

## Reflected Brownian Bridge Area Conditioned on its Local Time at the Origin

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### Abstract

Using properties of the Airy functions, we analyze the reflected Brownian bridge area  $W_b$  conditioned on its local time  $b$  at the origin. We give a closed form expression of the Laplace transform of  $W_b$ , a recurrence equation for the moments, leading to an efficient computation algorithm and an asymptotic form for the density  $f(x, b)$  of  $W_b$  for  $x \rightarrow 0$ .

### 1. Introduction

Let us first introduce the standard Brownian motion denoted by  $x(t)$  and a few classical variants: the reflected Brownian motion  $x^+(t) = |x(t)|$ ; the Brownian bridge  $B(t)$ ; the reflected Brownian bridge  $B^+(t)$  on  $[0, 1]$ ; the Brownian excursion  $e(t)$ .

The object of interest in this talk is  $W_b := \int_0^1 B^+(t) dt$ , the area of the reflected Brownian bridge conditioned on having a local time at the origin equal to  $b$ . This random variable appeared in [4] as the limit law for  $m^{-3/2} D_{m, m-b\sqrt{m}}$ , where  $D_{m, m-b\sqrt{m}}$  denotes the total displacement for a hash table with  $m$  locations and  $b\sqrt{m}$  empty locations, using linear probing. It also represents the limit law for the total height of random forests with  $b\sqrt{m}$  trees and  $m$  nodes or leaves. The only description of it was given by its moments, related to the classical Airy function  $\text{Ai}(z) := \frac{1}{\pi} \int_0^{+\infty} \cos(\frac{1}{3}t^3 + zt) dt$  (recall  $\text{Ai}'' = z\text{Ai}$ ) in the following way:

$$\mathbf{E}[W_b^k] = k! \sum_{j=1}^k \left( \sum_{k_1, \dots, k_j \geq 1, \Sigma k_i = k} \prod_{i=1}^j \omega_{k_i} \right) \frac{b^{j-1}}{j!} q_{3k-j-2}(b),$$

where the  $\omega_k$  are defined by the asymptotic expansion  $\frac{\text{Ai}'(z)}{\text{Ai}(z)} \underset{z \rightarrow +\infty}{\sim} \sum_{k=0}^{+\infty} \omega_k \frac{(-1)^k z^{-3(k-1)/2}}{2^k}$ , and  $q_r(b) := \int_0^{+\infty} \frac{x^r}{r!} e^{-bx-x^2/2} dt$ .

We will provide a closed form expression for the Laplace transform of  $W_b$ , a better way to compute its moments, and an asymptotic form for the density  $f(x, b)$  of  $W_b$  when  $x \rightarrow 0$ .

### 2. Laplace Transform of $W_b$

Computing the Laplace transform of  $W_b$  essentially requires using Kac's formula [3] and a few technicalities. Eq. (30) in [5, p. 491] states that, if we denote by  $t^+(t, a)$  the local time of  $x(t)$  at  $a$ ,

$$(1) \quad \int_0^\infty e^{-\alpha t} \mathbf{E}_0 \left[ \exp \left( - \int_0^t x^+(u) du - \delta t^+(t, 0) \right) \middle| x(t) = 0 \right] \frac{dt}{\sqrt{2\pi t}} = \left( \delta - \frac{2^* \text{Ai}'(2^* \alpha)}{\text{Ai}(2^* \alpha)} \right)^{-1},$$

where  $2^* := 2^{1/3}$ . From it we can derive the following theorem:

**Theorem 1.** *The Laplace transform  $\Theta(z, b)$  of  $W_b$  has the closed form expression*

$$\Theta(z, b) = \mathbf{E}[e^{-zW_b}] = \frac{-z^{1/3}e^{b^2/2}}{i2^{1/6}\sqrt{\pi}} \int_{-i\infty}^{i\infty} e^{bz^{1/3}2^{1/3}Ai'(u)/Ai(u)} (Ai'(u)/Ai(u))' e^{uz^{2/3}/2^{1/3}} du.$$

