Algorithms Seminar 2001–2002,
F. Chyzak (ed.), INRIA, (2003), p. 95.

Available online at the URL
http://algo.inria.fr/seminars/.

L-Series of Squares of Squares

Jonathan Borwein
Department of Mathematics and Statistics, SFU (Canada)

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The book by Hardy and Wright records elegant closed forms for the Dirichlet generating functions of the divisor functions $\sigma_k(n) = \sum_{d|n} d^k$ and $\sigma_k^2(n)$ in terms of the Riemann zeta function $\zeta(s)$. This talk extends such evaluations by providing a general identity for Dirichlet convolutions of completely multiplicative sequences. If f_1, f_2, g_1, and g_2 are completely multiplicative, then the formula

$$\sum_{n=1}^{\infty}(f_1 * g_1)(n) \times (f_2 * g_2)(n) \times n^{-s} = L_{f_1 f_2 g_1 g_2}(2s)^{-1} L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)$$

holds, where for a sequence f,

$$L_f(s) = \sum_{n=1}^{\infty} f(n) n^{-s}.$$

Applications are given to the number of representations of integers as sums of squares. Let $r_N(n)$ be the number of integer solutions of $x_1^2 + \cdots + x_N^2 = n$ and $r_{2,P}(n)$ be the number of integer solutions of $x^2 + Py^2 = n$. Closed forms in terms of $\zeta(s)$ and Dirichlet L-functions are obtained for the generating functions of $r_N(n)$, $r_N(n)^2$, $r_{2,P}(n)$, and $r_{2,P}(n)^2$ and certain N and P.

The talk is based on joint work with Stephen Choi. See CECM Preprint 01:167, which can be obtained at `http://www.cecm.sfu.ca/preprints`.