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A Hyperasymptotic Approach of the Multi-Dimensional Saddle-Point Method

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Abstract

This summary is a presentation of the saddle-point method and of one application. It concludes with an attempt to give some intuition towards problems that can arise (for example the Stokes phenomenon), and gives references for further exploration.

1. Introduction

The content of this summary is only an introduction to the presentation of Éric Delabaere on multi-dimensional saddle-point method. We will only consider the one-dimensional problem on an example: the Airy function. For a general study of the problem, see [3]. The first section briefly presents the saddle-point method for oscillating integrals, the second section presents an application where the Airy function appears, and the last section shows the consequences of the Stokes phenomenon on the saddle-point method. Finally we give references to the resurgence theory which is a general (and difficult) approach for this type of problems.

2. The Saddle-Point Method

The saddle-point method is a way to get an approximate value of an integral by a good deformation of the integration contour. Consider an integral of the type

$$I(x) = \int g(z)e^{ixf(z)}\,dz. \tag{1}$$

We suppose that the integrand is analytic in some domain of the complex plane, so that a deformation of the integration contour is allowed. If we view the complex plane with a parameter height equal to $|e^{ixf(z)}|$ then it is convenient to talk about valleys and hills to describe the integrand. In order to get a good approximation for large x, it is interesting to keep the contour in the zones where the integrand is as small as possible. This is realized by staying in the valleys, except when going from one valley to another, where the contour should cross saddle points by using steepest-descent paths. These saddle points are characterized by $f'(z) = 0$, and are the points that contribute the most to the integral I. Suppose that f' has only one zero located at z_0 (the case with multiple zeros is very similar), and expand g up to order 0 and f up to order 2, then I is rewritten as (neglecting error terms):

$$I(x) = \int g(z_0)e^{ixf(z_0)}e^{ix\frac{(z-z_0)^2}{2}f''(z_0)}\,dz.$$

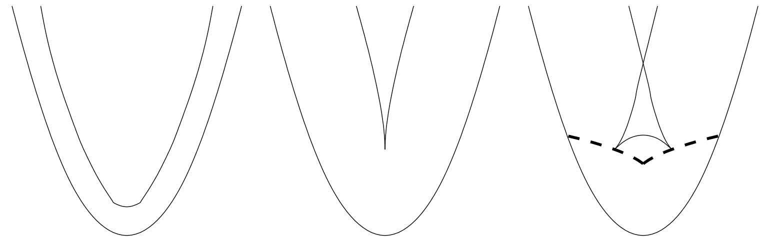

FIGURE 1. A parabolic light source, with the evolution of Σ_t.

The change of variable $u = (z - z_0)\left(\frac{xf''(z_0)}{2i}\right)^{1/2}$ gives the following approximation for I:

$$I(x) \sim g(z_0)e^{ixf(z_0)}\left(\frac{2\pi i}{xf''(z_0)}\right)^{1/2}.$$

When there are several saddle points (not too close), the result is obtained by adding the contributions of all of the saddle points.

3. **One Application in Optics**

In this section, we study an example from optics where an integral of type (1) occurs. This application is treated in [7, 8]. Consider a monochromatic source of light produced by a curve Σ. Following geometrical optic rules, the space is split into two distinct zones, one luminous and the other totally dark, but this model does not fit totally well with reality.

The Huygens principle describes light propagation in the space (filled with ether according to Huygens) by an analogy with the propagation of sound in the air, or waves on water. It says that each point of the light source is a punctual source that emits a spherical wave. The wave surface Σ_t at a time t is thus the envelope of the spherical wave surfaces of all the punctual sources. We easily deduce from the Huygens principle that Σ_t is the location of the points at distance ct of the source (in the proper direction), where c is the speed of light. An example is given for a parabolic luminous source in Figure 1. As t grows, the curve Σ_t starts to have cusps. The location of all these cusps is called the *caustic* (represented with a dotted line on Figure 1). Another way of observing the phenomenon is to trace all the normal lines to the luminous source, the caustic appears naturally as the accumulation of lines, or for a real luminous source, by an accumulation of light, see Figure 2.

