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Engel Expansions of q-Series

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1. Engel Expansions

A real number $A > 0$ has a unique expansion of the form

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \dots,$$

where the a_i are positive integers with $a_{i+1} \geq a_i$ for $i \geq 1$. These expansions were called *Engel expansions* by Perron and their study goes back to Lambert around 1770. Uniqueness of the expansion is not difficult to see, together with the following recurrences from which an iterative algorithm derives:

$$a_k = \lfloor r_k \rfloor + 1, \quad \frac{1}{r_k} = \frac{1}{a_k} + \frac{1}{a_k r_{k+1}}, \qquad k \geq 1.$$

The initial conditions are given by $a_0 < A \leq a_0 + 1$ and $A - a_0 = 1/r_1$. Rational numbers are characterized by the ultimate stationarity of the sequence (a_i). An obvious example of Engel expansion of a non-rational number is provided by $e = \exp(1)$ for which $a_0 = 2$ and $a_i = i + 1$ for $i > 0$.

Arnold and John Knopfmacher defined in [4, 5] an analogous notion for Laurent series.

Definition 1. Given a Laurent series $A = \sum_{n \geq \nu} c_n q^n \in \mathbb{C}((q))$, and an integer $\rho \geq 0$, the *q-Engel sequence* associated with A and ρ is the unique sequence (a_i) of polynomials in q^{-1} such that

$$A = a_0 + \sum_{n \geq 1} \frac{q^{-\rho n}}{a_1 \cdots a_n},$$

with the degrees of the a_i obeying $\deg(a_{i+1}) \geq \deg(a_i) + \rho + 1$.

This definition is motivated by the numerous q-identities involving such expansions. A sample is given in Table 1, using the classical notations

$$(a;q)_0 = 1, \qquad (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \quad \text{for } k > 0, \qquad (a;q)_\infty = \prod_{k \geq 0}(1 - aq^k).$$

Again, uniqueness is not difficult to check and an iterative algorithm follows from

$$A_{k+1} := q^\rho (a_k A_k - 1), \quad a_k = \left[\frac{1}{A_k}\right], \qquad k \geq 1, \tag{1}$$

with $A_0 := A$, $a_0 = [A]$ and $A_1 = q^\rho (A_0 - a_0)$. The bracket notation corresponds to the *integral part* of a Laurent series defined by $[A] := \sum_{\nu \leq n \leq 0} c_n q^n \in \mathbb{C}[q^{-1}]$.

(2)	$\displaystyle\sum_{k\geq 0} \frac{q^{\binom{k+1}{2}}}{(q;q)_k} = \prod_{k>0}(1+q^k),$	(Euler)
(3)	$\displaystyle\sum_{k\geq 0} \frac{z^k q^{k^2}}{(q;q)_k (zq;q)_k} = \frac{1}{(z;q)_\infty},$	(Cauchy)
(4)	$\displaystyle\sum_{k\geq 0} \frac{q^{k^2}}{(-q;q^2)_k} = \sum_{k\geq 0} \frac{(-1)^{k+1} q^k}{(-q;q)_k},$	(Fine)
(5)	$\displaystyle\sum_{k\geq 0} \frac{q^{k(3k-1)/2}}{(q;q)_k (q;q^2)_k} = \prod_{k\geq 1} \frac{(1-q^{10k-6})(1-q^{10k-4})(1-q^{10k})}{1-q^k},$	(Rogers)
(6)	$\displaystyle\sum_{k>0} \frac{q^{k(3k-1)/2}}{(q;q)_{k-1} (q;q^2)_k} = \prod_{k\geq 1} \frac{(1-q^{10k-8})(1-q^{10k-2})(1-q^{10k})}{1-q^k},$	(Rogers)
(7)	$\displaystyle\sum_{k\geq 0} \frac{q^{k(2k-1)}}{(q;q)_{2k}} = \prod_{k>0}(1+q^k),$	(Slater)
(8)	$\displaystyle\sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$	(1st Rogers–Ramanujan)
(9)	$\displaystyle\sum_{k\geq 0} \frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$	(2nd Rogers–Ramanujan)
(10)	$\displaystyle\sum_{k\geq 0} \frac{q^{2k^2}}{(q;q)_{2k}} = \prod_{k>0,\quad k\equiv\pm 2,\pm 3,\pm 4,\pm 5 \pmod{16}} \frac{1}{1-q^k},$	(Slater)
(11)	$\displaystyle\sum_{k\geq 0} \frac{q^{2k^2+2k}}{(q;q)_{2k+1}} = \prod_{k>0\quad k\equiv\pm 1,\pm 4,\pm 6,\pm 7 \pmod{16}} \frac{1}{1-q^k},$	(Slater)

TABLE 1. q-identities involving q-Engel expansions.

2. **Engel Guessing**

Equipped with (1), it is very natural to implement a package computing q-Engel sequences of Laurent series. Such a package opens the way to experimental mathematics with q-Engel expansions [2]. For instance, starting from a truncation of the series expansion of the right-hand side of (2) (a special case of an identity due to Euler) and using $\rho = 0$, the package outputs

$$1 + \frac{q}{(q;q)_1} + \frac{q^3}{(q;q)_2} + \frac{q^6}{(q;q)_3} + O(q^{10}),$$

from which the left-hand side is easily guessed. The task of *proving* such an identity still requires human work.

