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On Jackson's q-Bessel Functions

Changgui Zhang
Université de La Rochelle (France)

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Summary by Bruno Salvy

Abstract

An analytic study of linear q-difference equations leads to a simple derivation of some connection formulae, generalizing the asymptotic expansion of the Bessel J_ν functions.

1. Differential and q-Difference Equations

Linear differential operators are polynomials in x and $\partial_x = d/dx$. These operators can be discretized using q-difference operators expressed in terms of q, x, and σ_q where $\sigma_q(f)(x) := f(qx)$. When $q \to 1$, $(\sigma_q - 1)(f)(x)/(q-1)$ tends to $xf'(x)$. This discretization is not unique. It gives rise to several generalizations of classical functions and identities relating them. C. Zhang's work is an *analytic* study of these operators, of the asymptotics of their solutions and the divergence of their series expansions.

A simple example of a q-difference equation is given by $(x\sigma_q - 1)y(x) = 0$. For $|q| < 1$ and $x \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, a solution of this equation is the Jacobi function

$$\theta_q(x) := \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n = (q;q)_\infty (-x;q)_\infty (-q/x;q)_\infty$$

where the last equality is *Jacobi's triple product identity*, using the notation

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots .$$

The product form shows that $\theta_q(x)$ is analytic in $\mathbb{C}^*$, and that its set of zeroes is $-q^{\mathbb{Z}}$.

Another important solution of the same equation is $e_q(x) := q^{\log_q x(\log_q x - 1)/2}$, equivalent to $\theta_q(x)$ when $x \to 0$. In the asymptotic behaviour of solutions in the neighbourhood of irregular singular points, the function e_q plays the same role as the exponential in the differential case. Another simple equation is $(\sigma_q - x)y(x) = 0$, which has $q^{-\log_q x(\log_q x - 1)/2}$ and $1/\theta_q(x)$ as solutions. As opposed to the differential case, inverses of these analogues of the exponential are not obtained by changing x into $-x$.

A complete classification of the possible formal local behaviours of solutions of linear q-difference equations was obtained by Carmichael in 1912. For an equation of order m in σ_q with analytic coefficients at the origin, there exists a family of m formal solutions, each of which is of the form

$$y_j(x) = x^{r_j} e_q^{-k_j}(x) \sum_{\nu=0}^{m_j - 1} (\log_q x)^\nu f_{j,\nu}(x), \qquad j = 1, \ldots, m, \tag{1}$$

where $r_j \in \mathbb{C}$, $k_j \in \mathbb{Q}$, $m_j \in \mathbb{N}^*$, and $f_{j,\nu}(x) \in \mathbb{C}[[x^{1/d}]]$ for some $d \in \mathbb{N}^*$. Each of these can be computed from the equation.

2. Hypergeometric and q-Hypergeometric Connection Formulae

The connection problem lies in expressing (the analytic continuation of) one of the above y_j's that are defined at the origin as a linear combination in terms of a similar basis at another singular point. There is no general method to compute "closed forms" for these constants, except in special cases such as the hypergeometric case.

Hypergeometric series in the classical (differential) case are series $F(x) = \sum_{n\geq 0} a(n)x^n$ such that $a(n+1)/a(n) =: r(n) = P(n)/Q(n)$ is a fixed rational function in n. In terms of the shift operator S_n this means that the sequence $a(n)$ cancels $Q(n) - P(n)S_n^{-1}$ from which it follows that the generating series F cancels the linear differential operator $Q(x\partial_x) - P(x\partial_x)x$. Introducing the roots of P and Q, hypergeometric series are classicaly denoted

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\middle|\, x\right) := \sum_{n\geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}, $$

where $(a)_n = a(a+1)\cdots(a+n-1)$. This series is convergent for $q > p$ and has only regular singularities if and only if $p = q+1$.

The q-analogue of this function is known as the ${}_r\phi_s$ *basic hypergeometric series.* In this case the ratio of two consecutive coefficients is a fixed rational function in q^n. The general form is

$$ {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x\right) := \sum_{n\geq 0} \frac{(a_1;q)_n \cdots (a_p;q)_n}{(b_1;q)_n \cdots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r} x^n, $$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$.

