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Reflected Brownian Bridge Area Conditioned on its Local Time at the Origin

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Abstract

Using properties of the Airy functions, we analyze the reflected Brownian bridge area W_b conditioned on its local time b at the origin. We give a closed form expression of the Laplace transform of W_b, a recurrence equation for the moments, leading to an efficient computation algorithm and an asymptotic form for the density $f(x, b)$ of W_b for $x \to 0$.

1. Introduction

Let us first introduce the standard Brownian motion denoted by $x(t)$ and a few classical variants: the reflected Brownian motion $x^+(t) = \left|x(t)\right|$; the Brownian bridge $B(t)$; the reflected Brownian bridge $B^+(t)$ on $[0,1]$; the Brownian excursion $e(t)$.

The object of interest in this talk is $W_b := \int_0^1 B^+(t)\,dt$, the area of the reflected Brownian bridge conditioned on having a local time at the origin equal to b. This random variable appeared in [4] as the limit law for $m^{-3/2}D_{m,m-b\sqrt{m}}$, where $D_{m,m-b\sqrt{m}}$ denotes the total displacement for a hash table with m locations and $b\sqrt{m}$ empty locations, using linear probing. It also represents the limit law for the total height of random forests with $b\sqrt{m}$ trees and m nodes or leaves. The only description of it was given by its moments, related to the classical Airy function $\mathrm{Ai}(z) := \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{1}{3}t^3 + zt\right)\,dt$ (recall $\mathrm{Ai}'' = z\mathrm{Ai}$) in the following way:

$$\mathbf{E}\left[W_b^k\right] = k!\sum_{j=1}^{k}\left(\sum_{k_1,\dots,k_j\geq 1,\ \Sigma k_i = k}\ \prod_{i=1}^{j}\omega_{k_j}\right)\frac{b^{j-1}}{j!}q_{3k-j-2}(b),$$

where the ω_k are defined by the asymptotic expansion $\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)} \underset{z\to+\infty}{\sim} \sum_{k=0}^{+\infty}\omega_k \frac{(-1)^k z^{-3(k-1)/2}}{2^k}$, and $q_r(b) := \int_0^{+\infty}\frac{x^r}{r!}e^{-bx-x^2/2}\,dt$.

We will provide a closed form expression for the Laplace transform of W_b, a better way to compute its moments, and an asymptotic form for the density $f(x, b)$ of W_b when $x \to 0$.

2. Laplace Transform of W_b

Computing the Laplace transform of W_b essentially requires using Kac's formula [3] and a few technicalities. Eq. (30) in [5, p. 491] states that, if we denote by $t^+(t, a)$ the local time of $x(t)$ at a,

$$(1)\qquad \int_0^{\infty} e^{-\alpha t}\,\mathbf{E}_0\left[\exp\Bigl(-\int_0^t x^+(u)\,du - \delta t^+(t,0)\Bigr)\,\middle|\, x(t) = 0\right]\frac{dt}{\sqrt{2\pi t}} = \left(\delta - \frac{2^\star\mathrm{Ai}'(2^\star\alpha)}{\mathrm{Ai}(2^\star\alpha)}\right)^{-1},$$

where $2^\star := 2^{1/3}$. From it we can derive the following theorem:

Theorem 1. *The Laplace transform $\Theta(z,b)$ of W_b has the closed form expression*

$$\Theta(z,b)=\mathbf{E}\big[e^{-zW_b}\big]=\frac{-z^{1/3}e^{b^2/2}}{i\,2^{1/6}\sqrt{\pi}}\int_{-i\infty}^{i\infty}e^{bz^{1/3}2^{1/3}Ai'(u)/Ai(u)}(Ai'(u)/Ai(u))'e^{uz^{2/3}/2^{1/3}}\,du.$$

Proof. Given $[\int_0^t x^+(u)\,du\big|x(t)=0]\overset{\mathcal{D}}{\equiv}t^{3/2}Y$ and $t^+(t,0)\overset{\mathcal{D}}{\equiv}\sqrt{t}t^+(1,0)$ (scaling property), Eq. (1) leads to

$$\mathbf{E}_0\int_0^\infty e^{-\alpha t}\int_0^\infty e^{-t^{3/2}W_b}be^{-b^2/2}e^{-\delta\sqrt{t}b}\frac{db\,dt}{\sqrt{2\pi t}}=[\delta-2^\star\Lambda(\alpha)]^{-1},$$

where $\Lambda(\alpha):=\frac{\mathrm{Ai}'(2^\star\alpha)}{\mathrm{Ai}(2^\star\alpha)}$. The change of variable $v=\sqrt{t}b$ and an inversion on δ delivers

$$\int_0^\infty e^{-b^2/2}e^{-\alpha v^2/b^2}\mathbf{E}\left[e^{-v^3/b^3W_b}\right]\frac{2\,db}{\sqrt{2\pi}}=e^{v2^\star\Lambda(\alpha)}. \tag{2}$$

