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Irrationality Measures of $\log 2$ and $\pi/\sqrt{3}$

Nicolas Brisebarre
Université de St-Étienne

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Summary by Bruno Salvy

Abstract

1. Irrationality Measures

An *irrationality measure* of $x \in \mathbb{R} \setminus \mathbb{Q}$ is a number μ such that

$$\forall \epsilon > 0, \exists C > 0, \forall (p,q) \in \mathbb{Z}^2, \left| x - \frac{p}{q} \right| \geq \frac{C}{q^{\mu+\epsilon}}.$$

This is a way to measure how well the number x can be approximated by rational numbers. The measure is *effective* when $C(\epsilon)$ is known. We denote $\inf \{ \mu \mid \mu \text{ is an irrationality measure of } x \}$ by $\mu(x)$, and we call it *the* irrationality measure of x.

By definition, rational numbers do not have an irrationality measure. Given two irrationality measures for a number, the smaller one is more precise, since it shows the number to be further "away" from rational numbers. For all $x \in \mathbb{R} \setminus \mathbb{Q}$, the inequality $\mu(x) \geq 2$ holds and gives the minimal possible value. This inequality follows from a pigeon-hole principle: for any integer $n > 1$, the fractional parts $\{qx\}$, $0 \leq q < n$ together with the number 1, are $n+1$ real numbers in the interval $[\,0,1\,]$; therefore two of them must be at distance less than or equal to $1/n$; their difference is of the form $qx - p$, so that $|x - p/q| < 1/nq < 1/q^2$. A more explicit construction of these rational approximations is given by continued fraction expansions. The periodicity of continued fraction expansions of irrational quadratic numbers implies that they have an (effective) measure equal to 2. This result was generalized by Liouville in 1844, when he obtained the first practical criterion for constructing transcendental numbers.

Theorem 1 (Liouville)**.** *An algebraic number α of degree n has effective irrationality measure n.*

Proof. Let P be the minimal polynomial of α. This is a polynomial of degree n with integer coefficients. By the mean value theorem,

$$P(\alpha) - P(p/q) = -P(p/q) = (\alpha - p/q)P'(\xi),$$

for some ξ between α and p/q. Since P is irreducible, $P(p/q) \neq 0$ and $\left|q^n P(p/q)\right|$ is an integer which is therefore at least 1. It is sufficient to restrict attention to p/q at distance less than 1 from α. Then $P'(\xi)$ has a lower bound on this interval and this proves the measure. The bound is made effective in terms of the *height* H of P (the largest absolute value of its coefficients), as $\left|P'(\xi)\right| < n^2 H\bigl(1 + |\alpha|\bigr)^{n-1}$. $\square$

Number	$\log 2$	π	$\pi/\sqrt{3}$	$\zeta(2)$	$\zeta(3)$
measure	3.8913997	8.016045	4.601579	5.441243	5.513891
author	Rukhadze (1987)	Hata (1993)		Rhin & Viola (1996)	

Table 1. Irrationality measures and their authors.

Using this result, Liouville constructed so-called *Liouville numbers* whose smallest measure is infinite. These numbers are therefore transcendental, since their measure cannot be bounded by any integer as demanded by the above theorem. A family of such numbers is

$$\sum_{n\geq 0} a^{-n!}, \qquad a \in \mathbb{N} \setminus \{0,1\}.$$

Indeed, truncating after the kth term gives a rational approximation p_k/q_k with $q_k = a^{k!}$ and a simple computation on the tail of the series shows that it is less than q_k^{-k}.

In the twentieth century, a sequence of results improved on Liouville's theorem, this was ended by Roth, who showed in 1955 that all algebraic numbers have irrationality measure exactly 2 (this result is not effective). In a different direction, Khintchine showed that almost all real numbers (in the sense of Lebesgue) have irrationality measure 2. However, not all reals have measure 2: apart from Liouville numbers, for every $\mu \in [2,\infty)$ the following gives a family of numbers with measure exactly μ:

$$[a] + \cfrac{1}{[a^b] + \cfrac{1}{[a^{b^2}] + \cfrac{1}{[a^{b^3}] + \dots}}}, \quad a > 1,\ b = \mu - 1,$$

where $[a]$ denotes the integer part of a.

2. Padé–Hermite Approximants

Very few actual values of the irrationality measure are known. Techniques have been developed to derive upper bounds for given numbers. A summary of the current best known upper bounds for a few constants is given in Table 1. Note that in each case, the mere existence of a bound is a proof of irrationality.

The basis for several of these bounds lies in sequences of approximants of the form

$$q_n x - p_n = \epsilon_n, \tag{1}$$

where p_n and q_n are integers. Then if q_n does not grow too fast with n, while ϵ_n tends to 0 fast enough, an effective irrationality measure can be found. More precisely, several lemmas of the following type are available.

Lemma 1 (G. V. Chudnovsky)**.** *If there exist positive real numbers* σ *and* τ *such that*

$$\limsup_{n\to\infty} \frac{\log q_n}{n} \leq \sigma, \qquad \lim_{n\to\infty} \frac{\log|\epsilon_n|}{n} = -\tau,$$

then $\mu = 1 + \sigma/\tau$ *is an effective measure of irrationality for* x.

An important tool to obtain approximants of type (1) is the use of more general *Padé–Hermite approximants*. (See the summary of Rivoal's talk in these proceedings for a similar use in transcendence theory.) In the case of $\log 2$ and $\pi/\sqrt{3}$, the approximants that will be considered are of the form

$$Q_n(z)\log(1-z) - P_n(z) = E_n(z),$$

where Q_n and P_n are polynomials while E_n is an analytic function. Setting $z=-1$ in this equation gives a relation involving $\log 2$, while setting $z = \exp(i\pi/3) = 1-\exp(-i\pi/3) = 1 + i\sqrt{3}/2$ gives a relation involving π and $\sqrt{3}$.

