

## On the Group of a Sandpile

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### Abstract

The abelian sandpile model is a cellular automaton. Its rules generalize the sandpile rules for general graphs. This model has been introduced by Bak, Tang, and Wiesenfeld [1] in 1987. Dhar [9] showed that the set of recurrent configurations of this automaton has the structure of a finite abelian group.

In this talk, we describe several algorithms to determine the identity in the group. This element presents fractal aspects that we are not able yet to explain. These algorithms allow us to introduce relationships between the sandpile group and well-known algebraic or combinatorial objects.

Details may be found in the recent works of R. Cori, D. Rossin, and B. Salvy [6], and D. Rossin [12]. The papers [1, 10], the book [2], and the thesis [13] are good introductions to sandpiles.

### 1. Introduction

Let  $G = (V, E)$  be a non-oriented and connected *multi-graph* with  $V = \{1, \dots, n\}$  its set of vertices and  $E$  a symmetric  $n \times n$  matrix whose entry  $e_{i,j}$  is the number of edges with endpoints  $i, j$ . It is assumed that for any  $i$ ,  $e_{i,i} = 0$  so that the multi-graph has no loops. Frequently,  $G$  is a graph, and hence  $e_{i,j}$  is either 0 or 1. The *degree* of vertex  $i$  in  $G$  is  $d_i = \sum_{j=1}^n e_{i,j}$ . A multi-graph is *rooted* if one of its vertices is distinguished, it is called the *sink* and is numbered  $n$ .

A *configuration*  $u = (u_1, \dots, u_n) \in \mathbb{N}^n$  of  $G$  is a vector of non-negative integers. In the context of the sandpile model, the vertices of the graph are cells, and the number  $u_i$  may be interpreted as the height of a pile of grains of sand standing in cell  $i$ . In the rest of this talk, the number of grains in the sink is not taken into account. Thus two configurations  $u$  and  $v$  which differ only in position  $n$  are considered as equal; we write  $u = v$  if  $u_i = v_i$  for all  $1 \leq i < n$ . This translates the fact that the sink collects all grains of sand getting out of the system.

A *toppling* of the vertex  $i$ ,  $1 \leq i < n$ , in configuration  $u$  consists in decreasing the number of grains in this vertex by its degree while the number of those of each of its neighbours  $j$  increases by  $e_{i,j}$ . This is equivalent to the addition to  $u$  of the vector  $\Delta_i$  such that  $(\Delta_i)_i = -d_i$  and  $(\Delta_i)_j = e_{i,j}$  for  $j \neq i$ . The notation  $u \rightarrow v$  means that  $v$  is obtained from  $u$  by *toppling* a vertex, so that there exists an  $1 \leq i < n$  such that  $v = u + \Delta_i$ . The transitive closure of the toppling operation  $\rightarrow$  is denoted  $\xrightarrow{*}$ :  $u \xrightarrow{*} v$  if  $v$  is obtained from  $u$  by a sequence of topplings. An *avalanche* is a sequence of topplings (see Figures 1, 2 and 3).

The sandpile model has been introduced by Bak, Tang, and Wiesenfeld [1] in 1987. In a recent book, Bak [2] gives an overview of many physical problems—earthquakes and solar flares for

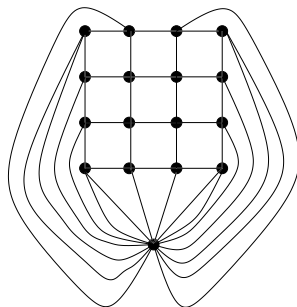


FIGURE 1. Multi-graph corresponding to the  $4 \times 4$  grid.

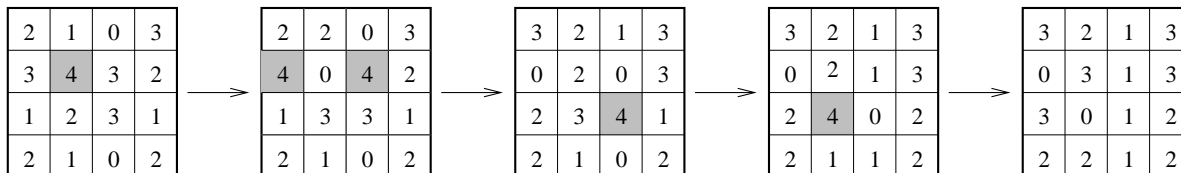


FIGURE 2. Topplings and avalanche on the  $4 \times 4$  grid.