After setting $b=\frac{v}{\sqrt{2^\star\sigma}}$, $u=2^\star\alpha$, differentiating with respect to u and using $\left(\frac{\mathrm{Ai}'}{\mathrm{Ai}}\right)'=u-\left(\frac{\mathrm{Ai}'}{\mathrm{Ai}}\right)^2$:

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-u\sigma}\mathbf{E}\left[e^{-\sqrt{2}\sigma^{3/2}W_{v/\sqrt{2^\star\sigma}}}\right]e^{-v^2/(2^{4/3}\sigma)}\frac{d\sigma}{\sqrt{2\sigma}}=-e^{v2^\star\mathrm{Ai}'(u)/\mathrm{Ai}(u)}(\mathrm{Ai}'(u)/\mathrm{Ai}(u))'.$$

The inversion formula for Laplace transforms then writes:

$$\mathbf{E}\left[e^{-\sqrt{2}\sigma^{3/2}W_{v/\sqrt{2^\star\sigma}}}\right]e^{-v^2/(2^{4/3}\sigma)}/\sqrt{4\pi\sigma}=\frac{-1}{2\pi i}\int_{-i\infty}^{i\infty}e^{v2^\star\mathrm{Ai}'(u)/\mathrm{Ai}(u)}(\mathrm{Ai}'(u)/\mathrm{Ai}(u))'e^{u\sigma}\,du. \tag{3}$$

Now set $v=b\sqrt{2^\star\sigma}$, $z=\sqrt{2}\sigma^{3/2}$, $\Theta(z,b)=\mathbf{E}[e^{-zW_b}]$. Eq. (3) becomes

$$\frac{2^{1/6}\Theta(z,b)e^{-b^2/2}}{2\sqrt{\pi}}=\frac{-z^{1/3}}{2\pi i}\int_{-i\infty}^{i\infty}e^{bz^{1/3}2^\star\mathrm{Ai}'(u)/\mathrm{Ai}(u)}(\mathrm{Ai}'(u)/\mathrm{Ai}(u))'e^{uz^{2/3}/2^\star}\,du$$

which proves the theorem. □

3. Recurrence Formulae

Using Laplace transforms and inversions of Laplace transforms, we show here how to find an algorithm to compute the moments $\psi_k(b):=\mathbf{E}[W_b^k]$ by recurrence. We first need:

Lemma 1. *Define $G(\eta):=2^\star\Lambda(\alpha)/\sqrt{\alpha}$ and $s=1/b^2$; we have*

$$\int_0^\infty e^{-1/(2s)}e^{-ws}(-1)^ks^{3/2k}\psi_k(b)\frac{ds}{s^{3/2}\sqrt{2\pi}k!}=[\eta^k]\frac{e^{\sqrt{w}G_0}}{w^{3/2k}}\sum_{i=1}^\infty\frac{\left(\sqrt{w}\big(G(\eta)-G_0\big)\right)^i}{i!}. \tag{4}$$

Proof. Set $s:=1/b^2$, $w=\alpha v^2$, and $\eta=\alpha^{-3/2}$. Eq. (2) becomes

$$\int_0^\infty e^{-1/(2s)}e^{-ws}\mathbf{E}\left[e^{-\eta w^{3/2}s^{3/2}W_b}\right]\frac{ds}{s^{3/2}\sqrt{2\pi}}=e^{\sqrt{w}G(\eta)},$$

Set $G_0:=G(0)$. Eq. (3) leads to

$$\int_0^\infty e^{-1/(2s)}e^{-ws}\mathbf{E}\left[e^{-\eta w^{3/2}s^{3/2}W_b}-1\right]\frac{ds}{s^{3/2}\sqrt{2\pi}}=e^{\sqrt{w}G(\eta)}-e^{\sqrt{w}G_0}$$

$$=e^{\sqrt{w}G_0}\sum_{i=1}^\infty\frac{\left(\sqrt{w}\big(G(\eta)-G_0\big)\right)^i}{i!}. \tag{5}$$

Upon expanding both sides of (5) with respect to η, this gives the desired formula. □

To invert the Laplace transforms of the form $e^{-\sqrt{2w}}/w^{(j+1)/2}$, we will use the following lemmas:

