Algorithms Seminar 2001–2002,
F. Chyzak (ed.), INRIA, (2003), pp. 37–42.

Available online at the URL
http://algo.inria.fr/seminars/.

Transseries Solutions of Algebraic Differential Equations

Joris van der Hoeven
CNRS, Université Paris-Sud (France)

May 27, 2002

Summary by Anne Fredet

Abstract

Transseries are series defined using exponential and logarithmic variables. They were first introduced to describe very general types of strongly monotonic asymptotic behaviour. The functions that are considered do not present any oscillatory phenomenon. An algorithm is presented that computes transseries solutions of algebraic differential equations with transseries coefficients.

1. Introduction to Transseries

1.1. **Well-ordered and grid-based transseries.** The transseries are a generalization of the usual formal power series, allowing the recursive introduction of exponential and logarithmic variables (see [1] or [3] and references).

Example. The following series are transseries:
- $1 + x^{-1} + x^{-2} + x^{-e} + x^{-3} + x^{-e-1} + \cdots = \frac{1}{1-x^{-1}-x^{-e}}$,
- $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + e^{-x} + \frac{e^{-x}}{x} + \cdots + e^{-2x} + \cdots$,
- $\frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{e^{\log^2 x}} + \frac{1}{e^{2\log^2 x}} + \cdots + \frac{1}{e^{\log^4 x}} + \frac{1}{e^{2\log^4 x}} + \cdots$ solution of $f(x) = \frac{1}{x} + f(x^2) + f(e^{\log^2 x})$,
- $1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots$,
- $x^{-1} + x^{-\pi} + x^{-\pi^2} + x^{-\pi^3} + \cdots$,
- $x + \sqrt{x} + \sqrt{\sqrt{x}} + \sqrt{\sqrt{\sqrt{x}}} + \cdots$,
- $e^{e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots} + x^{-1} e^{e^x + \frac{e^x}{x} + \frac{e^x}{x^2} + \cdots} + \cdots$,
- $\Gamma(x-\pi) + \log \Gamma(e^{\Gamma(x^2)}) x^{x^{x^x}}$,
- $e^{\sqrt{x} + e^{\sqrt{\log x} + e^{\sqrt{\log\log x} + \cdots}}}$.

An *ordered ring* is a ring A, together with an order $\leq$ which is compatible with the ring structure. This means that: *(i)* $(x \leq y$ and $x' \leq y') \Rightarrow x + x' \leq y + y'$, *(ii)* $0 \leq 1$, and *(iii)* $(0 \leq x$ and $0 \leq y) \Rightarrow 0 \leq xy$. The *absolute value* $|x|$ of $x \in A$ is defined by $|x| = x$ if $x \geq 0$ and $-x$ otherwise. One writes $x \prec y$ if $|\lambda x| \leq |\mu y|$ for some $\mu \in A$ and all $\lambda \in A$ and one says that x is *negligible* with respect to y.

More generally, let C be a constant field with a total order (i.e., either $\alpha = \beta$ or $\alpha < \beta$ or $\alpha > \beta$ for all α and β in C—see [7]) and $\mathfrak{M}$ be a group with a total order $\succcurlyeq$. A *well-ordered* transseries is a mapping $f : \mathfrak{M} \to C$ with well-ordered support (this means that every nonempty subset of the support of f has a least element—see [7]). The elements of C are called *coefficients* and the elements of $\mathfrak{M}$ are *monomials*. If $f = \sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ is a well-ordered transseries and $\mathfrak{m} \in \mathfrak{M}$ then one says that $f_{\mathfrak{m}}$ is the *coefficient* of $\mathfrak{m}$ in f and $f_{\mathfrak{m}}\mathfrak{m}$ is a *term* occurring in f. Since the support of f is well-ordered, it admits a maximal element $\mathfrak{d}_f$ which is called the *dominant monomial*. If $f = c_f\mathfrak{d}_f(1+\delta_f)$ then $c_f\mathfrak{d}_f$ is the *leading term* of f and one denotes that $f \preccurlyeq g$ if and only if $\mathfrak{d}_f \preccurlyeq \mathfrak{d}_g$. One decomposes

$$f = f^{\uparrow} + f^{=} + f^{\downarrow}$$

with

$$f^{\uparrow} = \sum_{\mathfrak{m}\succ 1} f_{\mathfrak{m}}\mathfrak{m}, \qquad f^{=} = f_1, \qquad f^{\downarrow} = \sum_{\mathfrak{m}\prec 1} f_{\mathfrak{m}}\mathfrak{m}.$$

