

Recent Algorithms for Solving Second-Order Differential Equations[†]

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We consider a second order ordinary linear differential equation

$$(1) \quad \tilde{L}y \equiv \partial^2 y + A_1(x)\partial y + A_2(x)y = 0$$

with rational coefficients $A_1, A_2 \in C(x)$, over a constant field C which is assumed to be of characteristic zero and algebraically closed. We denote $\partial = d/dx$ and $K = C(x)$.

After a suitable change of variable $y \mapsto ye^{\int -a/2}$ Equation (1) is changed into the *reduced form*

$$(2) \quad L_r y \equiv \partial^2 y - r(x)y = 0 \quad \text{where} \quad r(x) = \frac{A_1^2}{4} + \frac{A_1'}{2} - A_2.$$

Given two linearly independent solutions of (1), say y_1, y_2 , either formal or actual, the differential field generated by K, y_1 , and y_2 is called a *Picard-Vessiot extension* of (1). The group of K -differential automorphisms (i.e., of field automorphisms leaving K pointwise fixed and commuting with ∂) is called the *differential Galois group of (1) over K* . We denote it by $G(\tilde{L}) = \text{Gal}_K(\tilde{L})$ and by $PG(\tilde{L}) = G(\tilde{L})/\mathcal{Z}(G(\tilde{L})) \simeq G(\tilde{L})/(G(\tilde{L}) \cap C^*)$ the corresponding projective group.

A differential Galois group is a linear algebraic group over C ; it can then be represented as a subgroup of $GL(2, C)$. In the case of an operator in reduced form L_r the differential Galois group is a special linear algebraic group over C and it can thus be represented as a subgroup of $SL(2, C)$.

The Galois correspondence states the link between properties of solutions and the form of the differential Galois group. Equation (2), hence (1), has no *Liouvillian solutions* (also called solutions *in closed form*) if and only if the differential Galois group $G(L_r)$ is isomorphic to $SL(2, C)$. At the opposite end, all solutions of (2) are algebraic if and only if the differential Galois group $G(L_r)$ is a finite group. In the case when $G(L_r) \neq SL(2, C)$ since the order is only 2, then all solutions are Liouvillian.

The Kovacic algorithm [4] is an algorithm to effectively determine whether or not a second-order linear differential equation has Liouvillian solutions with a computation of those. It can be extended to the calculation of the differential Galois group of the equation in reduced form. What follows applies to general second order differential equations in form (1) as well as form (2).

This talk is concerned with the case when the solutions are algebraic and an explicit direct computation of those. The idea consists in referring to a small amount of standard equations whose solutions have been computed once for all. Using a theorem of Klein each equation is seen as an adequate pullback of one of the standard equations. Our aim is to make this pullback explicit.

[†]Joint work with Maint Berkenbosch and Mark Van Hoeij.

1. Standard Equations

The possible projective differential Galois groups in this case are the dihedral groups \mathbf{D}_n for all $n \in \mathbb{N}$, the tetrahedral group \mathbf{A}_4 , the octahedral group \mathbf{S}_4 , and the icosahedral group \mathbf{A}_5 .

The standard equations in reference are hypergeometric equations

$$\text{St}_G y \equiv \partial^2 y + \left(\frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{x(x-1)} \right) y = 0$$

where the coefficients a, b, c are related to the differences λ, μ, ν of the exponents at 0, 1, and ∞ by the relations $4a = 1 - \lambda^2$, $4b = 1 - \mu^2$, and $4c = 1 - \nu^2 + \lambda^2 + \mu^2$. We choose $(\lambda, \mu, \nu) = (1/2, 1/2, 1/n)$ for $PG = \mathbf{D}_n$, $(1/3, 1/2, 1/3)$ for $PG = \mathbf{A}_4$, $(1/3, 1/2, 1/4)$ for $PG = \mathbf{S}_4$, and $(1/3, 1/2, 1/5)$ for $PG = \mathbf{A}_5$.

The index G refers to the differential Galois group of the equation $\text{St}_G y = 0$ corresponding to the chosen values of a, b, c . The solutions of these hypergeometric equations are Legendre functions.

2. Klein's Theorem

Definition 1. Let $L_1 \in C(z)[d/dz]$ and $L_2 \in C(x)[d/dx]$ be linear differential operators.

1. L_2 is a *proper pullback* of L_1 under $f \in C(x)$ if the change of variable $z = f(x)$ changes L_1 into L_2 .
2. L_2 is a *pullback* of L_1 under $f \in C(x)$ if there exists $v \in C(x)$ such that $L_2 \otimes (\partial + v)$ is a proper pullback of L_1 under f .