*Proof.* Given  $[\int_0^t x^+(u) du | x(t) = 0] \stackrel{\mathcal{D}}{=} t^{3/2}Y$  and  $t^+(t, 0) \stackrel{\mathcal{D}}{=} \sqrt{tt^+}(1, 0)$  (scaling property), Eq. (1) leads to

$$\mathbf{E}_0 \int_0^\infty e^{-\alpha t} \int_0^\infty e^{-t^{3/2}W_b} b e^{-b^2/2} e^{-\delta\sqrt{tb}} \frac{db dt}{\sqrt{2\pi t}} = [\delta - 2^*\Lambda(\alpha)]^{-1},$$

where  $\Lambda(\alpha) := \frac{Ai'(2^*\alpha)}{Ai(2^*\alpha)}$ . The change of variable  $v = \sqrt{tb}$  and an inversion on  $\delta$  delivers

$$(2) \quad \int_0^\infty e^{-b^2/2} e^{-\alpha v^2/b^2} \mathbf{E} \left[ e^{-v^3/b^3 W_b} \right] \frac{2 db}{\sqrt{2\pi}} = e^{v2^*\Lambda(\alpha)}.$$

After setting  $b = \frac{v}{\sqrt{2^*\sigma}}$ ,  $u = 2^*\alpha$ , differentiating with respect to  $u$  and using  $(\frac{Ai'}{Ai})' = u - (\frac{Ai'}{Ai})^2$ :

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-u\sigma} \mathbf{E} \left[ e^{-\sqrt{2}\sigma^{3/2}W_{v/\sqrt{2^*\sigma}}} \right] e^{-v^2/(2^{4/3}\sigma)} \frac{d\sigma}{\sqrt{2\sigma}} = -e^{v2^*Ai'(u)/Ai(u)} (Ai'(u)/Ai(u))'.$$

The inversion formula for Laplace transforms then writes:

$$(3) \quad \mathbf{E} \left[ e^{-\sqrt{2}\sigma^{3/2}W_{v/\sqrt{2^*\sigma}}} \right] e^{-v^2/(2^{4/3}\sigma)} / \sqrt{4\pi\sigma} = \frac{-1}{2\pi i} \int_{-i\infty}^{i\infty} e^{v2^*Ai'(u)/Ai(u)} (Ai'(u)/Ai(u))' e^{u\sigma} du.$$

Now set  $v = b\sqrt{2^*\sigma}$ ,  $z = \sqrt{2}\sigma^{3/2}$ ,  $\Theta(z, b) = \mathbf{E}[e^{-zW_b}]$ . Eq. (3) becomes

$$\frac{2^{1/6}\Theta(z, b)e^{-b^2/2}}{2\sqrt{\pi}} = \frac{-z^{1/3}}{2\pi i} \int_{-i\infty}^{i\infty} e^{bz^{1/3}2^{1/3}Ai'(u)/Ai(u)} (Ai'(u)/Ai(u))' e^{uz^{2/3}/2^*} du$$

which proves the theorem.  $\square$

### 3. Recurrence Formulae

Using Laplace transforms and inversions of Laplace transforms, we show here how to find an algorithm to compute the moments  $\psi_k(b) := \mathbf{E}[W_b^k]$  by recurrence. We first need:

**Lemma 1.** *Define  $G(\eta) := 2^*\Lambda(\alpha)/\sqrt{\alpha}$  and  $s = 1/b^2$ ; we have*

$$(4) \quad \int_0^\infty e^{-1/(2s)} e^{-ws} (-1)^k s^{3/2k} \psi_k(b) \frac{ds}{s^{3/2}\sqrt{2\pi k!}} = [\eta^k] \frac{e^{\sqrt{w}G_0}}{w^{3/2k}} \sum_{i=1}^\infty \frac{(\sqrt{w}(G(\eta) - G_0))^i}{i!}.$$

*Proof.* Set  $s := 1/b^2$ ,  $w = \alpha v^2$ , and  $\eta = \alpha^{-3/2}$ . Eq. (2) becomes

$$\int_0^\infty e^{-1/(2s)} e^{-ws} \mathbf{E} \left[ e^{-\eta w^{3/2}s^{3/2}W_b} \right] \frac{ds}{s^{3/2}\sqrt{2\pi}} = e^{\sqrt{w}G(\eta)},$$

Set  $G_0 := G(0)$ . Eq. (3) leads to

$$(5) \quad \int_0^\infty e^{-1/(2s)} e^{-ws} \mathbf{E} \left[ e^{-\eta w^{3/2}s^{3/2}W_b} - 1 \right] \frac{ds}{s^{3/2}\sqrt{2\pi}} = e^{\sqrt{w}G(\eta)} - e^{\sqrt{w}G_0} \\ = e^{\sqrt{w}G_0} \sum_{i=1}^\infty \frac{(\sqrt{w}(G(\eta) - G_0))^i}{i!}.$$

