Algorithms Seminar 2001–2002,
F. Chyzak (ed.), INRIA, (2003), pp. 23–26.

Available online at the URL
http://algo.inria.fr/seminars/.

Symmetric Functions and P-Recursiveness

Marni Mishna
LACIM, Université du Québec à Montréal (Canada)

October 15, 2001

Summary by Henry Crapo

In his 1990 paper [1], Ira Gessel introduced a notion of D-finite for symmetric functions, and showed how it could be used to determine D-finiteness of combinatorial generating functions.

In this context, a *symmetric function* is a polynomial function of finite degree (here, n) in infinitely many variables x_1, x_2, ..., invariant with respect to arbitrary permutations of finite subsets of the variables. Typical symmetric functions are indexed either by the degree n itself, or by a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, and include the following (by way of illustration, we set $n = 3$, and take $(2, 1)$ as a typical partition of n):

1. the homogeneous symmetric functions,
$$h_n = \sum_{1 \le i_1 \le \cdots \le i_k} x_1^{i_1} \dots x_1^{i^k}$$
so
$$h_3 = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + \cdots + x_1 x_2 x_3 + \cdots ,$$
$$h_{[2,1]} = h_2 h_1 = x_1^3 + 2x_1^2 x_2 + \cdots + 3x_1 x_2 x_3 + \cdots .$$
2. the elementary symmetric functions,
$$e_n = \sum_{1 \le i_1 < \cdots < i_k} x_1^{i_1} \dots x_1^{i^k}$$
so
$$e_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots ,$$
$$e_{[2,1]} = e_2 e_1 = x_1^2 x_2 + \cdots + 3x_1 x_2 x_3 + \cdots .$$
3. the power symmetric functions,
$$p_n = \sum_{i>0} x_i^n$$
so
$$p_3 = x_1^3 + x_2^3 + \cdots ,$$
$$p_{[2,1]} = p_2 p_1 = x_1^3 + x_1^2 x_2 + \cdots + x_2^2 x_1 + \cdots .$$
4. the monomial symmetric functions,
$$m_\lambda = \sum_\sigma x_{\sigma(1)}^{\lambda_1} \dots x_{\sigma(n)}^{\lambda_n}$$
where the sum ranges over all permutations of $\{1, \dots, n\}$, so
$$m_{[3]} = x_1^3 + x_2^3 + \cdots ,$$
$$m_{[2,1]} = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_2^2 x_1 + \cdots .$$

5. the Schur functions,

$$s_\lambda = \det(h_{j-i+\lambda_i}) \text{ where } 1 \le i, j \le k$$

so

$$\begin{aligned} s_{[3]} &= h_3 = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + \cdots + x_1 x_2 x_3 + \cdots, \\ s_{[2,1]} &= \begin{vmatrix} h_2 & h_3 \\ h_0 & h_1 \end{vmatrix} \\ &= m_{[1]}(m_{[2]} + m_{[1,1]}) - (m_{[3]} + m_{[2,1]} + m_{[1,1,1]}) \\ &= m_{[2,1]} + 2m_{[1,1,1]}. \end{aligned}$$

The sets $\{p_\lambda\}$, $\{h_\lambda\}$, $\{e_\lambda\}$, $\{s_\lambda\}$, where λ ranges over all partitions of n, are bases for the vector space Λ^n of symmetric functions of degree n. The basic tool in what follows is a scalar product $\langle \cdot, \cdot \rangle$ introduced by Redfield, and characterized by the condition that the monomial and homogeneous symmetric functions be dual bases for Λ^n, that is

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$$

and with the property

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}, \qquad \text{where } z_\lambda = n!/(\lambda_1!\, 1^{\lambda_1} \ldots \lambda_n!\, n^{\lambda_n})$$

A formal power series $y \in K[[x]]$ in one variable x is *differentiably finite* or simply *D-finite*, if y and all its derivatives $y^{(n)} = \frac{d^n y}{dx^n}$ span a finite dimensional subspace over the field of rational functions over K, that is, if and only if they satisfy a non-trivial polynomial relation of the form:

$$p_n(x) y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_0(x) y = 0.$$

Rational functions, algebraic functions, and the exponential function are D-finite. D-finite functions are closed under addition, multiplication, and the Hadamard product. If f is D-finite and g is algebraic, then the composite function $f(g)$ is D-finite. (Are there weaker conditions on g that guarantee $f(g)$ D-finite?)

A function $f : N \to K$ defined on the positive integers is *polynomially recursive*, in short *P-recursive*, if it satisfies a homogeneous linear recurrence of finite degree. For example, $f(n) = n!$ satisfies $f(n) - nf(n-1) = 0$, with polynomial coefficients 1 and $-n$. A power series $\sum_n f(n)x^n$ with coefficient sequence $f(n)$ is D-finite if and only if the sequence f is P-recursive. The speaker was interested in conditions for the existence of such recurrences, rather than in their precise construction.

