqrt{\frac{(1-x-y)^2}{4} - xy}\,\frac{\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}{\alpha^{1/2}\varepsilon^{-1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3})} \times \{1 + O(\varepsilon)\}$$

where

$$\frac{4}{3}\alpha^{3/2} = \log(x)\log\frac{z_m - \sqrt{d}}{z_m + \sqrt{d}} + 2\mathrm{Li}_2(z_m - \sqrt{d}) - 2\mathrm{Li}_2(z_m + \sqrt{d})$$

and

$$z_{1,2} = z_m \pm \sqrt{d}\,, \quad z_m = \frac{1+y-x}{2} \quad \textit{and} \quad d = z_m^2 - y\,.$$

Proof: We recall

$$G(x,y,q) = y\left(\frac{H(q^2x, qy, q)}{H(qx, qy, q)} - 1\right)$$

Using Lemma 6, we arrive at an asymptotic expression of the form

$$\frac{p_0^{(1)}\varepsilon^{1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3}) + q_0^{(1)}\varepsilon^{2/3}\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}{p_0^{(2)}\varepsilon^{1/3}\mathrm{Ai}(\alpha\varepsilon^{-2/3}) + q_0^{(2)}\varepsilon^{2/3}\mathrm{Ai}'(\alpha\varepsilon^{-2/3})}\,,$$

and all that is left is to determine the coefficients. ∎

Summary and Outlook

- Statistical mechanics of vesicle models
- Phase diagram, tricritical scaling
- Solvable models, functional equations
- Asymptotic analysis via
  - q-functional equation
  - q-series solution
  - contour integral
  - saddle-point analysis
- Can one do the asymptotics without solving the functional equation?
  - Achieved for some q-linear equations (Richard and Guttmann 2001)
  - Unsolved for q-algebraic equations
    Some heuristic results (Richard, 2002)
- There are indications that the unrestricted vesicle problem also involves the Airy function as scaling function (Cardy 2001; Richard, Guttmann and Jensen 2001)