Upon expanding both sides of (5) with respect to  $\eta$ , this gives the desired formula.  $\square$

To invert the Laplace transforms of the form  $e^{-\sqrt{2w}}/w^{(j+1)/2}$ , we will use the following lemmas:

**Lemma 2.** Set  $\phi^{(1)}(x) := \phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$  (classical Gaussian distribution function) and  $\phi^{(j+1)}(x) := \int_{-\infty}^x \phi^{(j)}(u) du$ . Then

$$\int_0^\infty \phi^{(j)}(-b) e^{-ws} \frac{(2s)^{(j+1)/2}}{s} ds = \frac{e^{-\sqrt{2w}}}{w^{(j+1)/2}}, \quad j \geq 1, \quad \text{where } b = 1/\sqrt{s}.$$

*Sketch of proof.* One proves the lemma by induction and uses an integration by part and an integration with respect to  $w$  to prove it at rank  $k+1$  from rank  $k$ .  $\square$

**Lemma 3.** The  $\phi^{(j)}(x)$  can be expressed in the form:

$$\phi^{(k)}(z) = p_1(k, z)\phi(z) + p_2(k, z)e^{-z^2/2}/\sqrt{2\pi},$$

where  $p_1(k, z)$  is of degree  $k-1$ ,  $p_2(k, z)$  is of degree  $k-2$ .

Using integration by parts on  $\int_{-\infty}^z x^j \phi(x) dx$  and identification of coefficients, it is possible to prove the following proposition, enabling us to compute nice expressions of the  $\phi^{(j)}(x)$ :

**Proposition 1.** Define, for  $k \geq 1$ ,  $j \geq 0$ ,  $P_1[k, j] := [z^j]p_1(k, z)$ , and  $P_2[k, j] := [z^j]p_2(k, z)$ . Then the sequences  $(P_1[k, j])_{k \geq 1, j \geq 0}$  and  $(P_2[k, j])_{k, j \geq 0}$  are defined by the initial values  $P_1[1, 0] = 1$ ,  $P_2[1, 0] = 0$ ,  $P_1[1, j] = P_2[1, j] = 0$  for  $j \geq 1$ , and the recurrence relations, for  $k \geq 1$ :

$$\begin{aligned} P_1[k+1, j] &:= P_1[k, j-1]/j, \quad j = 1, \dots, k, \\ P_2[k+1, j] &:= \sum_{l=0}^{\lfloor (k-1-j)/2 \rfloor} P_1[k, j+2l]/(j+2l+1)(j+2l+1)_l \\ &\quad - \sum_{l=0}^{\lfloor (k-3-j)/2 \rfloor} P_2[k, j+2l+1](j+2l+1)_l, \quad j = 0, \dots, k-1, \\ P_1[k+1, 0] &:= - \sum_{l=1,3,\dots,k-1} P_1[k, l]/(l+1)(l+1)_{(l+1)/2} + \sum_{l=0,2,\dots,k-2} P_2[k, l](l)_l. \end{aligned}$$

Determining a recurrence relation for the moments  $\psi_k(b)$  hence amounts to determining a recurrence relation for the  $Z_j$  defined by (see (4)):

$$(-1)^j b^{-3j} \frac{Z_j}{j!} = [\eta^j] \frac{1}{w^{3/2j}} \sum_{i=1}^{\infty} \frac{(\sqrt{w}(G(\eta) - G_0))^i}{i!}.$$

Indeed, along the mechanical transfer rule  $\frac{1}{w^{(l+1)/2}} \rightarrow \frac{\phi^{(l)}(-b)}{b^{l+1}} b^2 2^{(l+1)/2}$ ,  $\psi_j(b)$  is equivalent to  $Z_j \sqrt{2\pi} e^{b^2/2}/b^3$ . To get a recurrence formula giving  $Z_k$  in function of the  $Z_1, \dots, Z_j$ , we introduce

$$S_k(\eta) := \sum_{j=1}^k (-1)^j b^{-3j} \frac{Z_j}{j!} w^{3j/2} \eta^j = \sum_{j=1}^k \eta^j [\eta^j] \left( \frac{\sum_{l=1}^k (-1)^l (d_l - c_l) \left(\frac{3\eta}{2^{3/2}}\right)^l}{\sum_{l=0}^k (-1)^l c_l \left(\frac{3\eta}{2^{3/2}}\right)^l} \right)^j \frac{(-\sqrt{2w})^j}{j!},$$