A simple example is Heine's ${}_2\phi_1(a,b;c;q,x)$, which has Gauss's ${}_2F_1(\alpha,\beta;\gamma;x)$ as a limit when $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, and $q \to 1$. Heine's function satisfies a second-order q-difference equation. This equation has no irregular singularity (it is a *Fuchsian* equation). A general technique to relate solutions of such equations at 0 and infinity in the classical hypergeometric case is based on a Mellin–Barnes integral representation. This approach was extended to the q-difference case by Watson in 1910, who found that for $ab \neq 0$,

$$ {}_2\phi_1(a,b;c;q,x) = C_1(x)\, {}_2\phi_1(a, aq/c; aq/b; q, cq/abx) + C_2(x)\, {}_2\phi_1(b, bq/c; bq/a; q, cq/abx), \tag{2} $$

where

$$ C_1(x) = \frac{(b, c/a; q)_\infty (ax, q/ax; q)_\infty}{(c, b/a; q)_\infty (x, q/x; q)_\infty}, \qquad C_2(x) = \frac{(a, c/b; q)_\infty (bx, q/bx; q)_\infty}{(c, a/b; q)_\infty (x, q/x; q)_\infty}. $$

This method is presented in detail in Slater's book [4]. The connection "constants" $C_1(x)$ and $C_2(x)$ are annihilated by $\sigma_q - 1$ and are uniform (they satisfy $C_k(xe^{2i\pi}) = C_k(x)$). Thus they are *elliptic*, since when expressed in (u,τ) defined by $x = \exp(2i\pi u)$ and $q = \exp(-2i\pi\tau)$ with $\Im(\tau) > 0$ they are doubly periodic.

3. Jackson's q-Bessel Functions

Bessel functions are classically defined as solutions of the Bessel equation

$$ \left((x\partial_x - \nu)(x\partial_x + \nu) + x^2\right) y(x) = 0. $$

When $\nu \notin \mathbb{Z}$, a basis of solutions is given by the Bessel $J_\nu(x)$ and $J_{-\nu}(x)$ functions, which can be expressed in terms of the hypergeometric series by

$$ J_{\pm\nu}(x) = \frac{(x/2)^{\pm\nu}}{\Gamma(\pm\nu + 1)} {}_2F_1(1, 1; \pm\nu + 1; -x^2/4). $$

The Bessel equation can be derived from the differential equation of the ${}_2F_1$ by *confluence*: this is achieved by considering ${}_2F_1(\nu+1/2,\beta;2\nu+1;x/\beta)$ and letting β tend to infinity. In this process, the singularity at infinity becomes irregular.

Similarly, Jackson introduced in 1905 two q-analogues of the Bessel functions,

$$
\begin{aligned}
J_\nu^{(1)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_2\phi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right),\\
J_\nu^{(2)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(;q^{\nu+1};q,-\frac{x^2q^{\nu+1}}{4}\right).
\end{aligned}
\tag{3}
$$

The classical J_ν function is recovered in two ways by letting q tend to 1 in $J_\nu^{(k)}\big(x(1-q);q\big)$ for $k\in\{1,2\}$. The radiuses of convergence of the basic hypergeometric series (in q) given here are respectively finite for $J_\nu^{(1)}$ (provided $|x|<2$) and infinite for $J_\nu^{(2)}$.

These functions are solutions of two q-difference equations of order 2 in σ_p with $p=\sqrt{q}$ that are easily derived from (3). These equations can be seen as arising from the equation of the ${}_2\phi_1$ by confluence, but it is not clear how to use this process in order to obtain a connection formula by a limiting process from (2). As in the classical case, both $J_\nu^{(k)}$ and $J_{-\nu}^{(k)}$ are independent solutions of their respective q-difference equation, for $k=1,2$. The equations have a regular singularity at the origin and an irregular singularity at infinity.

4. **Derivation of Connection Formulae**

Connection formulae between the series expansions (3) and the (unique) basis of formal solutions at infinity of the form given by (1) generalize the classical asymptotic expansion

$$
J_\nu(x) = \frac{e^{-i\frac{\pi}{4}(2\nu+1)}}{\sqrt{2\pi x}}e^{ix}{}_2F_0\left(-\nu+\frac12,\nu+\frac12;;\frac{2i}{x}\right) + \frac{e^{i\frac{\pi}{4}(2\nu+1)}}{\sqrt{2\pi x}}e^{-ix}{}_2F_0\left(-\nu+\frac12,\nu+\frac12;;-\frac{2i}{x}\right).
\tag{4}
$$

(A nice application of this formula is the derivation of an asymptotic expansion of the location of the zeroes of $J_\nu(x)$; this generalizes to those of $J_\nu^{(2)}$.)