The concept of D-finite was extended to several variables by Zeilberger and Lipschitz, and to infinitely many variables by Gessel. Viewing f as a formal power series in infinitely many variables, p_1, p_2, $\ldots$, the power sum symmetric functions, one applies the Gessel theory to the algebra of symmetric functions. In this way we find that h_n, e_n and $\sum_\lambda s_\lambda$ are D-finite. If $f(x_1, \ldots, x_n)$ is D-finite in $x_1, \ldots, x_n$ and for each i, r_i is a polynomial in the variables $y_1, \ldots, y_m$, then $f(r_1, \ldots, r_n)$ is D-finite in $y_1, \ldots, y_m$, as long as it is well defined as a power series. In particular, if $P(x)$ is a polynomial in $p_1, \ldots, p_n$ then $e^{P(x)}$ is D-finite. In the case of infinitely many variables, Gessel proved:

Theorem 1 ([1, Theorem 8])**.** *Let f and g be symmetric functions D-finite in the p_i and t, and suppose that g involves only finitely many p_i. Then $\langle f, g \rangle$ is D-finite in t as long as it is well-defined as a power series.*

Symmetric functions f and g can be *composed* by substituting the infinite set of monomials of g for the variables of f. Thus

$$e_2(h_2) = e_2(x_1^2, x_2^2, \dots, x_1x_2, \dots) = x_1^2x_2^2 + x_1^3x^2 + x_1x_2^3 + \cdots$$
$$p_k(g) = g(p_k), \quad g(p_1) = g, \quad p_m(p_n) = p_{mn}, \quad (f_1f_2)(g) = f_1(g)f_2(g)$$

Such symmetric functions are said to arise by *plethysm*.

The speaker asks, under what conditions does composition preserve D-finiteness? Are plethysms and D-finiteness friends or enemies? (Apparently, these days, it is necessary to choose.) Gessel proved

Theorem 2 ([1, Theorem 10]). *If g is a polynomial in the power sum symmetric functions p_i, then $h(g)$ and $e(g)$ are D-finite.*

What are the weakest conditions for f and g that retain D-finiteness?

The inner product can be used to extract coefficients of specified monomials: the coefficient of $x_1^{\lambda_1} \dots x_k^{\lambda_k}$ in f is $\langle f, h_\lambda \rangle$. To evaluate this inner product, expand both f and h_λ in power sum symmetric functions. Gessel's following resultshows how certain sums of coefficients are D-finite.

Theorem 3 ([1, Corollary 9]). *Let f be a D-finite symmetric function and S a finite set of integers. Define a sequence s_n to be the sum, for all n-tuples $(\lambda_1, \dots, \lambda_n)$ in S^n, of the coefficient of $x_1^{\lambda_1} \dots x_n^{\lambda_n}$ in f. Then $s(t) = \sum_n s_n t^n$ is D-finite.*

As an application of this method:

Theorem 4 ([1, Theorem 1]). *Define $\theta : \Lambda \to K[[x]]$ by $\theta(p_k) = \delta_{1,k}X$. Then for any symmetric function f,*

$$\theta(f) = \sum_n a_n \frac{X^n}{n!},$$

where a_n is the coefficient of $x_1 \dots x_n$ in f. In particular, $\theta(h_n) = X_n/n!$.

The speaker provided numerous applications to graph theory, Young tableaux, and suggested further applications to nonnegative integer matrices and to permutations with forbidden sequences. The applications to graph enumeration begin with the generating function

$$G = \prod_{i<j}(1 + x_ix_j) = e(e_2),$$

which is D-finite. The coefficient of $x_1^{\lambda_1}, \dots, x_n^{\lambda_n}$ in G is the number of graphs on n vertices with specified degrees $\lambda_1, \dots, \lambda_n$. By Theorem 3 above, the generating function for graphs on n vertices with certain specified classes of degree sequences are D-finite, settling a problem of Goulden and Jackson. For example, taking $S = \{1\}$ counts matchings, $S = \{2\}$ counts (disjoint unions of) circuits, $S = \{k\}$ counts k-regular graphs.

Since the coefficient of $x_1^{\lambda_1}, \dots, x_n^{\lambda_n}$ in the Schur function s_μ is the number of tableaux of shape μ and content λ, when λ is equal to 1^n this counts the number of standard tableaux of size n. Using the fact that $\sum_\lambda s_\lambda = h(e_1 + e_2)$, we have

Theorem 5. *The number y_n of standard tableaux with n entries is P-recursive.*

A similar approach yields:

Theorem 6. *Let $B_k = \sum_\lambda s_\lambda$, the sum over all partitions λ with at most k parts. Then B_k is D-finite.*

This leads us to the conclusion that the number $y_k(n)$ of standard tableaux with n entries and at most k rows is P-recursive.

The speaker presented a long and interesting list of suggestions for further work. She suggested extending the inner product to functions of several sets of variables, in order, for instance, to handle problems concerning directed graphs. She suggested the study of *morphisms* arising in the context of Theorem 3 above, and the definition of other D-finiteness-preserving morphisms.

Further: what are the q-analogues of D-finiteness? Do these concepts make sense in other symmetry classes of functions (skew-symmetric, quasi-symmetric)? When such morphisms exist, what are their corresponding differential equations?

We admire the speaker's skill and courage in undertaking new work in a field already harvested by Gessel, Stanley, and Zeilberger.

Bibliography

[1] Gessel (Ira M.). – Symmetric functions and P-recursiveness. *Journal of Combinatorial Theory, Series A*, vol. 53, n° 2, 1990, pp. 257–285.