Theorem 1. Let L be a second order linear differential operator over $C(x)$ in reduced form with projective differential Galois group PG . Then, $PG \in \{\mathbf{D}_n, \mathbf{A}_4, \mathbf{S}_5, \mathbf{A}_5\}$ if and only if L is a pullback of St_G .

Let L have a finite projective differential Galois group PG and suppose the standard equation with differential Galois group G has (H_1, H_2) as a C -basis of solutions. The theorem of Klein says that L is a pullback of St_G . Suppose we know f and v . Then, a C -basis of solutions of $Ly = 0$ is given by $H_1(f(x))e^{\int v}$ and $H_2(f(x))e^{\int v}$.

H_1 and H_2 are known for all standard equations. To get the solutions in explicit form one should then determine the projective differential Galois group and, in case it is finite, determine the pullback functions f and v . The idea is to build these quantities using semi-invariants of the equation.

3. Invariants and Semi-Invariants

Definition 2. 1. A polynomial $I(Y_1, Y_2) \in C[Y_1, Y_2]$ is called an *invariant* with respect to a differential operator L if its evaluation on a C -basis (y_1, y_2) of solutions is invariant under the action of the differential Galois group $G(L)$ of L . The rational function $h(x) = I(y_1(x), y_2(x))$ is called the *value* of the invariant polynomial I .

2. A polynomial $I(Y_1, Y_2) \in C[Y_1, Y_2]$ is called a *semi-invariant* with respect to a differential operator L if the logarithmic derivative h'/h of its evaluation $h(x) = I(y_1(x), y_2(x))$ on any C -basis (y_1, y_2) of solutions is rational, i.e., in $C(x)$.

The invariant polynomials (in short invariants) of degree m of a differential equation $Ly = 0$ are elements of the m th symmetric power $\text{Sym}^m(\text{Sol}(L))$. Their values are elements of the space $\text{Sol}(\text{Sym}^m(L))$. An isomorphism between these two spaces preserving the Galois representations allows to identify an invariant to its value. As a consequence, determining the invariants or the semi-invariants of degree m of L is equivalent to determining the rational solutions of the m th

symmetric power $\text{Sym}^m(L)$ of L . On the other hand, we know the full set of possible m since we know the list of invariants and semi-invariants of the finite groups $\mathbf{D}_n, \mathbf{A}_4, \mathbf{S}_4, \mathbf{A}_5$.

This provides us with a perfectly effective procedure to determine the invariants or semi-invariants of L and consequently the type of its differential Galois group.

Now, suppose L has a differential Galois group G with semi-invariant \mathcal{S} of degree m and value $\sigma(x)$. And suppose the value of \mathcal{S} with respect to the standard operator St_G equals σ_0 (modulo C^*). Then, the value of \mathcal{S} with respect to both differential operators

$$S_G = \text{St}_{PG} \otimes \left(\partial_z + \frac{\sigma'_0}{m\sigma_0} \right) \quad \text{and} \quad L = \tilde{L} \otimes \left(\partial_x + \frac{\sigma'}{m\sigma} \right)$$

is equal to 1 and the following property holds.

Proposition 1. *L is a proper pullback of S_G under $z = f(x)$.*

A direct examination in each case will provide the pullback function f .

4. Pullback Formulas

Primitive case: $PG \in \{\mathbf{A}_4, \mathbf{S}_4, \mathbf{A}_5\}$. The standard equation in reference is $\text{St}_G y = 0$ where the differences of exponents are $\lambda = 1/3$ at $x = 0$, $\mu = 1/2$ at $x = 1$, and $\nu = 1/3$ for \mathbf{A}_4 , $1/4$ for \mathbf{S}_4 , and $1/5$ for \mathbf{A}_5 at infinity.

The differential Galois group of this equation has a semi-invariant S of degree $m = 4$ in the case of \mathbf{A}_4 , $m = 6$ in the case of \mathbf{S}_4 and $m = 12$ in the case of \mathbf{A}_5 with value $s(x) = x^{-m/3}(x-1)^{-m/4}$. The new standard equation

$$S_G = \text{St}_G \otimes \left(\partial + \frac{1}{3z} + \frac{1}{4(z-1)} \right)$$

reads

$$S_G = \partial^2 + \frac{7z-4}{6z(z-1)}\partial - \frac{1}{144} \frac{(6\nu-1)(6\nu+1)}{z(z-1)}.$$

It has exponents $(0, 1/3)$ at 0, $(0, 1/2)$ at 1, and $(\frac{6\nu+1}{12}, \frac{-6\nu+1}{12})$ at infinity where ν has the previous value in each case. The semi-invariant \mathcal{S} of degree m now has value 1. The coefficients of the pullback equation $\partial^2 y + a_1 \partial y + a_0 y = 0$ satisfy

$$a_1 = \frac{f''}{f'} + f' \frac{7f-4}{6f(f-1)} \quad \text{and} \quad a_0 = -\frac{(6\nu-1)(6\nu+1)f'^2}{144f(f-1)}.$$

Algorithm. Input: \tilde{L} with finite primitive group.