We start with $J_\nu^{(1)}$ and its q-difference equation

$$\left(\sigma_p^2-(p^\nu+p^{-\nu})\sigma_p+(1+x^2/4)\right)y(x)=0.$$

By changing x into $1/t$ and $y(x)$ into $z(1/t)$, the equation becomes

$$\left(\left(1+\frac{1}{4p^4t^2}\right)\sigma_p^2-(p^\nu+p^{-\nu})\sigma_p+1\right)z(t)=0.$$

The exponential part of the behaviour (see Eq. (1)) is sought in terms of $\mathcal{E}_\alpha(t)=1/\theta_p(-\alpha t)$, which is cancelled by $\sigma_p+\alpha t$. The change of unknown function $z(t)=\mathcal{E}_\alpha(t)f_\alpha(t)$ leads to

$$\left(\left(1+\frac{1}{4p^4t^2}\right)\alpha^2pt^2\sigma_p^2-\alpha t(p^\nu+p^{-\nu})\sigma_p-1\right)f_\alpha(t)=0.$$

Thus, by choosing α such that $\alpha^2=4p^3$, one gets an equation for f_α which has power series solutions. A further simplification is achieved by considering the "p-Borel transform" of the series f_α:

$$g_\alpha(\tau):=\mathcal{B}_pf_\alpha(\tau)=\sum_{n\ge0}a_np^{-n(n-1)/2}\tau^n, \tag{5}$$

where a_n are the coefficients of f_α. By the commutation rule $\mathcal{B}_p(t^m \sigma_p^\ell) = p^{-m(m-1)/2} \tau^m \sigma_p^{\ell - m} \mathcal{B}_p$, g_α is solution of a two-term q-difference equation. This is easily solved to find

$$g_\alpha(\tau) = \frac{1}{(-\alpha p^\nu \tau; q)_\infty (-\alpha p^{-\nu} \tau; q)_\infty}.$$

It follows that g_α is meromorphic in $\mathbb{C}$ with (simple) poles at $\{-p^{\nu-2n}/\alpha, -p^{-\nu-2n}/\alpha\}$ for $n \in \mathbb{N}$, which implies that f_α is an entire function.

In order to recover f_α from g_α, the p-Borel transform of (5) is reverted by means of a Hadamard product of g_α with θ_p. This leads to a Cauchy integral representation from which a residue computation yields the connection formula. The Cauchy integral is

$$f_\alpha(t) = \frac{1}{2\pi i} \int_{|\tau|=r} g_\alpha(\tau) \theta_p(t/\tau) \frac{d\tau}{\tau},$$

where $r < \min\bigl(|p^\nu/\alpha|, |p^{-\nu}/\alpha|\bigr)$. The only residues come from the poles of g_α. The asymptotic behaviour of g_α implies that this integral is equal to the sum of the residues and an actual computation of these residues leads to

$$f_\alpha(t) = \frac{\theta_p(-\alpha q^{\nu/2} t)}{(q;q)_\infty (q^{-\nu};q)_\infty} {}_2\phi_1(0,0;q^{\nu+1};q,-x^2/4) + \frac{\theta_p(-\alpha q^{-\nu/2} t)}{(q;q)_\infty (q^{\nu};q)_\infty} {}_2\phi_1(0,0;q^{-\nu+1};q,-x^2/4),$$

where $xt = 1$ and $|x| < 2$. With very little rewriting, this is the desired connection formula. The limiting behaviour of this formula when $q \to 1$ is studied in [5].

The second family of q-Bessel functions is actually related to the first one by a relation discovered by Hahn in 1949:

$$J_\nu^{(2)}(x;q) = (-x^2/4;q)_\infty J_\nu^{(1)}(x;q), \qquad |x| < 2.$$

Another way of viewing the relation between these functions is through the p-Laplace transform that sends x^n to $p^{n(n-1)/2} x^n$. Then the transform of the ${}_2\phi_1$ in the definition of $J_\nu^{(1)}$ is the ${}_0\phi_1$ in that of $J_\nu^{(2)}$. From there, a Cauchy integral representation follows and again a residue computation gives the connection formula, thanks to extra considerations about the asymptotic behaviour of the integrand.

5. Comments

It has been observed that the connection "constants" possess the nice property that they are elliptic in the case of Heine's function. This is a general phenomenon [3]. The formulae in the q-world imply important identities (after all, Jacobi's triple product can be seen as a connection formula). Recent work by Changgui Zhang shows that the limiting behaviour of these q-connection formulae when $q \to 1$ yields the Stokes phenomenon of the differential world.

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