If Q is a polynomial in $\mathbb{Z}_n[t]$ (polynomials with integer coefficients and degree at most n), then

$$I(z) = \int_0^1 \frac{Q(t)}{1-zt}\,dt = -\frac{Q(1/z)}{z}\log(1-z) + P(1/z)/d_n,$$

where $P(t) \in \mathbb{Z}_n[t]$ and $d_n = \operatorname{lcm}(1,2,\dots,n)$. Now, the idea is to look for "good" families of polynomials Q_n in order to reach both a small σ and a large τ in the lemma.

In 1980, Alladi and Robinson [1] used $Q_n(z) = \bigl(z^n(1-z)^n\bigr)^{(n)}/n!$ (these are related to the Legendre polynomials). It is easily seen that $Q_n(z) \in \mathbb{Z}_n[z]$ with coefficients

$$\frac{(n+k)!}{k!^2(n-k)!}, \qquad k = 0,\dots,n$$

whose absolute value is asymptotically of order $\bigl(3+2\sqrt{2}\,\bigr)^n$ (the maximal coefficient is reached for $k \sim n/\sqrt{2}$). By repeated integration by parts one gets

$$I_n(z) = (-z)^n \int_0^1 \frac{t^n(1-t)^n}{(1-zt)^{n+1}}\,dt.$$

Now, for $z=-1$, the integral is easily bounded by considering the maximum of $t(1-t)/(1+t)$ in the interval $[\,0,1\,]$, which gives $\bigl(3-2\sqrt{2}\,\bigr)^n$. Finally, it is a classical result from number theory that $d_n \simeq e^n$. Putting all this together gives

$$\mu(\log 2) \le 1 - \frac{1+\log\bigl(3+2\sqrt{2}\,\bigr)}{1+\log\bigl(3-2\sqrt{2}\,\bigr)} \approx 4.622.$$

Similarly, they get $\mu\bigl(\pi/\sqrt{3}\,\bigr) \le 8.310$.

3. Better Polynomials

In 1987, G. Rhin [4] replaced the polynomials $t^n(1-t)^n$ in the integral I_n by polynomials with integer coefficients but giving the integrand a lower upper bound. Using the factors

$$X, 1-6X+X^2, 1-6X, 1-5X, 2-11X, 1-7X+2X^2$$

with linear exponents that are computed by an optimization process, he obtains $\mu(\log 2) \le 4.0765$ and $\mu(\pi/\sqrt{3}) \le 4.97$.

The following family of polynomials was considered by N. Brisebarre [3]. It generalizes the polynomials of Alladin & Robinson, but also more general families that had been used by M. Hata, E. A. Rukhadze and A. Dubitskas as well as D. V. and G. V. Chudnovsky to obtain the bounds for $\log 2$ and $\pi/\sqrt{3}$ in Table 1.

$$Q_{n,m,m'} = \frac{\bigl(z^{n+m'}(1-z)^{n+m}\bigr)^{(n+m+m')}}{(n+m+m')!} = \sum_{j=0}^{n} (-1)^{m+j}\binom{n+m}{m+j}\binom{n+m+m'+j}{n+m+m'} z^j.$$

One-parameter families are obtained by considering $Q_{an,bn,cn}$, with a,b,c integers, a being restricted to be positive. As shown by the formula above, these polynomials have integer coefficients. Moreover, it turns out that the *content* of these polynomials (the gcd of their coefficients) is quite large and can be exploited to some extent.

Proposition 1. *Let* c_n *be the content of* $Q_{an,bn,cn}$ *when* $a > -\min(b,c,b+c,0)$, *then*

$$e(b/a,c/a) := \lim_{n\to\infty} \frac{\log c_n}{n} = \int_{E_{a,b,c}} \frac{dx}{x^2},$$

where

$$E = \left\{ x \;\middle|\; x > 0,\ 0 < \{x\} + \left\{\frac{c}{a}x\right\} - 1 < 1 - \left\{\frac{b}{a}x\right\} < \{x\} \right\}.$$

The proof of this lemma consists in exhibiting sufficiently many intervals containing prime divisors of each of the coefficients of the polynomial, see [3]. The computation of the integral starts by slicing the interval $[\,0,1\,]$ in a finite number of subintervals, bounded by the rationals j/a, $j/|b|$, $j/|c|$, for $j \in \mathbb{N}$. On each subinterval, the value of the fractional parts in the definition of E are then studied in more detail, which leads to a more or less explicit formula for $e(b/a,c/a)$. For specific values of b and c, the formula becomes completely explicit, and for instance one recovers a few special cases due to Hata, like

$$e(a^{-1},a^{-1}) = \log\left(\frac{a+1}{(a+2)^{(a+2)/(2a+2)}a^{a/(2a+2)}}\right) + \frac{\pi}{2a+2}\big(\chi(a+2)-\chi(a)\big), \quad \chi(a) := \sum_{r=1}^{[a/2]} \cot(r\pi/a).$$

As before, the next steps consist in bounding the coefficients of $Q_{an,bn,cn}$ and the maximum of $Q_{a,b,c}(t)/(1+t)$ in the interval $[\,0,1\,]$ so as to get an irrationality measure. These are achieved without too much difficulty. The final result is in terms of $e(b/a,c/a)$ and one is left with an optimization problem in $\mathbb{R}^2$. Experiments show that the optimal result is reached at several values of (a,b,c), namely $(8,-1,-1)$, $(7,1,-1)$, $(6,1,1)$, and $(7,-1,1)$. The corresponding polynomials have been considered by Hata and Rukhadze, they lead to the bound from Table 1. Similar considerations apply for $\pi/\sqrt{3}$, see [3].

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