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Mac Mahon's Partition Analysis Revisited

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Abstract

The purpose of this talk is to present the Ω operator introduced by Mac Mahon in 1915 and to show its power in current combinatorial and partition-theoretic research. This operator is implemented in the Mathematica Package Omega which was developpped by A. Riese. This is joint work with G. E. Andrews (Penn State University) and A. Riese (RISC-Linz).

1. Introduction

Mac Mahon devoted many pages of his famous book "Combinatorial Analysis" [9] to Ω-calculus. Netherveless this method was not used for 85 years except by Stanley in 1973 [10]. The purpose of this talk is to present the Ω operator and to show its power in current combinatorial and partition-theoretic research [1, 2, 3, 4, 5]. In this summary, we define the Ω operator and exhibit a few of its elimination rules, before giving two problems where this operator is a powerful tool: lecture hall partitions and k-gons of integer length.

2. The Omega Operator

Let us now define the operator and present a few rules.

Definition 1. [9] The Omega operator $\underset{\geq}{\Omega}$ is defined as follows:

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\dots,s_r} \lambda_1^{s_1} \dots \lambda_r^{s_r} = \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\dots,s_r}.$$

To evaluate this operator, Mac Mahon proposed a list of elimination rules. The proof of each is straightforward as it uses the simple identity

$$\sum_{n\geq 0} x^n = 1/(1-x).$$

We list a few of them only:

$$\underset{\geq}{\Omega} \frac{\lambda^{-s}}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)} = \frac{x^s}{(1-x)(1-xy)}, \qquad s \geq 0,$$

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda}\right)\left(1-\frac{z}{\lambda}\right)} = \frac{1}{(1-x)(1-xy)(1-xz)},$$

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)} = \frac{1}{(1-x)(1-x^s y)}, \qquad s > 0,$$

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda^s x)\left(1-\frac{y}{\lambda}\right)} = \frac{1 + xy\frac{1-y^{s-1}}{1-y}}{(1-x)(1-xy^s)}, \qquad s > 0.$$

For example to find the generating function of the partitions with three parts and whose parts differ by at least two, we use the first rule:

$$f_3(q) = \underset{\geq}{\Omega} \sum_{a1,a_2,a_3 \geq 1} \lambda_1^{a_1-a_2-2} \lambda_2^{a_2-a_3-2} q^{a_1+a_2+a_3} = \underset{\geq}{\Omega} \frac{\lambda_1^{-2}\lambda_2^{-2}q^3}{(1-\lambda_1 q)\left(1-\frac{\lambda_2 q}{\lambda_1}\right)\left(1-\frac{q}{\lambda_2}\right)}$$

$$= \underset{\geq}{\Omega} \frac{q^2\lambda_2^{-2}q^3}{(1-q)(1-\lambda_2 q^2)\left(1-\frac{q}{\lambda_2}\right)} = \frac{q^2 q^4 q^3}{(1-q)(1-q^2)(1-q^3)}.$$

It is also possible to generalize this result for partitions with k parts and whose parts differ by at least two for any $k > 0$, that is

$$f_k(q) = \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}.$$

3. Lecture Hall Partitions

The lecture hall partition theorem is one of the most elegant recent result in partition analysis [6, 7]. Let us state the refinement of this theorem [8].

Theorem 1. *The number of partitions of n of the form $(b_j, b_{j-1}, \dots, b_1)$ with $\frac{b_j}{j} \geq \frac{b_{j-1}}{j-1} \geq \dots \geq \frac{b_1}{1} \geq 0$ and $b_j - b_{j-1} + \dots + (-1)^{j-1} b_1 = m$ is equal to the number of partitions of n into m odd parts less than $2j$.*

This theorem can also be proved with the Omega operator [1], which is what motivated G. E. Andrews to resuscitate the Omega operator. The proof mainly uses the elimination rule

$$\underset{\geq}{\Omega} \frac{1}{(1-\lambda x)\left(1-\frac{y}{\lambda^s}\right)} = \frac{1}{(1-x)(1-x^s y)}$$

Let us illustrate it for $j = 3$.

$$\sum_{\frac{b_3}{3} \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0} x^{b_3-b_2+b_1} q^{b_3+b_2+b_1} = \underset{\geq}{\Omega} \sum_{b_3,b_2,b_1 \geq 0} \lambda_1^{2b_3-3b_2} \lambda_2^{b_2-2b_1} x^{b_3-b_2+b_1} q^{b_3+b_2+b_1}$$

$$= \underset{\geq}{\Omega} \frac{1}{(1-\lambda_1^2 qx)\left(1-\frac{\lambda_2 q}{\lambda_1^3 x}\right)\left(1-\frac{qx}{\lambda_2^2}\right)} = \underset{\geq}{\Omega} \frac{1}{(1-xq)(1-xq^3)(1-xq^5)}.$$