One focuses on particular transseries:

Definition 1. A series f is *grid-based* if there are $\mathfrak{m}_1, \ldots, \mathfrak{m}_k \prec 1$ and $\mathfrak{n} \in \mathfrak{M}$ such that

$$\operatorname{supp} f \subset \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}^* \mathfrak{n}$$

One denotes by $C[\![\mathfrak{M}]\!]$ the set of mappings from $\mathfrak{M}$ to C with grid-based support and one calls it the set of grid-based transseries. One remarks that $C[\![\mathfrak{M}]\!] \neq C[[\mathfrak{M}]]$.

Example. If $f = x^2 + x + 1 + x^{-1} + \cdots$ then $\operatorname{supp} f \subseteq \{x^{-1}\}^* x^2$

The field of the grid-based transseries in x over C is denoted by $\mathbb{T}$ and is stable under derivation, composition and functional inversion as proved in [3]. Ways to construct $\mathbb{T}$ are presented in [3].

1.2. **Transbasis.**

Definition 2. An ordered set of transseries $\mathfrak{B} = (\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ is a *transbasis* if the following conditions are satisfied:

1. $\mathfrak{b}_1$ is an iterated logarithm or exponential: $\mathfrak{b}_1 = \exp_l x$ for some $l \in \mathbb{Z}$ (l is the *level* of the transbasis);
2. $1 \prec \mathfrak{b}_1 \prec \cdots \prec \mathfrak{b}_n$;
3. $\mathfrak{b}_i \in \exp C[\![\mathfrak{b}_1, \ldots, \mathfrak{b}_{i-1}]\!]$ for $i > 1$.

Example. The sets $\mathfrak{B}_1 = \{x^{-1}, e^{-x}, e^{-x^2}, e^{-x^3}\}$ and $\mathfrak{B}_2 = \{\log^{-1} x, x^{-1}, e^{-\log^2 x}, e^{-x}, e^{-e^x/(1+x^{-1})}\}$ are transbasis but $\mathfrak{B}_3 = \{x^{-1}, e^{-x+e^{-x}}\}$ is not because $e^{-x+e^{-x}}$ is not in $\exp C[\![x^{-1}]\!]$.

One says that a transseries f can be expanded with respect to $\mathfrak{B}$ if $f \in C[\![\mathfrak{b}_1, \ldots, \mathfrak{b}_n]\!]$. Equivalently, one says that $\mathfrak{B}$ is a transbasis for f.

Example. $\log(x + e^{\frac{-x^2}{1-x^{-1}}}) \in C[\![\log x; x; e^{x^2+x}]\!]$ and then $\mathfrak{B} = \{\log x; x; e^{x^2+x}\}$ is a transbasis for f.

For any $f \in C[\![\mathfrak{b}_1, \ldots, \mathfrak{b}_n]\!]$, one can recursively expand f: $f = \sum_{\alpha_n} f_{\alpha_n} \mathfrak{b}_n^{\alpha_n}$ where $f_{\alpha_n} = \sum_{\alpha_{n-1}} f_{\alpha_n,\alpha_{n-1}} \mathfrak{b}_{n-1}^{\alpha_{n-1}}$, where…, where $f_{\alpha_n,\ldots,\alpha_2} = \sum_{\alpha_1} f_{\alpha_n,\ldots,\alpha_1} \mathfrak{b}_1^{\alpha_1}$.