Lemma 2. *Set* $\phi^{(1)}(x) := \phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$ *(classical Gaussian distribution function) and* $\phi^{(j+1)}(x) := \int_{-\infty}^{x} \phi^{(j)}(u)\, du$. *Then*

$$\int_0^\infty \phi^{(j)}(-b) e^{-ws} \frac{(2s)^{(j+1)/2}}{s}\, ds = \frac{e^{-\sqrt{2w}}}{w^{(j+1)/2}}, \quad j \geq 1, \qquad \textit{where } b = 1/\sqrt{s}.$$

Sketch of proof. Ones proves the lemma by induction and uses an integration by part and an integration with respect to w to prove it at rank $k+1$ from rank k. □

Lemma 3. *The* $\phi^{(j)}(x)$ *can be expressed in the form:*

$$\phi^{(k)}(z) = p_1(k,z)\phi(z) + p_2(k,z) e^{-z^2/2}/\sqrt{2\pi},$$

where $p_1(k,z)$ *is of degree* $k-1$, $p_2(k,z)$ *is of degree* $k-2$.

Using integration by parts on $\int_{-\infty}^{z} x^j \phi(x)\, dx$ and identification of coefficients, it is possible to prove the following proposition, enabling us to compute nice expressions of the $\phi^{(j)}(x)$:

Proposition 1. *Define, for* $k \geq 1$, $j \geq 0$, $P_1[k,j] := [z^j] p_1(k,z)$, *and* $P_2[k,j] := [z^j] p_2(k,z)$. *Then the sequences* $(P_1[k,j])_{k\geq 1, j\geq 0}$ *and* $(P_1[k,j])_{k,j\geq 0}$ *are defined by the initial values* $P_1[1,0] = 1$, $P_2[1,0] = 0$, $P_1[1,j] = P_2[1,j] = 0$ *for* $j \geq 1$, *and the recurrence relations, for* $k \geq 1$*:*

$$\begin{aligned}
P_1[k+1,j] &:= P_1[k,j-1]/j, \quad j = 1,\dots,k,\\
P_2[k+1,j] &:= \sum_{l=0}^{\lfloor (k-1-j)/2 \rfloor} P_1[k,j+2l]/(j+2l+1)(j+2l+1)_l\\
&\quad - \sum_{l=0}^{\lfloor (k-3-j)/2 \rfloor} P_2[k,j+2l+1](j+2l+1)_l, \quad j = 0,\dots,k-1,\\
P_1[k+1,0] &:= -\sum_{l=1,3,\dots,k-1} P_1[k,l]/(l+1)(l+1)_{(l+1)/2} + \sum_{l=0,2,\dots,k-2} P_2[k,l](l)_{l/2}.
\end{aligned}$$

Determining a recurrence relation for the moments $\psi_k(b)$ hence amounts to determining a recurrence relation for the Z_j defined by (see (4)):

$$(-1)^j b^{-3j} \frac{Z_j}{j!} = [\eta^j] \frac{1}{w^{3/2j}} \sum_{i=1}^{\infty} \frac{\left(\sqrt{w}\left(G(\eta) - G_0\right)\right)^i}{i!}.$$

Indeed, along the mechanical transfer rule $\frac{1}{w^{(l+1)/2}} \to \frac{\phi^{(l)}(-b)}{b^{l+1}} b^2 2^{(l+1)/2}$, $\psi_j(b)$ is equivalent to $Z_j \sqrt{2\pi} e^{b^2/2}/b^3$. To get a recurrence formula giving Z_k in function of the $Z_1,\dots,Z_j$, we introduce

$$S_k(\eta) := \sum_{j=1}^{k} (-1)^j b^{-3j} \frac{Z_j}{j!} w^{3j/2} \eta^j = \sum_{j=1}^{k} \eta^j [\eta^j] \left(\frac{\sum_{l=1}^{k} (-1)^l (d_l - c_l) \left(\frac{3\eta}{2^{3/2}}\right)^l}{\sum_{l=0}^{k} (-1)^l c_l \left(\frac{3\eta}{2^{3/2}}\right)^l} \right)^j \frac{(-\sqrt{2w})^j}{j!},$$

where the coefficients c_l and d_l are defined in [1, Eq. (10.4.59) and (10.4.61)] by asymptotic expansions of Ai and Ai′ for $|z|$ large, $\left|\arg(z)\right| < \pi$:

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k c_k \zeta^{-k}, \quad \mathrm{Ai}'(z) \sim -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k d_k \zeta^{-k},$$

with $\zeta := \frac{2}{3}z^{3/2}$. More explicitly: $c_0 = 1$, $c_k = \Gamma(3k + 1/2)/\big(\Gamma(k + 1/2) \cdot 54^k k!\big)$, $d_0 = 1$, $d_k = -\frac{6k+1}{6k-1}c_k$. The relation

$$[\eta^k] \sum_{j=1}^{k} (-1)^j b^{-3j} \frac{Z_j}{j!} w^{3j/2} \eta^j \left(\sum_{l=0}^{k} (-1)^l c_l \left(\frac{3\eta}{2^{3/2}} \right)^l \right)^k$$

$$[\eta^k] \sum_{j=1}^{k} \left(\frac{-\sqrt{2}}{z} \right)^j \frac{1}{j!} \left(\sum_{l=1}^{k} (-1)^l (d_l - c_l) \left(\frac{3\eta}{2^{3/2}} \right)^l \right)^j \left(\sum_{l=0}^{k} (-1)^l c_l \left(\frac{3\eta}{2^{3/2}} \right)^l \right)^{k-j}$$

provides an algorithm that can easily be implemented in Maple and proves more tractable than the general expressions of the moments given by Janson.

4. Asymptotic Form of Density

4.1. **Asymptotics of $f(x,b)$ as $b \to \infty$.** Using $\mathbf{E}[W_b] \sim \frac{1}{2b}$ and $\mathbf{Var}[W_b] \sim \frac{1}{4b^4}$ as $b \to \infty$, already mentioned in [4], asymptotics of $(\log \mathrm{Ai})'$ and $(\log \mathrm{Ai})'$, and a saddle point method, we recover the fact that we obtain a density of a Gaussian distribution when $b \to \infty$.

4.2. **Asymptotics of $\Theta(z,b)$ as $|z| \to \infty$.** Using a saddle point again, setting $z = \kappa^6$, we obtain

$$\Theta \sim e^{\kappa^3 \mu_1} e^{-\alpha_1 \kappa^4 / 2^\star} \left(\frac{2^{1/2}\kappa^{3/2}}{2b^{3/4}} + \frac{b^{1/4} 2^{1/6} \alpha_1}{4\kappa^{1/2}} + \mathcal{O}\left(\frac{1}{\kappa^{3/2}} \right) \right).$$

4.3. **Asymptotics of $f(x,b)$ as $x \to 0$.** The formula $f(x,b) = \frac{1}{2\pi i} \Re \int_{c-i\infty}^{c+i\infty} e^{xz} \Theta(z,b)\, dz$, $c > 0$, the former asymptotics and a saddle point method lead to:

$$f(x,b) \sim e^{\mu_2/x^2} \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{3^{1/4}\alpha_1^{9/4}}{9x^{11/4} b^{3/4}} - \frac{3^{3/4}\alpha_1^{3/4}}{3x^{9/4} b^{1/4}} + \frac{b^{1/4} 3^{1/4} (27 + 16\alpha_1^3)}{x^{7/4} \alpha_1^{3/4}} + \mathcal{O}\left(\frac{1}{x^{5/4}} \right) \right).$$

5. Open Questions

It remains to find an asymptotic form for the density $f(x,b)$ as $x \to \infty$—this not even known for the classical Airy density—and an explicit form for the density $f(x,b)$. Are also missing an analysis of the local time $t^+(t,a)$ of $B^+(t)$ at a, conditioned on its local time b at the origin, and some analytic variations on W_b (see [2] for the classical Airy distribution).

Bibliography

[1] Abramowitz (Milton) and Stegun (Irene A.) (editors). – *Handbook of mathematical functions, with formulas, graphs, and mathematical tables.* – Dover Publications Inc., New York, 1966, xiv+1046p.

[2] Flajolet (P.) and Louchard (G.). – Analytic variations on the Airy distribution. *Algorithmica*, vol. 31, n° 3, 2001, pp. 361–377. – Mathematical analysis of algorithms.

[3] Itô (Kiyosi) and McKean, Jr. (Henry P.). – *Diffusion processes and their sample paths.* – Springer-Verlag, Berlin, 1974, xv+321p. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

[4] Janson (Svante). – Asymptotic distribution for the cost of linear probing hashing. *Random Structures & Algorithms*, vol. 19, n° 3-4, 2001, pp. 438–471.

[5] Louchard (G.). – Kac's formula, Levy's local time and Brownian excursion. *Journal of Applied Probability*, vol. 21, n° 3, 1984, pp. 479–499.