The Omega operator can also give a bijective proof of the theorem [5]. Let us show how to proceed for $j = 3$:

$$\sum_{\frac{b_3}{3} \geq \frac{b_2}{2} \geq \frac{b_1}{1} \geq 0} q_3^{b_3} q_2^{b_2} q_1^{b_1} = \underset{\geq}{\Omega} \sum_{b_3,b_2,b_1 \geq 0} \lambda_1^{2b_3-3b_2} \lambda_2^{b_2-2b_1} q_3^{b_3} q_2^{b_2} q_1^{b_1} = \underset{\geq}{\Omega} \frac{1 + q_2 q_3^2}{(1-q_3)(1-q_2^2 q_3^3)(1-q_1 q_2^2 q_3^3)}.$$

From the previous equation we get that there is a bijection between the lecture hall partitions (b_3, b_2, b_1) of n and the partitions of n into parts $\{1,3,5\}$ with multiplicity m_i for the part i. This bijection becomes:

$$b_3 = 3m_5 + 2m_3 - \left\lfloor \frac{m_3}{2} \right\rfloor + m_1, \qquad b_2 = 2m_5 + m_3, \qquad b_1 = \left\lfloor \frac{m_3}{2} \right\rfloor.$$

4. k-Gons with Integer Length

The problem can be defined as follows. The number $|T_k(n)|$ of k-gons with length n is equal to the number of solutions of

$$a_k \geq a_{k-1} \geq \cdots \geq a_1 \geq 1, \quad a_1 + a_2 + \cdots + a_k = n, \quad a_1 + a_2 + \cdots + a_{k-1} > a_k. \tag{1}$$

Let $F_k(q) = \sum_n |T_k(n)| q^n$ be the associated generating function. For triangles ($k = 3$) we get

$$F_3(q) = \sum_n |T_3(n)| q^n = \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}.$$

This is easy to prove as conditions (1) give

$$F_3(q) = \underset{\geq}{\Omega} \sum_{\substack{a_1 \geq 1 \\ a_2, a_3 \geq 0}} \lambda_1^{a_3 - a_2} \lambda_2^{a_2 - a_1} \lambda_3^{a_1 + a_2 - a_3 - 1} q^{a_1 + a_2 + a_3}$$

$$= \underset{\geq}{\Omega} \frac{q\lambda_1^{-1}}{\left(1 - \frac{q\lambda_2}{\lambda_3}\right)\left(1 - \frac{q\lambda_1\lambda_3}{\lambda_2}\right)\left(1 - \frac{q\lambda_3}{\lambda_1}\right)} = \frac{q^3}{(1-q^2)(1-q^3)(1-q^4)}$$

We can even be more specific

$$F_3(q_1, q_2, q_3) = \sum_{\substack{a_3 \geq a_2 \geq a_1 \geq 1 \\ a_1 + a_2 > a_3}} q_1^{a_1} q_2^{a_2} q_3^{a_3} = \underset{\geq}{\Omega} \sum_{\substack{a_1 \geq 1 \\ a_2, a_3 \geq 0}} \lambda_1^{a_3 - a_2} \lambda_2^{a_2 - a_1} \lambda_3^{a_1 + a_2 - a_3 - 1} q_1^{a_1} q_2^{a_2}$$

$$= \underset{\geq}{\Omega} \frac{q\lambda_1^{-1}}{\left(1 - \frac{q\lambda_2}{\lambda_3}\right)\left(1 - \frac{q\lambda_1\lambda_3}{\lambda_2}\right)\left(1 - \frac{q\lambda_3}{\lambda_1}\right)} = \frac{q_1 q_2 q_3}{(1 - q_2 q_3)(1 - q_1 q_2 q_3)(1 - q_1 q_2 q_3^2)}.$$

This shows there is a bijection between the 3-tuples (u_1, u_2, u_3) of $\mathbb{N}^3$ and the triangles whose sides have length $u_1 + u_2 + 1$, $u_1 + u_2 + u_3 + 1$ and $u_1 + 2u_2 + u_3 + 1$.

Thanks to the Omega operator we can compute the generating function for larger k:

$$F_4(q) = \frac{q^4(1 + q + q^5)}{(1-q^2)(1-q^3)(1-q^4)(1-q^6)},$$

$$F_5(q) = \frac{q^5(1 - q^{11})}{(1-q)(1-q^2)(1-q^4)(1-q^5)(1-q^6)(1-q^8)},$$

$$F_6(q) = \frac{q^6(1 - q^4 + q^5 + q^7 - q^8 - q^{13})}{(1-q)(1-q^2)(1-q^4)(1-q^6)(1-q^8)(1-q^{10})}.$$

We then can see that no pattern can be found and the Omega operator was a quick tool to show that the solutions of this k-gon problem do not have "nice" generating functions.

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