Theorem 1. *Let f be a transseries and let $\mathfrak{B}_0$ be a transbasis. Then there exists a transbasis $\mathfrak{B}$ for f which contains $\mathfrak{B}_0$.*

1.3. **Differentiation and shifting.** Right compositions by exp (resp. log) are referred to by *upward shifting* (resp. *downward shifting*). The upward (resp. downward) shifting of $f \in \mathbb{T}$ is denoted by $f \circ \exp = f{\uparrow}$ (resp. $f \circ \log = f{\downarrow}$). One observes that $\uparrow$ and $\downarrow$ are scale changes which preserve the set of transmonomials. Note that $f{\uparrow} \neq f^{\uparrow}$ and $f{\downarrow} \neq f^{\downarrow}$. These compositions are used to consider transbasis starting with level one ($\mathfrak{b}_1 = x$) which is particularly useful for differential calculus (see below).

1.4. **A conjecture of Hardy.** In [2] a conjecture states that the functional inverse of $\log x \log\log x$ is not equivalent to any exp-log function over $\mathbb{R}$ for $x \to \infty$. Theorem 1.2 of [3] illustrate the interest of transseries by a proof of this conjecture.

2. Differential Algebraic Polynomials

Let $P = \sum_d P_d$ be a differential algebraic polynomial where

$$P_d = \sum_{i_0+\cdots+i_r=d} P_{i_0,\dots,i_r} f^{i_0} \dots f^{(r)^{i_r}}.$$

One defines:

- the *degree* of P, $\deg P = \max\{\, i_0 + \cdots + i_r \mid P_{i_0,\dots,i_r} \neq 0 \,\}$,
- the *additive conjugate*, $P_{+h}(f) = P(f+h)$,
- the *multiplicative conjugate*, $P_{\times h}(f) = P(fh)$,
- the *upward shifting*, $P{\uparrow}(f{\uparrow}) = P(f){\uparrow}$,
- the *dominant monomial*, $\mathfrak{d}_P = \max\{\, \mathfrak{d}_{P_{i_0,\dots,i_r}} \mid \mathfrak{d}_{P_{i_0,\dots,i_r}} \neq 0 \,\}$,
- the *dominant coefficient*, $D_P = \sum_{i_0,\dots,i_r} P_{i_0,\dots,i_r,\mathfrak{d}_P} c^{i_0} \dots c^{(r)^{i_r}}$, where c is a variable.

2.1. **Differential Newton polynomials.** One now describes an algorithm for the resolution of algebraic differential equations with transseries coefficients like

$$P(f) = 0 \qquad (f < \mathfrak{v}) \tag{1}$$

where $P \in \mathbb{T}[f, f', \dots, f^{(r)}]$ is a differential polynomial with transseries coefficients and $\mathfrak{v} \in \mathfrak{M}$ a transmonomial. The first step is to construct an analogue of the Newton polygon and polynomial method in this setting, enabling us to compute the successive terms of solutions one by one.

The following theorem shows how D_P looks like after sufficiently many upward shifting.

Theorem 2. *There exist an integer $k \le \deg(P)$ and a polynomial N_P depending only on the variables c and c' such that for any $l \ge k$, $D_{P\uparrow_l} = N_P$.*

Example. If one considers $P = ff'' - f'^2$ and one denotes $\tilde f(x) = f{\uparrow} = f(e^x)$ then one has $\tilde f'(x) = e^x f'(e^x)$ and $\tilde f''(x) = e^x f'(e^x) + e^{2x} f''(e^x)$. This implies that $f(e^x) = \tilde f(x)$, $f'(e^x) = e^{-x}\tilde f'(x)$, $f''(e^x) = e^{-2x}\left(\tilde f''(x) - \tilde f'(x)\right)$ and $P(f){\uparrow} = e^{-2x}\left(\tilde f \tilde f'' - \tilde f \tilde f' - \tilde f'^2\right) = P{\uparrow}(\tilde f) = P{\uparrow}(f{\uparrow})$. So one deduces that

$$P{\uparrow} = e^{-2x}\left(ff'' - ff' - f'^2\right).$$

Using the same method, one finds that $P{\uparrow\uparrow} = e^{-2e^x - x}(ff') + e^{-2e^x-2x}(ff'' - ff' - f'^2)$. This implies that $N_P = cc'$.