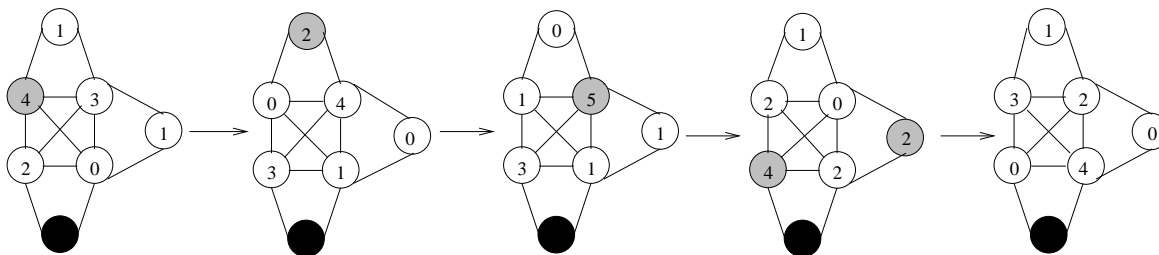


FIGURE 3. Topplings and avalanche on a graph.

example—whose models are based on the sandpile one. All these models follow the Gutenberg–Richter law:  $\log N = a - bM$ ,  $\log E = c + dM$ , and  $N \sim E^{1-\tau}$  ( $\tau \approx 2$ ) where  $M$  is the magnitude,  $N$  is the number of topplings, and  $E$  is the energy. In three dimensions,  $N \sim E^{1-\tau}$  ( $\tau \approx 2.5$ ). A very similar automaton was introduced independently by other authors under the name of the chip-firing game [4, 11]. Biggs [3] found many algebraic and combinatorial properties of the chip-firing game, some of which correspond to Dhar’s results on sandpiles [10]. In [5], we also showed a close relationship between recurrent configurations of the complete graph and the parking functions.

## 2. The Sandpile Group

A vertex is *stable* if it contains a number of grains less than its degree, otherwise this vertex is *unstable*. A *stable configuration* is a configuration where all vertices are stable. It is not difficult to prove that for every configuration  $u$  there exists a stable configuration  $\hat{u}$  such that  $u \xrightarrow{*} \hat{u}$ . Moreover this configuration is unique, and the number of topplings is independent of the way in which  $\hat{u}$  is obtained from  $u$  [9].

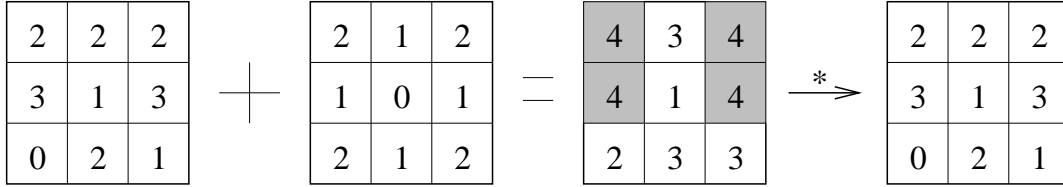


FIGURE 4. A recurrent configuration.

Let  $u, v$  be two configurations. Let  $u_i$  (resp.  $v_i$ ) be the number of grains on vertex  $i$  in configuration  $u$  (resp.  $v$ ). We will denote by  $u+v$  the configuration  $w$  such that  $w_i = u_i + v_i$ . A configuration  $u$  is *recurrent* if it is stable and if there exists a configuration  $v \neq 0$  such that  $u + v \xrightarrow{*} u$  (see Figure 4). The simplest example of a recurrent configuration is  $\delta = (d_1 - 1, d_2 - 1, \dots, d_{n-1} - 1, 0)$ . The set of recurrent configurations is isomorphic to the set of equivalence classes defined by the symmetric closure  $\equiv$  of  $\xrightarrow{*}$ .

