Algorithms Seminar 2001–2002,
F. Chyzak (ed.), INRIA, (2003), pp. 3–6.

Available online at the URL
http://algo.inria.fr/seminars/.

The Site Perimeter of Bargraphs

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May 13, 2002

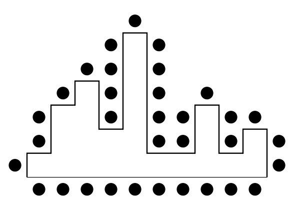
Summary by Sylvie Corteel

Abstract

The site perimeter enumeration of polyominoes that are both column- and row-convex is a well-understood problem that always yields algebraic generating functions. Counting more general families of polyominoes is a far more difficult problem. Here Mireille Bousquet-Mélou and Andrew Rechnitzer enumerate (by their site perimeter) the simplest family of polyominoes that are not fully convex—bargraphs. The generating function they obtain is of a type that has rarely been encountered so far in the combinatorics literature: a q-series into which an algebraic series has been substituted.

1. Introduction

A *polyomino* is a finite connected union of cells on a regular planar lattice (here the square lattice). The enumeration of polyominoes is a long-standing "elementary" combinatorial problem [1, 2, 3] that has some motivations in physics. The authors study the site perimeter of bargraphs. The *site perimeter* of a polyomino the number of nearest-neighbour vacant cells. A *bargraph* is a column-convex polyomino, such that its lower edge lies on the horizontal axis. It is uniquely defined by the heights of its columns. Here is a bargraph whose site perimeter is 33:



The site perimeter parameter is of considerable interest to physicists and probabilists. The enumeration of polyominoes according to their area and their site perimeter is equivalent to solve the problem of the site percolation.

This abstract is in four parts: functional equation, generating function, analysis of the generating function and application to chemistry (self-avoiding polymers). We will not give the details of the proofs. The article [1] can be found on Bousquet-Mélou's web page.

2. Functional Equation

Let $B(s; x, y, p)$ be the generating function of bargraphs enumerated by the height of the last column, the horizontal and vertical perimeter and the site perimeter.

Proposition 1. *The generating function of bargraphs satisfies the functional equation*

$$B(s) = a(s) + b(s)B(1) + c(s)B(sq) + d(s)B(s),$$

where $B(s)$ *denotes* $B(s; x, y, p)$, $q = py$, *and*

$$a(s) = \frac{xsqp^3}{1 - spq}, \qquad c(s) = \frac{x^2sqp^3(1-p)}{(1-q)(1-sq)(1-spq)},$$
$$b(s) = \frac{xsp\big((1-sq)(1-spq) + xs^2p^2q^2(1-p)\big)}{(1-s)(1-sq)(1-spq)},$$
$$d(s) = -\frac{xp\Big((1-q)(1-p)(1+s^2pq) + sp\big((1-q)(1+pq-2q) + xpq^2(1-p)\big)\Big)}{(1-q)(1-s)(1-sqp)}.$$

Let us briefly explain that equation. A bargraph:

- may consist of a single column, whence the term $xsqp^3/(1 - spq)$;
- may be *steady*. It can then be (uniquely) constructed by duplicating the last column of some bargraph, whence the term $xp^2B(s)$;
- may be *descending*. It can then be constructed by appending a shorter column to the right of some bargraph, whence the term $xp\big(sB(1) - B(s)\big)/(1 - s)$;
- may be *ascending*, and then there are two cases:
  - ∗ the ascending bargraph is construted from a *ascending* or a *steady* one:

$$xsqp^2\Big(B(s) - xp\big(sB(1) - B(s)\big)/(1-s)\Big)/(1 - spq);$$

  - ∗ the ascending bargraph is construted from a *descending* one:

$$\frac{x^2sp^3B(1)}{(1-s)(1-sq)} - \frac{x^2sp^3q(1-pq)B(s)}{(1-s)(1-q)(1-spq)} + \frac{x^2sp^3q(1-p)B(sq)}{(1-q)(1-sq)(1-spq)}.$$

3. Site Perimeter Generating Functions

In this section the functional equation of Proposition 1 is solved. The method combines two different techniques that have appeared previously in the combinatorics literature, but which have so far been applied independently. One of them is a simple iteration technique, which aims to "kill" the $B(sq)$ term. It was the key tool in [3]. The other one is the so-called *kernel method* which has been known since the 70's and is currently undergoing something of a revival.

Iteration. Let

$$\alpha(s) = \frac{a(s)}{1 - d(s)}, \quad \beta(s) = \frac{b(s)}{1 - d(s)}, \quad \gamma(s) = \frac{c(s)}{1 - d(s)}.$$

Then $B(s) = \alpha(s) + \beta(s)B(1) + \gamma(s)B(sq)$ and when one iterates, the result is:

(1) $$B(s) = \sum_{k\geq 0} \gamma(s) \dots \gamma(sp^{k-1})\big(\alpha(sp^k) + \beta(sp^k)B(1)\big).$$

The denominators of all the summands have a commun factor: $\eta(s) = (1 + p - p^2)(1 + s^2p^2) - s(1 + 2p^3 - p^4 - p^5)$. Moreover $1 - d(s) = \eta(s)/\big((1 - s)(1 - sp^2)\big)$.

