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On the Convergence of Borel Approximants

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Abstract

For some "irregular singular" problems coming from differential equations, there exist formal power series solutions that are everywhere divergent. These power series turn out to make sense as asymptotic expansions of actual solutions. The Borel summation technique is used to recover convergent representations for these actual functions solutions.

1. Resummation

Some "irregular singular" problems coming from differential equations have formal power series solutions that are everywhere divergent. By resummation techniques, one can obtain convergent solutions [7, 10]. We consider a power series, solution of a linear differential equation, that is everywhere divergent, noted $\tilde{x}(z) = \sum_1^\infty x_n z^{-n}$. We assume that it has *Gevrey order* equal to one, which means that there exist constants A and c such that $|x_n| \leq Ac^n n!$. For a function $f(z)$, holomorphic in an angular sector S extending to infinity and containing the real positive axis, we say that $\tilde{x}(z)$ is the *Gevrey expansion of order 1* of $f(z)$ if there exist constants K and C such that

$$\left| f(z) - \sum_1^{N-1} x_n z^{-n} \right| \leq CK^N N!\, |z|^{-N} \quad \text{when } z \in S \text{ and } N \geq 0.$$

This function f is a resummation of $\tilde{x}$, and it exists if the opening angle of S is smaller than π.

The formal Borel transform of $\tilde{x}(z)$ is defined by $y(z) = \sum_1^\infty \frac{x_n z^{n-1}}{(n-1)!}$. It converges for $|z| < \frac{1}{c}$. We assume that the function y can be continued analytically along a line that does not meet a singularity. In the particular case when x is a solution of a linear differential equation with rationnal coefficients, so does y, as this property is stable under the Borel transform. Thus y has a finite number of singularities and verifies the above hypothesis. Up to a possible linear change of variable, we may assume that there is no singularity on the real axis, which implies that y can be continued analytically on the positive real axis. If y satisfy the expected growth conditions at infinity, we apply the Laplace transform. This transform is defined by

$$x(z) = \mathcal{L}(y) = \int_0^\infty e^{-zt} y(t)\, dt,$$

and is convergent for $\Re(z^a)$ greater than a certain positive constant, the constant a being made precise later. The asymptotic expansion of $x(z)$ when $z \to 0^+$ is equal to $\sum_1^\infty x_n z^{-n}$. The function x is a solution of the initial differential equation [2, 8].

2. Balser, Lutz, and Schäfke's Technique

The next step is to find a way to compute this function x quickly and in a large domain. For this, Lutz *et al.* [1] reformulate x as a convergent series of the type $x(z) = \sum_0^\infty d_n q_n(z)$. This series is obtained by introducing a mapping function ϕ that maps $[0,1]$ onto $[0,\infty]$, so as to write the equation

$$x(z) = \int_0^\infty e^{-zt} y \circ \phi \circ \phi^{-1}(t)\, dt = \int_0^\infty e^{-zt} \sum_0^\infty d_n \phi^{-1}(t)^n\, dt, \tag{1}$$

where for the second equality we have used the re-expansion $y \circ \phi(u) = \sum_0^\infty d_n u^n$ in terms of the sequence d_n. The sequence q_n is thus determined by $q_n = \int_0^\infty e^{-zt} \phi^{-1}(t)^n\, dt$, under the assumption that the interversion of the integral and the sum holds, permitting termwise integration. We observe that q_n does not depend on x and on the initial problem, but only on the mapping function ϕ. This means that these coefficients can be precomputed. On the other hand the coefficients d_n correspond to a composition of the function ϕ with the Borel transform y. This is formalized in the following theorem.

Theorem 1 (Balser, Lutz and Schäfke). *Let $x(z) = \int_0^\infty e^{(-zt)} y(t)\, dt$ where the function y is holomorphic in the domain*

$$\mathcal{D} \supset \Big\{ \big|\operatorname{Arg}(1 + t/a)\big| < \pi/2p \Big\}$$

and satisfies $\big|y(t)\big| e^{-b|t|} \to 0$ as $|t| \to \infty$ in $\mathcal{D}$. Choose ϕ holomorphic in $\Delta = \big\{|\tau| < 1\big\}$ so that $\phi(\Delta) \subset \mathcal{D}$, $\phi\big([0,1]\big) = [0,\infty]$, and $(1-\tau)^c \phi(\tau) \to A$ as $\tau \to 1$ in Δ. Define (d_n) by its generating series $y\big(\phi(\tau)\big) = \sum_0^\infty d_n \tau^n$, and (q_n) by

$$q_n(z) = \int_0^1 e^{-z\phi(\tau)} \tau^n \phi'(\tau)\, d\tau \quad \textit{for } z \textit{ such that } \big|\operatorname{Arg}(z)\big| < \pi(1+c)/2.$$

Then for suitable positive constants (independent of n)

$$|d_n| \le K e^{L n^{c/(c+1)}} \qquad \textit{and} \qquad |q_n(z)| \le \tilde{K} e^{-A n^{c/(c+1)} \Re\left(z^{1/(c+1)}\right)}.$$

So we have $x(z) = \displaystyle\sum_0^\infty d_n q_n(z)$ for $\Re\left(z^{1/(c+1)}\right)$ large.