On the caustic itself, there appears some interference fringes that cannot be explained by the sole Huygens principle. At the beginning of the 19th century, Fresnel completed the Huygens principle into the Huygens–Fresnel principle by adding an amplitude and and a phase to the wave, both depending on the position and on the time. So up to a multiplicative constant, the electro-magnetic amplitude $\Psi(p, t)$ is

$$\Psi(p,t) \propto \int_\Sigma \frac{e^{ikd(p,q)}}{d(p,q)}\,dq, \tag{2}$$

where k is the wave number, and $d(x, y)$ the distance between x and y. The luminous intensity is then proportional to the square of the electro-magnetic amplitude.

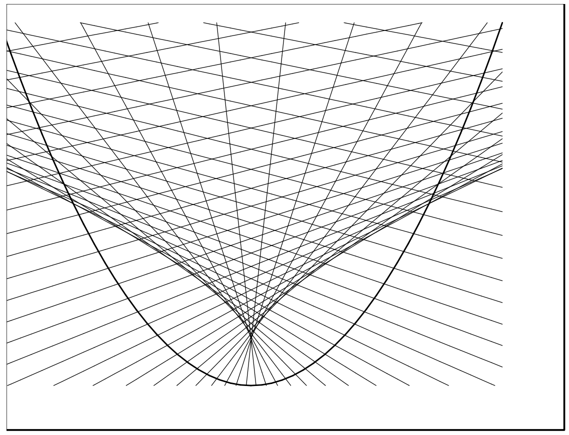

FIGURE 2. A parabolic source of light, and some of its normal rays.

For a point p located near the caustic and far enough from the source Σ, Equation (2) can be approximated by

$$\mathrm{Ai}\left(\left(\frac{2k^2}{\rho}\right)^{1/3} y\right),$$

with ρ the curving ray of the caustic, y the distance between p and the caustic, and Ai the Airy function defined by $\mathrm{Ai}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(z^3/3+wz)}\,dz$.

4. **The Stokes Phenomenon**

This section shows how the Stokes phenomenon appears on the example of the Airy function introduced in the previous section. A more detailed study can be found in [1].

The Airy function is approximated using the saddle-point method, that consists in choosing the integration path along the steepest descent lines of $\Re\big(i(z^3/3 + wz)\big)$, passing by the saddle points. This path depends on the value of w, and in fact only on the argument of w, so we assume that $|w| = 1$.

Figure 3 shows various integration paths represented by a thick line and oriented from left to right, depending on the argument of w. The saddle points and the lines of steepest descent are also drawn. The Stokes phenomenon can be seen on this figure. First when $\operatorname{Arg} w = \pi/3$, only one

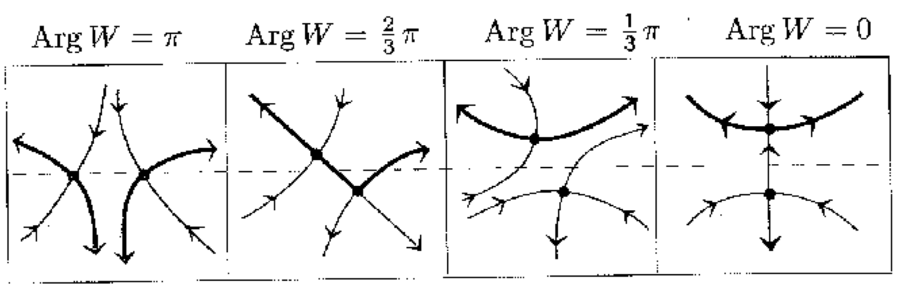

FIGURE 3. The integration path for various w

saddle point contributes to the asymptotic (as the integration path goes only through one saddle point). When the integration path goes through the second saddle point, for $\operatorname{Arg} w = 2\pi/3$, both saddle points compete for the asymptotic (this occurs when a line of steepest descent descending from a saddle point goes through another saddle point). The added contribution is negligible, so the asymptotic remains the same, but the Borel summability is lost. This is the Stokes phenomenon. And the line defined by $\operatorname{Arg} w = 2\pi/3$ is a Stokes line. Then when $\operatorname{Arg} w$ grows up to π, both saddle points contribute. The change of Borel summabillity is handled very well by the theory of resurgent functions due to Écalle [4, 5, 6] (see [2] for an introduction).

The resurgent point of view can be generalized to oscillating integrals of higher dimension, and has the interesting property of giving an exact coding of the integral by resurgent symbols, and not only an asymptotic expansion.

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