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The Structure of Multivariate Hypergeometric Terms

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Abstract

The structure of multivariate hypergeometric terms is studied. This leads to a proof of (a special case of) a conjecture formulated by Wilf and Zeilberger in 1992.

A function $u(n_1,\dots,n_d)$ with values in a field $\mathbb{K}$ is called a *hypergeometric term* if there exist rational functions $R_i \in \mathbb{K}(n_1,\dots,n_d)$, $i=1,\dots,d$, such that u is solution of a system of d first-order recurrences $S_i \cdot u = R_i(n_1,\dots,n_d)u$, $i=1,\dots,d$, where S_i denotes the shift operator with respect to n_i (e.g., $S_1 \cdot u(n_1,\dots,n_d) = u(n_1+1,n_2,\dots,n_d)$).

In the univariate case, the numerator and denominator of R_1 factor into linear factors over the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. This factorization induces an explicit form for univariate hypergeometric terms as $C\rho^n \prod_{i=1}^{I}(n+\alpha_i)^{k_i}$, where C is a constant, $\rho \in \mathbb{K}$, $\alpha_i \in \overline{\mathbb{K}}$, $k_i \in \{1,-1\}$, and I is the sum of the degrees of the numerator and denominator of R_1. These terms thus express the Taylor coefficients of generalized hypergeometric series, whence their name.

In the multivariate case, no such simple factorization exists, but the rational functions are related through the identities $R_j(S_jR_i) = R_i(S_iR_j)$, $1 \le i,j \le d$. A non-obvious consequence of these relations is the following theorem from an entirely elementary Appendix of [5] (see also [2]). The bivariate case was proved by Ore in [4].

Theorem 1 (Ore–Sato)**.** *Hypergeometric terms can be written*

$$R(n_1,\dots,n_d)\prod_{i=1}^{d}\rho_i^{n_i}\cdot\prod_{i=1}^{p}\prod_{k=0}^{e_i(n_1,\dots,n_d)-1}\psi_i\big(e_i(n_1,\dots,n_d)-k\big), \tag{1}$$

where R is a rational function, and for $i=1,\dots,d$, $\rho_i \in \mathbb{K}$, the e_i's are linear forms with integer coefficients and the ψ_i are univariate rational functions.

Definition 1. An expression of the form (1) where $R=1$ is called a *proper hypergeometric term.*

Proper hypergeometric terms have the property of forming *holonomic* sequences. These are defined as follows.

Definition 2. A (multivariate) sequence is *holonomic* when the set of partial derivatives of its generating series spans a finite-dimensional vector space over the rational functions.

These series are sometimes called *D-finite*. An elementary proof of Kashiwara's equivalence between D-finiteness and holonomy in the sense of D-module theory is derived in [6, Appendix]. A characterization of holonomic sequences is provided by the following [3].

Theorem 2 (Lipshitz)**.** *A sequence $u_{n_1,\dots,n_d}$ is holonomic if and only if there exists $s \in \mathbb{N}$ such that*

1. *for each $i = 1, \dots, d$, u satisfies a linear recurrence of the form*

$$\sum_{\mathbf{h} \in \{0,\dots,s\}^d} p_{\mathbf{h},i}(n_i) u_{\mathbf{n}-\mathbf{h}} = 0, \qquad n_i \geq s, i = 1, \dots, d, \tag{2}$$

   where bold letters indicate multi-indices and the coefficients are univariate polynomials;
2. *if $d \geq 2$, each of the specialized sequences $u_{n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_d}$ is holonomic, for $i = 1, \dots, d$, $k = 0, \dots, s-1$.*

The importance of holonomy in computer algebra comes from its use by Zeilberger [8] for an algorithmic proof of many identities. In this context, holonomy provides with a sufficient condition for several definite summation or integration algorithms to terminate. An algorithm specifically designed for the definite summation or integration of *hypergeometric* terms was given by Wilf and Zeilberger [7]. There, they give a conjecture which has the following as a special case.

Theorem 3. *Hypergeometric terms form holonomic sequences if and only if they are proper.*

This result is due to Abramov and Petkovšek [2]. The sketch of the proof is as follows.

First, it was shown by Lipshitz [3] that D-finite series are closed under Hadamard (i.e., termwise) product. In other words, holonomic sequences are closed under product. Lipshitz's proof relies on combinatorial considerations on the dimensions of the vector spaces that are involved.

Second, proper hypergeometric terms are holonomic. In view of the closure property above, it is sufficient to prove this for factorials of linear forms with integer coefficients and their reciprocals. In these cases, a linear recurrence with *constant* coefficients is found by shifting the argument along a vector with non-zero integer coordinates living in the kernel of the linear form. Then Theorem 2 can be applied. (One uses the same recurrence for each i.)

The holonomy of a hypergeometric term u in the form given by the Ore–Sato theorem is equivalent to that of the leading rational function R: u can be multiplied by the inverse of its proper part, itself proper and therefore holonomic. The problem is thus reduced to the study of which *rational* sequences are holonomic. The conclusion follows from considering the *univariate* constraints (2).

It should be mentioned that holonomy is only a sufficient condition for Zeilberger's algorithm to terminate. In the bivariate case, a necessary and sufficient condition was given recently in [1].

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