Let  $T_G(x, y)$  be the Tutte polynomial of the graph  $G$ . Then  $T_G(1, y)$  is the the generating function (a polynomial) of the recurrent configurations according to the number of sand grains.

We can associate to the set of recurrent configurations the operator  $\oplus$  defined by  $u \oplus v = \widehat{u + v}$  where  $u$  and  $v$  are two recurrent configurations. The set of recurrent configurations with the operation  $\oplus$  is an abelian group  $\mathcal{G}$  [8], this group is equal to the product  $\mathcal{G} = \prod_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$  and the group structure does not depend on the sink choice in the graph  $G$ .

Let  $u = (u_1, \dots, u_n)$  be a recurrent configuration. We denote  $\bar{u}$  the recurrent configuration  $(d_1 - 1 - u_1, d_2 - 1 - u_2, \dots, d_{n-1} - 1 - u_{n-1}, 0)$ . Then the identity of the sandpile group is  $\text{Id} = \delta \oplus (\bar{\delta} \oplus \delta)$  and the opposite of a recurrent configuration  $u$  is  $\ominus u = \text{Id} \oplus (\bar{\delta} \oplus u)$ .

### 3. Toppling Ideal, Set Topplings and Minimal Gröbner Basis

Configurations and topplings are easily translated from the linear algebra setting into polynomial operations by associating to a configuration  $u = (u_1, u_2, \dots, u_n) \in \mathbb{N}^n$  a monomial  $x_u = x_1^{u_1} x_2^{u_2} \dots x_n^{u_n} \in \mathbb{Q}[x_1, \dots, x_n]$ . To a toppling  $\Delta_i$  is associated the binomial  $T(x_i) = x_i^{d_i} - \prod_j x_j^{e_{i,j}}$ . The addition of two configurations translates into the multiplication of the corresponding monomials and toppling vertex  $i$  in  $u$  translates into the division of  $x_u$  by  $x_i^{d_i}$  followed by the multiplication by  $\prod_{j=1}^n x_j^{e_{i,j}}$ . We define the *toppling ideal*  $\mathcal{I}_G$  as the ideal generated by  $x_n - 1$  and the toppling polynomials  $T(x_i)$  for  $i \in \{1, \dots, n\}$ .

A toppling polynomial can also be associated to a subset  $X$  of the set  $V$  of vertices as follows. For a vertex  $i$  of  $V$ , define

$$d_i(X) = \sum_{j \in X} e_{i,j},$$

the number of edges with endpoints  $i$  and a vertex of  $X$ . The *set toppling* of the set  $X$  in configuration  $u$  consists in adding the vector  $\Delta_X$  to  $u$ , where

$$(\Delta_X)_i = \begin{cases} -d_i(\bar{X}), & \text{for } i \in X, \\ d_i(X), & \text{for } i \in \bar{X}, \end{cases}$$

where  $\bar{X}$  denotes  $V \setminus X$ .

Accordingly, the *toppling polynomial* of the subset  $X$  of  $V$  is defined by

$$T(X) = \prod_{i \in X} x_i^{d_i(\bar{X})} - \prod_{i \in \bar{X}} x_i^{d_i(X)}.$$

Gröbner bases are a classical computational tool for dealing with polynomial ideals. Given an ordering on monomials which is compatible with the product (a so-called *admissible* ordering) and a set of generators of an ideal  $\mathcal{I}$ , one can compute a Gröbner basis for  $\mathcal{I}$  and from there test ideal membership and more generally compute normal forms in the quotient of the algebra by  $\mathcal{I}$ . The rest of this work makes use of the notation and basic results from [7, Chapter 2].

The *graded reverse lexicographic order* (grevlex) denoted  $\prec$ , is defined as follows. If  $A = \prod_{i=1}^n x_i^{\alpha_i}$  and  $B = \prod_{i=1}^n x_i^{\beta_i}$  are two monomials in the variables  $x_i$ ,  $i = 1, 2, \dots, n$ , then  $A \prec B$  if

$$|\alpha| = \sum_{i=1}^n \alpha_i < |\beta| = \sum_{i=1}^n \beta_i$$

or  $|\alpha| = |\beta|$  and in  $(\alpha_1, \dots, \alpha_n) - (\beta_1, \dots, \beta_n)$  the right-most non-zero entry is positive.