Proof. Starting from Equation (1), we obtain $x(z) = \int_0^\infty \sum_{n=0}^\infty e^{-zt} d_n \phi^{-1}(t)^n\, dt$. The saddle-point method gives upper bounds for d_n and q_n that allows us to interchange the order of integrand and summation in the equation above for $\Re(z^{1/(c+1)})$ large enough. This interchange yields the expected result $x(z) = \sum_{n=0}^\infty d_n q_n(z)$. ☐

Some other classical conformal mappings can be found in [6]. Here is an example. The mapping

$$\tau = 1 - \frac{2}{(1 + t/a)^p + 1} \quad \text{with} \quad a \in \mathbb{R} \text{ and } p \ge 1/2 \tag{2}$$

takes the sectorial domain defined by $\big|\operatorname{Arg}(1 + t/a)\big| < \pi/(2p)$ onto the unit disk. The choice of the conformal mapping ϕ is important because it has an effect on the speed of convergence and on the area of convergence.

In the particular case where the differential equation is linear with polynomial coefficients, some efficient computation can be done using recurrences. We also suppose now that the function ϕ is algebraic. The precomputation of the coefficients q_n is based on the fact that they follow a linear recurrence. We first note that the coefficients q_n are equal to $\int_0^1 e^{-z\phi(u)} u^n \phi'(u)\, du$ as shown by a simple change of variable $t = \phi(u)$. The function $e^{-z\phi(u)}\phi'(u)$ satisfies the first-order linear differential equation

$$G'(t) = \left(\frac{\phi''(t)}{\phi'(t)} - z\phi'(t)\right) G(t). \tag{3}$$

If we note $\sum_{k=0}^{K} p_k(n)a(n+k) = 0$ the linear recurrence satisfied by the Taylor coefficients at the origin $a(n)$ of a power series solution of the equation (3), then the integrals $q_n(z)$ satisfy the recurrence $\sum_{k=0}^{K} p_k(-n)q_{n-k-1}(z) = 0$. Once we have the recurrence satisfied by the coefficients q_n and the initial conditions that are given by $q_n = \int_0^\infty e^{-zt}\phi^{-1}(t)^n\,dt$, all the q_n can be computed quickly. A problem is that we seek for numerical and not exact computations, and so we have, on each example, to seek for numerical stability. This point uses a backward scheme which is developped on an example below.

The computation of the coefficients d_n can be done efficiently by finding a recurrence for example using the gfun package [9], because it is a composition of a known algebraic function ϕ and a function y known by its differential equation. The initial conditions for the d_n derive directly from the initial conditions of the differential equation satisfied by y and so from the initial conditions of the differential equation satisfied by $\tilde{x}$. This is illustrated by the example of the Heun equation.

3. Heun Equation

The Heun equation is the generic differential equation with four regular singular points located at 0, 1, c, and ∞; see [5]. The double confluent Heun equation is obtained by letting the singularity located at c tend to the one located at ∞, and the singularity located at 1 tend to 0. The equation obtained then has two irregular singular points located at 0 and ∞. The example we study [3] is the confluent Heun equation in the form

$$z^2 f''(z) + (z + \alpha z^2 + \alpha)f'(z) + \frac{(2\alpha z^2\beta_1 + \alpha z^2 + \alpha^2 z - 2\gamma z + 2\alpha\beta_{-1} - \alpha)}{2z} f(z) = 0. \tag{4}$$

The acceleration is realised by the function $\phi = \frac{1}{(1-z)^2} - 1$ which maps from $[0, 1]$ onto $[0, \infty]$. The recurrence satisfied by q_n is thus

$$q(n) = \frac{(-6+3n)q(n-1) + (-2z+6-3n)q(n-2) + (n-2)q(n-3)}{n-2}. \tag{5}$$

The initial conditions, that are easily computed, using the definition of q_n, correspond to a dominated solution, so any numerical error makes the dominating solution appear. A solution to this problem is to compute the recurrence backwards, which exchanges the roles of dominating and dominated regimes. The idea is to choose arbitrary values for $q_{N-d}, \ldots, q_N$ where d is the order of the recurrence and N is a sufficiently large integer. All the values of q_n for $n \le N$ are then computed from these "final" values backwards. This technique is developped in [11]. The dominating solution of Recurrence 5 disappears and so the initial values found differ only by a multiplicative constant λ from the actual initial values. The sequence q_n thus found has to be multiplied by this constant λ to give the expected sequence q_n.