N_P is the *differential Newton polynomial of P*. More generally, given a monomial $\mathfrak{m}$, $N_{P\times\mathfrak{m}}$ is the *differential Newton polynomial of P associated to* $\mathfrak{m}$. The *Newton degree* of (1) is the largest possible degree of $N_{P_{\times\mathfrak{m}}}$ for all the monomials $\mathfrak{m} \prec \mathfrak{v}$. In the algebraic case, the Newton degree measures the number of solutions to the asymptotic equation when counting with multiplicities. In the differential case, it only gives a lower bound (see Theorem 1 of [6]). Also, an equation of degree zero does not admit any solutions.

2.2. **Potential dominant monomials of solutions.** One is now interested by the le ading terms of a solution f to the asymptotic differential equation (1). One calls $\mathfrak{m} \prec \mathfrak{v}$ a *potential dominant monomial* if $N_{P_{\times \mathfrak{m}}} \notin C$. If $c \in C$ is such that $c\mathfrak{m} \prec \mathfrak{v}$ and $N_{P_{\times \mathfrak{m}}}(c) = 0$ then the corresponding term $c\mathfrak{m}$ is called a *potential dominant term.*

A potential dominant monomial $\mathfrak{m}$ is said to be *algebraic* if $N_{P_{\times \mathfrak{m}}} \in C[c] \setminus C$, *differential* if $N_{P_{\times \mathfrak{m}}} \in C[c']$. A potential dominant monomial involving both c and c' in $C[c, c'] \setminus (C[c] \cup C[c'])$ is said to be *mixed.*

The algebraic potential dominant monomials correspond to the slopes of the Newton polygon in a non differential setting. However, they can not be determined directly as a function of the dominant monomials of the P_i, because there may be some cancellation of terms in the different homogeneous parts during multiplicative conjugation. The algebraic potential dominant monomials are determined by successive approximation:

Proposition 1. *Let i, j be such that $P_i \neq 0$ and $P_j \neq 0$. There exists a unique monomial $\mathfrak{m}$ such that $N_{(P_i+P_j)_{\times \mathfrak{m}}}$ is non homogeneous.*

This unique monomial is called an *equalizer* or the (i, j)-equalizer for P. An algebraic potential dominant monomial is necessarily an equalizer (see [5]). Consequently, there are only a finite number of algebraic potential dominant monomials. In the proof of proposition 5.3 in [5], the author gives a method to compute such monomials.

Example. Consider the algebraic differential equation

$$P(f) = f + ff'' - f'^2 \tag{2}$$

One starts by computing the potential dominant monomials of f. One first has to find the $(1, 2)$-equalizer relative to 2. Since D_{P_2} must be in $c^{\mathbb{N}}(c')^{\mathbb{N}}$ one cannot have $N_{P_2} = P_2$ so one has to compute

$$P\uparrow = f + e^{-2x}\left(-ff' + ff'' - f'^2\right)$$

In order to equalize $P\uparrow_1$ and $P\uparrow_2$ one conjugates P multiplicatively with e^{2x}:

$$\begin{aligned} P\uparrow_{\times e^{2x}} &= fe^{2x} + e^{-2x}\left(-fe^{2x}(fe^{2x})' + fe^{2x}(fe^{2x})'' - (fe^{2x})'^2\right) \\ &= e^{2x}\left(f - 2f^2 - ff' + ff'' - f'^2\right) \end{aligned}$$

One has

$$P\uparrow_{\times e^{2x}}\uparrow = e^{2x}(f - 2f^2) - e^x(ff') + \left(ff'' - ff' - f'^2\right)$$

One observes that $D_{P\uparrow_{\times e^{2x}}\uparrow} = c - 2c^2 \in C[c]$ so one has found the $(1, 2)$-equalizer which is $\mathfrak{e} = e^{2x}\downarrow = x^2$. Since $N_{P_{\times \mathfrak{e}}} = c - 2c^2$ the corresponding algebraic potential dominant term of f is $\tau^{\text{alg}} = \frac{1}{2}x^2$.