where the coefficients  $c_l$  and  $d_l$  are defined in [1, Eq. (10.4.59) and (10.4.61)] by asymptotic expansions of  $\text{Ai}$  and  $\text{Ai}'$  for  $|z|$  large,  $|\arg(z)| < \pi$ :

$$\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k c_k \zeta^{-k}, \quad \text{Ai}'(z) \sim -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k d_k \zeta^{-k},$$

with  $\zeta := \frac{2}{3}z^{3/2}$ . More explicitly:  $c_0 = 1$ ,  $c_k = \Gamma(3k + 1/2)/(\Gamma(k + 1/2) \cdot 54^k k!)$ ,  $d_0 = 1$ ,  $d_k = -\frac{6k+1}{6k-1}c_k$ . The relation

$$[\eta^k] \sum_{j=1}^k (-1)^j b^{-3j} \frac{Z_j}{j!} w^{3j/2} \eta^j \left( \sum_{l=0}^k (-1)^l c_l \left( \frac{3\eta}{2^{3/2}} \right)^l \right)^k$$

$$[\eta^k] \sum_{j=1}^k \left( \frac{-\sqrt{2}}{z} \right)^j \frac{1}{j!} \left( \sum_{l=1}^k (-1)^l (d_l - c_l) \left( \frac{3\eta}{2^{3/2}} \right)^l \right)^j \left( \sum_{l=0}^k (-1)^l c_l \left( \frac{3\eta}{2^{3/2}} \right)^l \right)^{k-j}$$

provides an algorithm that can easily be implemented in Maple and proves more tractable than the general expressions of the moments given by Janson.

#### 4. Asymptotic Form of Density

**4.1. Asymptotics of  $f(x, b)$  as  $b \rightarrow \infty$ .** Using  $\mathbf{E}[W_b] \sim \frac{1}{2b}$  and  $\mathbf{Var}[W_b] \sim \frac{1}{4b^4}$  as  $b \rightarrow \infty$ , already mentioned in [4], asymptotics of  $(\log \text{Ai})'$  and  $(\log \text{Ai})''$ , and a saddle point method, we recover the fact that we obtain a density of a Gaussian distribution when  $b \rightarrow \infty$ .

**4.2. Asymptotics of  $\Theta(z, b)$  as  $|z| \rightarrow \infty$ .** Using a saddle point again, setting  $z = \kappa^6$ , we obtain

$$\Theta \sim e^{\kappa^3 \mu_1} e^{-\alpha_1 \kappa^4 / 2^*} \left( \frac{2^{1/2} \kappa^{3/2}}{2b^{3/4}} + \frac{b^{1/4} 2^{1/6} \alpha_1}{4\kappa^{1/2}} + \mathcal{O}\left(\frac{1}{\kappa^{3/2}}\right) \right).$$

**4.3. Asymptotics of  $f(x, b)$  as  $x \rightarrow 0$ .** The formula  $f(x, b) = \frac{1}{2\pi i} \Re \int_{c-i\infty}^{c+i\infty} e^{xz} \Theta(z, b) dz$ ,  $c > 0$ , the former asymptotics and a saddle point method lead to:

$$f(x, b) \sim e^{\mu_2/x^2} \frac{\sqrt{2}}{\sqrt{\pi}} \left( \frac{3^{1/4} \alpha_1^{9/4}}{9x^{11/4} b^{3/4}} - \frac{3^{3/4} \alpha_1^{3/4}}{3x^{9/4} b^{1/4}} + \frac{b^{1/4} 3^{1/4} (27 + 16\alpha_1^3)}{x^{7/4} \alpha_1^{3/4}} + \mathcal{O}\left(\frac{1}{x^{5/4}}\right) \right).$$

#### 5. Open Questions

It remains to find an asymptotic form for the density  $f(x, b)$  as  $x \rightarrow \infty$ —this not even known for the classical Airy density—and an explicit form for the density  $f(x, b)$ . Are also missing an analysis of the local time  $t^+(t, a)$  of  $B^+(t)$  at  $a$ , conditioned on its local time  $b$  at the origin, and some analytic variations on  $W_b$  (see [2] for the classical Airy distribution).

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