From there a *toppling order* is defined as follows: let  $\sigma$  be a permutation of  $\{1, \dots, n\}$  such that  $\sigma(n) = n$  and if the distance from vertex  $i$  to the sink is larger than the distance from vertex  $j$  to the sink, then  $\sigma(i) > \sigma(j)$ . The toppling order is the graded reverse lexicographic order on  $x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}$ .

**Theorem 1.** *A Gröbner basis of the ideal  $\mathcal{I}_G$  with respect to a toppling order is given by*

$$\mathcal{T} = \{ T(X) \mid X \subset \{1, \dots, n\} \} \cup \{x_n - 1\}.$$

A Gröbner basis is *minimal* when its elements have leading coefficient 1 and no leading monomial divides another leading monomial in the basis. A subset  $X$  of vertices of the graph  $G = (V, E)$  is well-connected if both subgraphs of  $G$  induced by  $X$  and  $\bar{X}$  are connected.

**Theorem 2.** *The set  $S_c$  of toppling polynomials corresponding to the sets  $X \subset \{1, 2, \dots, n-1\}$  which are well-connected is a minimal Gröbner basis for the toppling order.*

In the worst case, the minimal Gröbner basis still contains  $2^{n-1}$  elements for the complete graph.

As mentioned before, the quotient  $\mathbb{Q}[x_1, x_2, \dots, x_n]/\mathcal{I}_G$  is a  $\mathbb{Q}$ -vector space whose dimension is the order of the group of recurrent configurations. From a Gröbner basis for  $\mathcal{I}_G$ , a basis of this vector space is given by the set of monomials that do not reduce to 0 by the basis. We call these *reduced* monomials. Theorem 3 below gives a simple bijection between reduced monomials for the toppling order and recurrent configurations.

Let  $\Phi$  be the mapping from the set of stable configurations onto itself given by  $\Phi(u) = \delta - u$ . We also denote  $\Phi(M) = \Phi(a_1, a_2, \dots, a_n)$  for a monomial  $M = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ .

**Theorem 3.** *The mapping  $\Phi$  defines a bijection between the set of reduced monomials with respect to the toppling order and the set of recurrent configurations.*

For a configuration  $u$ , let  $\rho(u)$  denote the reduced configuration obtained from the monomial associated to  $u$  by performing reductions by the Gröbner basis of  $\mathcal{I}_G$  associated with the toppling order.

**Proposition 1.** *If  $u$  is a configuration then the recurrent configuration equivalent to  $u$  is*

$$\Phi\left(\rho\left(\Phi(\rho(u))\right)\right).$$

*The identity in the group of recurrent configurations is  $\Phi(\rho(\delta))$ .*

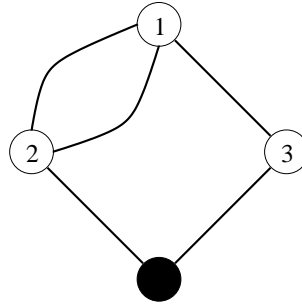


FIGURE 5. Multigraph with 4 vertices.