In order to find the differential potential dominant monomials, it suffices to consider P_i since $N_{P_{\times \mathfrak{m}},i} = N_{P_{i,\times \mathfrak{m}}}$ if $c'|N_{P_{\times \mathfrak{m}}}$ and $N_{P_{\times \mathfrak{m}}} \neq 0$. One rewrites $P_i = R_{P,i}(f^\dagger)f^i$ where the order of $R_{P,i}$ in $f^\dagger = f'/f$ is less than or equal to 1 and calls $R_{P,i}$ the *ith Riccati equation associated to P.*

Proposition 2. *The monomial $\mathfrak{m} \prec \mathfrak{v}$ is a potential dominant monomial of f w.r.t. $P_i(f) = 0$ if and only if*

$$R_{P,i,\frac{\mathfrak{m}'}{\mathfrak{m}}}(f^\dagger) \qquad \left(f^\dagger \prec \frac{1}{x \log x \log \log x \cdots}\right)$$

has strictly positive Newton degree.

Example. Consider the algebraic differential equation (2) given in the previous example. One has

$$R_{P,1} = 1, \qquad R_{P,2} = {f^{\dagger}}'.$$

$R_{P,1}$ has no roots and $R_{P,2}(f^{\dagger}) = 0$ has all constants $\lambda \in C$ as its solutions modulo $\frac{1}{x \log x \log\log x \cdots}$. Consequently $e^{\lambda x}$ is a potential dominant monomial of f for all $\lambda \in C$ such that $e^{\lambda x} \succ 1$. The corresponding differential dominant terms are of the form $\tau^{\text{diff}}_{\mu,\lambda} = \mu e^{\lambda x}$ with $\mu \neq 0$ and $e^{\lambda x} \succ 1$.

2.3. **Quasi-linear differential operators and distinguished solutions.** The equation (1) is *quasi-linear* if its Newton degree is one. A solution f to such an equation is said to be *distinguished* if $f_{\mathfrak{d}_{\tilde{f}-f}} = 0$ for all other solutions $\tilde{f}$ to (1).

Theorem 3 (Theorem 6.3 of [5])**.** *Assume that the equation (1) is quasi-linear. Then it admits a distinguished transseries solution.*

2.4. **Other terms of solutions.** Using the previous results, one knows how to determine the potential dominant terms of solutions to (1). One is now interested in obtaining more terms. A *refinement* is a change of variables together with an asymptotic constraint $f = \phi + \tilde{f} \quad (\tilde{f} \prec \tilde{\mathfrak{v}})$. Such refinement transforms (1) into

$$P_{+\phi}(\tilde{f}) = 0 \qquad (\tilde{f} \prec \tilde{\mathfrak{v}}). \tag{3}$$

Proposition 3. *Let τ be the dominant term of ϕ. The Newton degree of (3) is the multiplicity of τ as potential dominant term in (1).*

Example. In order to find more terms of the solution to (2) one has to refine the equation. First of all, consider the refinement associated to the algebraic potential dominant term,

$$f = \tau^{\text{alg}} + \tilde{f} \qquad (\tilde{f} \prec \tau^{\text{alg}}),$$

which transforms (2) into

$$2\tilde{f} - 2x\tilde{f}' + \frac{1}{2}x^2\tilde{f}'' + \tilde{f}\tilde{f}'' - \tilde{f}'^2 = 0 \qquad (\tilde{f} \prec x^2). \tag{4}$$

Since $P_0 = 0$ one first observes that $f = \frac{1}{2}x^2$ is actually a solution of (2). Since $\frac{1}{2}x^2$ is a potential dominant term of multiplicity 1 of f, the Newton degree of (4) is one. The only potential dominant monomials of $\tilde{f}$ therefore necessarily correspond to solutions modulo $\frac{1}{x \log x \log\log x}$ of the Riccati equation

$$2 - 2xf^{\dagger} + \frac{1}{2}x^2\left({f^{\dagger}}^2 + {f^{\dagger}}'\right) = 0$$