1. For $m \in \{4, 6, 12\}$ check for a semi-invariant of degree m and call v its logarithmic derivative.
2. If successful, let $L = \tilde{L} \otimes (\partial + \frac{1}{m}v)$ be a proper pullback of S_G with invariant value 1. Denote $L = \partial^2 + a_1 \partial + a_0$.
3. Let $s = \frac{(6\nu-1)(6\nu+1)}{144}$, where $\nu \in \{1/3, 1/4, 1/5\}$ is known.

Output: the pullback function $f = \left(1 + \frac{s}{a_0} (6a_1 + 3\frac{a'_0}{a_0})^2 \right)^{-1}$ and

for St_{A_4} , the basis of solutions $H_1 = {}_2F_1\left(\frac{-1}{12}, \frac{1}{4}, \frac{2}{3}; x\right)$ and $H_2 = \sqrt[3]{x} {}_2F_1\left(\frac{1}{4}, \frac{7}{12}, \frac{4}{3}; x\right)$;
 for St_{S_4} , the basis of solutions $H_1 = {}_2F_1\left(\frac{-1}{24}, \frac{5}{24}, \frac{2}{3}; x\right)$ and $H_2 = \sqrt[3]{x} {}_2F_1\left(\frac{7}{24}, \frac{13}{24}, \frac{4}{3}; x\right)$;
 for St_{A_5} , the basis of solutions $H_1 = {}_2F_1\left(\frac{11}{60}, -\frac{1}{60}, \frac{2}{3}; x\right)$ and $H_2 = \sqrt[3]{x} {}_2F_1\left(\frac{31}{60}, \frac{19}{60}, \frac{4}{3}; x\right)$.

Here ${}_2F_1$ denotes the hypergeometric function. In the case of St_{A_4} the solutions can also be given in terms of radicals or roots of an algebraic equation of degree 24.

Dihedral case: $PG = \mathbf{D}_n$ for $n \in \mathbb{N}$. The procedure is similar; however, one has to determine here the value of n .

For the sake of more symmetry, the standard equation in reference is chosen here with exponent differences $1/2$ at $+1$ and -1 and $1/n$ at infinity. It has a semi-invariant $\mathcal{S}_2 = Y_1 Y_2$ of degree 2 and two semi-invariants $\mathcal{S}_{n,a} = Y_1^n + Y_2^n$ and $\mathcal{S}_{n,b} = Y_1^n - Y_2^n$ of degree n . The new standard equation

$$S_{D_n} = \partial^2 - \frac{z}{z^2 - 1} \partial - \frac{1}{4n^2} \frac{1}{z^2 - 1}$$

has exponents $(0, 1/2)$ at $+1$ and -1 , and $(-1/2n, 1/2n)$ at infinity; it has a semi-invariant of degree 2 and value 1. The operator $L = \partial^2 + a_1 \partial + a_0$ is a pullback of S_{D_n} if

$$a_0 = -\frac{1}{4n^2} \frac{f'^2}{f^2 - 1} \quad \text{and} \quad a_1 = -\frac{1}{2} \frac{a'_0}{a_0}.$$

The equation $Ly = 0$ admits the solutions $\exp \int \pm \sqrt{-a_0} = \exp \int \frac{1}{2n} \frac{f'}{\sqrt{f^2 - 1}} dx$. The number n can thus be determined with the algorithm of integration on algebraic curves [3, 5, 6]; in fact, the authors give refinements of this part of the algorithm to compute a multiple of n .

Algorithm. Input: $\tilde{L} = \partial^2 + A_1(x)\partial + A_2(x)$ with finite differential Galois group.

1. Check for a semi-invariant of degree 2 and call v its logarithmic derivative.
2. If successful, let $L = \tilde{L} \otimes (\partial + \frac{1}{m}v)$ be a proper pullback of S_{D_n} with invariant value 1. Denote $L = \partial^2 + a_1 \partial + a_0$.
3. Determine a candidate for a multiple of n .
4. For an adequate n , the equation $L_n y \equiv \partial^2 y + a_1 \partial y + n^2 a_0 y = 0$ has solutions f and $\sqrt{f^2 - 1}$, hence f can be determined.
5. Let c be such that $c^2 = \frac{4n^2 a_0}{f'^2 + 4n^2 f^2 a_0}$.

Output: the pullback function $\pm cf$ and the solutions $(cf \pm \sqrt{c^2 f^2 - 1})^{1/n}$.

The procedure appears to be more efficient than the Kovacic algorithm. In addition, it provides the pullback and the solutions in simple form.

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