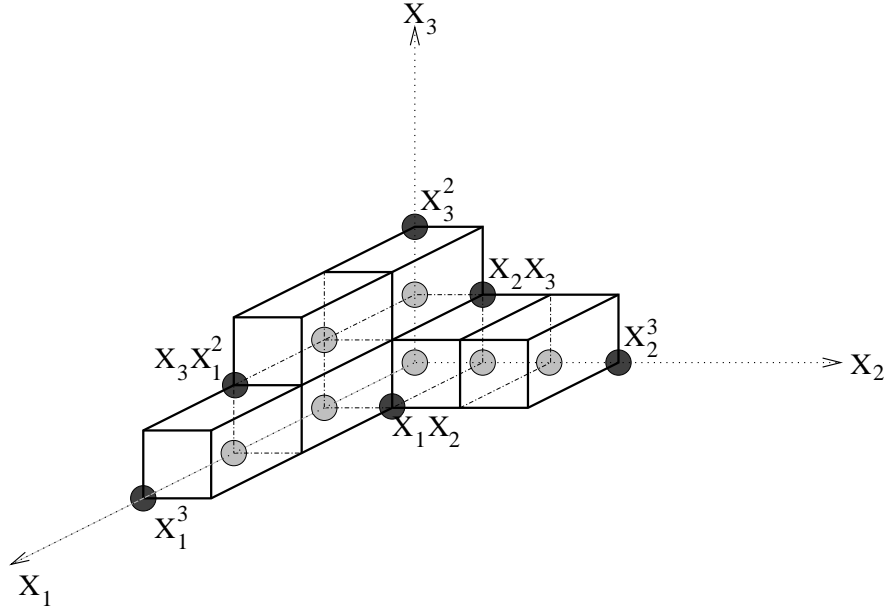


FIGURE 6. Representation of irreducible monomials.

**Corollary 1.** For two recurrent configurations  $u$  and  $v$ ,

$$u \oplus v = \Phi\left(\rho(\Phi(u) + \Phi(v))\right).$$

Proposition 1 yields the following algorithm to compute the identity on a graph  $G$  with sink  $s$ : beginning with the configuration  $\delta$ , perform the set topplings for all well-connected subgraph of  $G \setminus \{s\}$  (this is equivalent to reducing by the Gröbner basis for the toppling order). When no further set toppling can be performed, for each cell  $i$  replace its number of grains  $n_i$  with  $d_i - n_i$ . The resulting configuration is the identity.

#### 4. Examples

Our first example corresponds to the graph displayed on Figure 5. The structure of the graph is reflected by the toppling polynomials for the vertices:

$$x_1^3 - x_2^2x_3, \quad x_2^3 - x_1^2x_4, \quad x_3^2 - x_1x_4, \quad x_4^2 - x_2x_3, \quad x_4 - 1.$$

The minimal Gröbner basis for the graded reverse lexicographic order on monomials is

$$x_3^2 - x_1, \quad x_2^3 - x_1^2, \quad x_1^3 - x_2, \quad x_2x_3 - 1, \quad x_2x_1 - x_3, \quad x_3x_1^2 - x_2^2, \quad x_4 - 1.$$

Apart from the last, these polynomials correspond respectively to well-connected subgraphs with vertices

$$\{3\}, \quad \{2\}, \quad \{1\}, \quad \{1, 2, 3\}, \quad \{1, 2\}, \quad \{1, 3\}.$$

Given a Gröbner basis  $G = \{p_1, p_2, \dots, p_k\} \subset \mathbb{K}[x_1, x_2, \dots, x_n]$  for some field  $\mathbb{K}$ , it is usual to represent the leading monomials of the  $p_i$  on an integer lattice in  $n$  dimensions. Each polynomial  $p$  is associated to a point  $c(p)$  whose coordinates are the exponents of its leading monomial. The leading terms of the  $p_i$  generate the ideal of leading terms of polynomials in the ideal. These leading terms are thus exactly represented by  $\bigcup c(p_i) + \mathbb{N}^n$ . This removes from  $\mathbb{N}^n$  a staircase shape whose lattice points correspond to the quotient (see Figure 6). Their number is exactly the order of the group of recurrent configurations. Note that in our example, those seven monomials are  $\{1, x_1, x_1^2, x_2, x_2^2, x_3, x_1x_3\}$ , none of which correspond to a recurrent configuration. However, applying  $\Phi$  yields the recurrent configurations as explained above.

Our second example is the  $2 \times 2$  grid consisting of 4 cells, each connected twice to the sink. The sandpile group of this grid, computed for instance in [9], is the product of two cyclic group of orders 24 and 8.

After the computation of the Gröbner basis of the ideal generated by the toppling polynomials of vertices, it follows that  $x_4$  is of order 24 and that any element can be expressed as a product  $x_3^i x_4^j$  where  $0 \leq i \leq 7$  and  $0 \leq j \leq 23$ , which gives that the order of the group is 192. Also, since  $x_1$  and  $x_2$  can be expressed in terms of  $x_3$  and  $x_4$ , it is seen that the group has two generators.

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