For the coefficients d_n, the recurrence is found easily using gfun. For parameters $\alpha = -1$, $\beta_{-1} = 1/2$, $\beta_1 = 1/2$, and $\gamma = 1/3$, it is

$$\begin{aligned} 0 = (6n^2 + 3n^3)a_n &+ (-93n - 36 - 75n^2 - 18n^3)a_{n+1} \\ &+ (568n + 404 + 267n^2 + 42n^3)a_{n+2} + (-1193n - 1176 - 411n^2 - 48n^3)a_{n+3} \\ &+ (1042n + 1240 + 291n^2 + 27n^3)a_{n+4} + (-78n^2 - 336n - 480 - 6n^3)a_{n+5} \end{aligned}$$

with initial conditions $a_0 = 0$, $a_1 = 1$, $a_2 = 1/3$, $a_3 = -23/108$, and $a_4 = -2749/3888$.

Now for each fixed z, we can compute the value of $x(z)$ to arbitrary precision, by choosing the number of terms we take into account. The backwards computation for the q_n coefficients implies that the number of computable terms is limited by the starting point. If it is too low, we have to choose a larger starting point to get more terms. It is generally not possible to decide where a good starting point for the computation of the backward computation would be. This can be done on particular examples, but the starting point strongly depends on z.

4. Applications

Many problems related to differential equations yield formal power series of Gevrey order one. Whenever the Borel–Laplace transform applies, the results of Section 2 also applies. A concrete application coming from physics is the one-dimensional complex heat equation:

$$u_\tau(\tau, z) = u_{zz}(\tau, z), \quad u(0, z) = \phi(z).$$

The Cauchy data $\phi(z)$ is assumed to be holomorphic near the origin. A formal solution is

$$\tilde{u}(\tau, z) = \sum_0^\infty \phi^{(2n)}(z) \frac{\tau^n}{n!}.$$

Lutz *et al.* have shown that either $\tilde{u}(\tau, z)$ is convergent, or the method of Section 2 applies. If $v(\tau, z)$ is the Borel transform of $\tilde{u}(\tau, z)$ with respect to τ, then applying the Laplace transform in the variable τ to $v(\tau, z)$ for fixed z gives a convergent solution $u(\tau, z)$ of the Cauchy problem. Better knowledge on the function ϕ may easily lead to fast rate convergence possibly using the mapping function (2). Another application is about convergent Liouville–Green expansions for second order linear differential equations [4].

Bibliography

[1] Balser (W.), Lutz (D. A.), and Schäfke (R.). – On the convergence of Borel approximants. – Preprint.

[2] Borel (Émile). – Leçons sur les séries divergentes. In *Collection de monographies sur la théorie des fonctions, publiée sous la direction de M. Émile Borel.* – Gauthiers-Villars, Paris, 1901. Second edition, 1928. Reprinted by J. Gabay, 1988.

[3] Chyzak (Frédéric), Durand (Marianne), and Salvy (Bruno). – Borel resummation of divergent series using GFUN. – Maple worksheet. Available from `http://algo.inria.fr/libraries/autocomb/`.

[4] Dunster (T. M.), Lutz (D. A.), and Schäfke (R.). – Convergent Liouville–Green expansions for second-order linear differential equations, with an application to Bessel functions. *Proceedings of the Royal Society. London. Series A*, vol. 440, n° 1908, 1993, pp. 37–54.

[5] Duval (Anne) and Loday-Richaud (Michèle). – Kovačič's algorithm and its application to some families of special functions. *Applicable Algebra in Engineering, Communication and Computing*, vol. 3, n° 3, 1992, pp. 211–246.

[6] Kober (H.). – *Dictionary of conformal representations.* – Dover Publications, New York, N. Y., 1952, xvi+208p.

[7] Loday-Richaud (Michèle). – Solutions formelles des systèmes différentiels linéaires méromorphes et sommation. *Expositiones Mathematicae*, vol. 13, n° 2-3, 1995, pp. 116–162.

[8] Ramis (Jean-Pierre). – Séries divergentes et théories asymptotiques. *Bulletin de la Société Mathématique de France (Panoramas et Synthèses)*, vol. 121, 1993, p. 74.

[9] Salvy (Bruno) and Zimmermann (Paul). – Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, vol. 20, n° 2, 1994, pp. 163–177.

[10] Thomann (Jean). – Resommation des series formelles. Solutions d'équations différentielles linéaires ordinaires du second ordre dans le champ complexe au voisinage de singularités irrégulières. *Numerische Mathematik*, vol. 58, n° 5, 1990, pp. 503–535.

[11] Wimp (Jet). – *Computation with recurrence relations.* – Pitman (Advanced Publishing Program), Boston, MA, 1984, xii+310p.