Using $\rho = 1$ on the same series does not reveal any pattern. However, with $\rho = 2$, one gets

$$1 + \frac{q}{(q;q)_2} + \frac{q^6}{(q;q)_4} + \frac{q^{15}}{(q;q)_6} + O(q^{28}),$$

from which it is easy to conjecture the general formula (7).

3. Identities of Rogers–Ramanujan Type

In one of his independent proofs of the Rogers–Ramanujan identities (8–9), Schur introduced two sequences of polynomials

$$d_m = \sum_k (-1)^k q^{k(5k-3)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m+1-5k}{2} \rfloor \end{bmatrix}, \quad e_m = \sum_k (-1)^k q^{k(5k+1)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m-1-5k}{2} \rfloor \end{bmatrix}, \qquad m \geq 1,$$

with $e_0 = 0$ and $d_0 = 1$ in terms of the *Gaussian polynomials*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

The sequences d_m and e_m appear in the recent generalization of the Rogers–Ramanujan identities due to Garrett, Ismail and Stanton [3]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} d_m}{(q;q^5)_\infty (q^4;q^5)_\infty} - \frac{(-1)^m q^{-\binom{m}{2}} e_m}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{12}$$

Setting $m = 0$, $m = 1$ in this formula yields (8) and (9).

The left-hand side of (12) is the q-Engel expansion of the right-hand side for $\rho = 0$, which motivates [1] in looking for a q-Engel "proof" of this identity. For this, it is sufficient to prove that the sequence $a_n = q^{-(2n+m-1)} - q^{-(n+m-1)}$ is the corresponding q-Engel sequence. Defining

$$A_0 = A, \qquad A_n = (-1)^m q^{-\binom{m}{2}-(m-1)(n-1)} \sum_{j>m} q^{jn} (d_m e_j - d_j e_m) \quad \text{for } n \geq 1,$$

the proof consists in showing that $a_n A_n = 1 + A_{n+1}$ and $a_n = [1/A_n]$, together with correct initial conditions. In view of (14) below, this is not too difficult, but technical (see [1] for details).

Schur proved that both d_m and e_m satisfy the recurrence

$$c_{m+2} = c_{m+1} + q^m c_m, \qquad m \geq 0. \tag{13}$$

Nowadays, this identity is proved automatically by invoking the q-WZ algorithm [7] and this leads to the first purely automatic *elementary* proof of the Rogers–Ramanujan identity [6]. In view of this recurrence, d_m and e_m are nothing but q-analogues of the Fibonacci numbers. It turns out that a generalization of the Cassini identity, namely

$$F_{m-1} F_{m+k} - F_{m+k-1} F_m = (-1)^m F_k,$$

admits a q-analogue in terms of e_m and d_m:

$$d_m e_{m+k} - d_{m+k} e_m = (-1)^m q^{\binom{m}{2}} \sum_{j \geq 0} \begin{bmatrix} k-1-j \\ j \end{bmatrix} q^{j^2+mj}. \tag{14}$$

This identity can be proved automatically from (13) by univariate D-finite closure properties (m being fixed). In fact, a non-Engel proof of (12) follows from letting k tend to infinity in (14) in view of Schur's limit formulae

$$d_\infty = \frac{1}{(q;q)_\infty} \sum_k (-1)^k q^{k(5k-3)/2} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

$$e_\infty = \frac{1}{(q;q)_\infty} \sum_k (-1)^k q^{k(5k+1)/2} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

The infinite products are obtained by Jacobi's triple product identity, which also admits a simple computer proof [6].

4. A New Identity Discovered by Engel Guessing

The identities (10) and (11) can be conjectured by Engel guessing after first multiplying the product by $1-q$. An Engel proof is also available [2] using the *Santos polynomials* defined by

$$S_m = \sum_j q^{4j^2-j} \begin{bmatrix} m \\ \lfloor \frac{m+1-4j}{2} \rfloor \end{bmatrix}, \qquad T_m = \sum_j q^{4j^2-3j} \begin{bmatrix} m \\ \lfloor \frac{m+2-4j}{2} \rfloor \end{bmatrix},$$

whose limits S_∞ and T_∞ when $m \to \infty$ are precisely the right-hand sides of (10) and (11).

In view of (12), a natural idea consists in experimenting with q-Engel expansions of $S_n T_\infty - T_n S_\infty$ or variations of it. It turns out that a pattern readily emerges leading to conjecturing the following generalization of (10) and (11):

$$S_n T_\infty - T_n S_\infty = q^n (q;q^2)_n \sum_{k \geq 0} \frac{q^{2k^2+2(n+1)k}}{(q;q)_{2k+1}}.$$

Again, a possible proof [2] consists in relying on a finite version, namely

$$S_n T_{n+m} - T_n S_{n+m} = q^n (q;q^2)_n \sum_{k \geq 0} \begin{bmatrix} m \\ 2k+1 \end{bmatrix} q^{2k^2+2(n+1)k}.$$

5. Conclusion

Engel expansions are a new way of looking at q-identities which allows for easy computer experiments and hence should lead to many discoveries. A pending issue is to make q-Engel proving into an algorithmic task.

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