Kernel. The equation (1) is multiplied with $(1-d(s))$ and $B(s)$ is eliminated by taking $s=\sigma$ with $\eta(\sigma)=0$:

$$\sigma(p)=\frac{1+2p^3-p^4-p^5-\sqrt{(1+2p^3-p^4-p^5)^2-4p^2(1+p-p^2)^2}}{2p^2(1+p-p^2)}.$$

Then

$$B(1)=-\frac{a(\sigma)+c(\sigma)+\sum_{k\geq 0}\gamma(\sigma)\dots\gamma(\sigma p^{k-1})\alpha(\sigma p^k)}{b(\sigma)+c(\sigma)+\sum_{k\geq 0}\gamma(\sigma)\dots\gamma(\sigma p^{k-1})\beta(\sigma p^k)}.$$

The result is:

Theorem 1. *Let b_n be the number of bargraphs with site perimeter n. Let $\sigma=\sigma(p)$ be the following algebraic power series in p:*

$$\sigma(p)=\frac{1+2p^3-p^4-p^5-\sqrt{(1+2p^3-p^4-p^5)^2-4p^2(1+p-p^2)^2}}{2p^2(1+p-p^2)}.$$

Then the site perimeter generating function of bargraphs is

$$\sum_{n\geq 0} b_n p^n=\frac{-p^3\displaystyle\sum_{n\geq 0}\frac{\sigma^n p^{\binom{n+5}{2}}}{(p)_n(\sigma^2p^3)_n(1+p-p^2)^n}}{\displaystyle\sum_{n\geq 0}\frac{\sigma^n p^{\binom{n+5}{2}}}{(p)_n(\sigma^2p^3)_n(1+p-p^2)^n}\frac{(1-\sigma p^{n+1})(1-\sigma p^{n+2})+\sigma^2p^{2n+4}(1-p)}{(1-\sigma p^n)(1-\sigma p^{n+1})}},$$

where we use $(a)_n$ to denote the product $(1-a)(1-ap)\dots(1-ap^{n-1})$.

4. Analysis of the Generating Function

In this section two aspects of the generating function of Theorem 1 are analysed: the asymptotic behaviour of the number of bargraphs with site perimeter n, and the nature of the width and site perimeter generating function.

Asymptotic behavior. The asymptotic behavior of the number of bargraphs with site perimeter n is determined by analysing the singularity structure of the generating function of Theorem 1. An examination of this series shows that the possible sources of singularities are:

- divergence of the summands in the numerator and denominator,
- divergence of the numerator or denominator,
- a singularity arising from the square-root in $\sigma(p)$,
- poles given by the zeros of the denominator.

It is in fact the case that the dominant singularity is a square-root singularity arising from the square-root singularity in $\sigma(p)$.

Theorem 2. *The number of bargraphs with site perimeter n grows asymptotically like $Cp_c^{-n}n^{-3/2}$ for some positive constant C, where $p_c=0.45002...$ is the smallest positive solution of*

$$1-2p-2p^2+4p^3-p^4-p^5=0.$$

Nature of the width and site permeter generating function. By iterating the functional equation, the coefficient of x^n in the bargraph generating function $B(1;x,1,p)$ is a rational function of p, whose denominator is a product of cyclotomic polynomials.[1] They suggest that a new cyclotomic polynomial factor appears in the denominator of every second coefficient of x, so that more and more singularities accumulate on the unit circle $|p| = 1$.

Proposition 2. *For $n \geq 2$, the coefficient of x^{2n-3} in the bargraph generating function $B(1;x,1,p)$ is a rational function of p that is singular at any primitive n-th root of unity.*

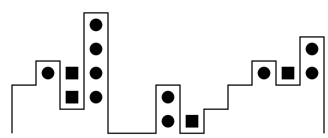
Such an accumulation of singularities indicates that the power series is not D-finite, so that:

Theorem 3. *The generating function $B(1;x,1,p)$ which counts bargraphs by width and site perimeter is not D-finite. Consequently, the series $B(1;x,y,p)$ and $B(s;x,y,p)$ are not D-finite either.*

Remark. The non-D-finitedness of $B(1;x,1,p)$ does not give any information about the nature of the power series $B(1;1,1,p)$. One can readily construct multivariate series that are not D-finite, whose specialisations are D-finite.

5. Self-Avoiding Polymers

Another model is the model of self-avoiding polymers. These polymers consist of walks above the horizontal axis that use North, East, and South steps. For this model, two parameters are important: the number of contacts with the horizontal axis (East step of height 0) and the number of interactions (circles and squares on the figure). Here is a polymer of length 41 with 3 contacts and 14 interactions.



Let $Z(t,w,q)$ be generating function of the walks enumerated by the length, the number of contacts and the number of interactions. It is conjectured that the phase diagram with coordinates q and w has three phases: confined, collapsed, and free.

By adding a North step at the beginning of the walk and a South step at the end, one gets a bargraph. The study of the parameters are possible if the interactions are separated into two classesé: internal (circles) and external (squares). For example the polymer on the figure has 9 internal interactions and 5 external interactions. It is possible to enumerate these bargraphs according to their length, contacts and internal or external interactions with the same techniques as before. For the external interactions, the generating function can be calculated explicitely and the authors can show that there is no collapsed phase. For the external interactions, the generating function can be calculated explicitely and is algebraic. Is it possible to obtain the generating function of these walks enumerated by internal and external interactions?

Bibliography

[1] Bousquet-Mélou (M.) and Rechnitzer (A.). – The site-perimeter of bargraphs. *Advances in Applied Mathematics*, vol. 31, n° 1, 2003, pp. 86–112.

[2] Bousquet-Mélou (Mireille). – *Combinatoire énumérative.* – Habilitation à diriger des recherches, LaBRI, Université Bordeaux 1, December 1996. 89 pages.

[3] Bousquet-Mélou (Mireille). – A method for the enumeration of various classes of column-convex polygons. *Discrete Mathematics*, vol. 154, n° 1-3, 1996, pp. 1–25.

[1]The cyclotomic polynomials $\Psi_d(x)$ are the factors of $(1-x^n)$, for $n \geq 1$. More precisely, $(1-x^n) = \prod_{d|n} \Psi_d(x)$.