These solutions are of the form $f^{\dagger} = \frac{1}{x} + \cdots$ and $f^{\dagger} = \frac{4}{x} + \cdots$ which leads to the potential dominant monomials x and x^4 from which one removes x^4 since $x^4 \not\prec x^2$. Expanding one term further, one sees that the generic solution to (4) is

$$\tilde{f} = \lambda x + \frac{\lambda^2}{2}$$

with $\lambda \in C$ where the case $\lambda = 0$ recovers the previous solution. So

$$f = \frac{1}{2}x^2 + \lambda x + \frac{\lambda^2}{2}$$

is the first type of generic solution to (2). As to the second case, we consider the refinement

$$f = \tau_{\mu,\lambda}^{\text{diff}} + \tilde{f} \qquad (\tilde{f} \prec \tau_{\mu,\lambda}^{\text{diff}})$$

which transforms (2) into

$$\mu e^{\lambda x} + \left(\lambda^2 f - 2\lambda f' + f''\right) \mu e^{\lambda x} + f + \tilde{f}\tilde{f}'' - \tilde{f}'^2 = 0 \qquad (\tilde{f} \prec \mu e^{\lambda x}) \tag{5}$$

This equation has Newton degree one and one observes that the linear part of this equation only admits solutions with dominant monomial $e^{\lambda x}$ or $xe^{\lambda x}$. Consequently (5) admits at most one solution. By Theorem 3 one knows that quasi-linear equations always admit at least one solution. This leads to the following second type of generic solution to (2):

$$f = \mu e^{\lambda x} - \frac{1}{\lambda^2} + \frac{1}{4\mu\lambda^4} e^{-\lambda x}$$

For this example, we found exact solutions but the expansion are infinite in general.

3. A Differential Intermediate Value Theorem

Theorem 4 ([4]). *Let P be a differential polynomial with coefficients in $\mathbb{T}$. Given $\varphi < \psi$ in $\mathbb{T}$ such that $P(\varphi)P(\psi) < 0$, there exists an $f \in (\varphi, \psi)$ with $P(f) = 0$.*

If there exists a differential polynomial with coefficients in $\mathbb{T}$ which admits a sign change on a non empty interval (φ, ψ) of transseries, one uses the differential Newton polygon method to shrink the interval further and further while preserving the sign change property. Ultimately, one ends up with an interval which is reduced to a point which will be seen as a zero of P.

Corollary 1. *Any algebraic equation of odd degree has at least one transseries solution.*

4. Conclusion

In [3] this approach of transseries was introduced, based on Écalle's works (see [1]). In [5] the approach is generalized to complex transseries. In particular, some results on the factorization of linear differential equation are presented. There remains some difficulties in this generalization, as to determine the differentially algebraic closure.

The transseries formalism could also be used to solve functional equations, and the multiple results should be extended to such operators.

Bibliography

[1] Écalle (Jean). – *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* – Hermann, Paris, 1992, *Actualités Mathématiques. [Current Mathematical Topics]*, ii+340p.

[2] Hardy (G. H.). – Properties of logarithmico-exponential functions. *Proceedings of the London Mathematical Society*, vol. 10, 1911, pp. 54–90.

[3] van der Hoeven (Joris). – *Asymptotique automatique.* – PhD thesis, Université Paris VII, 1997.

[4] van der Hoeven (Joris). – *A differential intermediate value theorem.* – Technical Report n° 2000-50, Université d'Orsay, 2000.

[5] van der Hoeven (Joris). – *Complex transseries solutions to algebraic differential equations.* – Technical Report n° 2001-34, Université d'Orsay, 2001.

[6] van der Hoeven (Joris). – A new zero-test for formal power series. In Mora (Teo) (editor), *ISSAC 2002 (July 7-10, 2002. Université de Lille, Lille, France)*. pp. 117–122. – ACM Press, New York, 2002. Conference proceedings.

[7] von zur Gathen (Joachim) and Gerhard (Jürgen). – *Modern computer algebra.* – Cambridge University Press, New York, 1999, xiv+753p.