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Enumeration of Sand Piles

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Summary by Michel Nguyễn-Thế

Abstract

Sand piles are integer partitions that can be obtained from a column of n grains by moving grains from left to right according to rules defined by a model. We try to better understand the structure of those objects by decomposing and counting them. For the model introduced by Goles, Morvan, and Phan, we fing generating functions according to area, height, and width. We establish a bound for the number of the sand piles consisting of n grains in IPM(k) for large n. We present the series according to area and height for Phan's model $L(\theta)$. We introduce a more general model, where grains can also go to the left, that we call Frobenius sand piles. (Joint work of S. Corteel with D. Gouyou-Beauchamps (LRI, Orsay)).

1. Preliminaries and SPM(k) Model

After the necessary basic concepts, we present here the simplest model of sand pile, i.e., the SPM(k) model, from which all other models are derived.

1.1. **Definitions.** A *sand pile* made of n grains is a partition of the integer n. A *partition* of an integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_l)$. The λ_i are called the *parts* of the partition. The *area* of the sand pile is the sum $|\lambda| = \lambda_1 + \cdots + \lambda_l = n$. The *height* of the sand pile is the number $h(\lambda) = l$ of parts of the partition. For any partition λ, we will consider that $\lambda_i = 0$ for $i < 1$ and $i > h(\lambda)$. The *width* $w(\lambda)$ of the sand pile λ is the largest part λ_1. The *Ferrers diagram* of a partition λ is a drawing of λ such that the ith column is a pile of λ_i packed squares (called grains). The rows are labelled from bottom to top. The *conjugate* λ' of λ is the partition whose ith part is the number of squares in the ith column of the Ferrers diagram of λ.

Let $\pi = (\pi_1, \dots, \pi_l)$ be a sand pile and $\pi' = \left(\pi'_1, \dots, \pi'_{\pi_1}\right)$ be its conjugate. The moves of the sand grains are of two types (see Figure 1):

1. *Vertical rule*: a grain can move from column i to column $i+1$ if $\pi'_i - \pi'_{i+1} \le 2$, so that
$$\left(\pi'_1, \cdots, \pi'_{\pi_1}\right) \quad \text{is replaced with} \quad \left(\pi'_1, \cdots, \pi'_i - 1, \pi'_{i+1} + 1, \cdots, \pi'_{\pi_1}\right).$$
2. *Horizontal rule*: a grain can move from column i to column j if $j > i+1$ and $\pi'_i - 1 = \pi'_{i+1} = \cdots = \pi'_j = \pi'_{j+1} + 1$, so that
$$\left(\pi'_1, \cdots, \pi'_{\pi_1}\right) \quad \text{is replaced with} \quad \left(\pi'_1, \cdots, \pi'_i - 1, \pi'_{i+1}, \cdots, \pi'_j, \pi'_{j+1} + 1, \cdots, \pi'_{\pi_1}\right).$$

The shift is said to have length 0 or or $j - i - 1$, respectively.

In the SPM(k) model (*Sand Pile Model*), introduced by Goles and Kiwi [3], the initial configuration is made of one column of n grains, and the only available rule is the vertical rule.

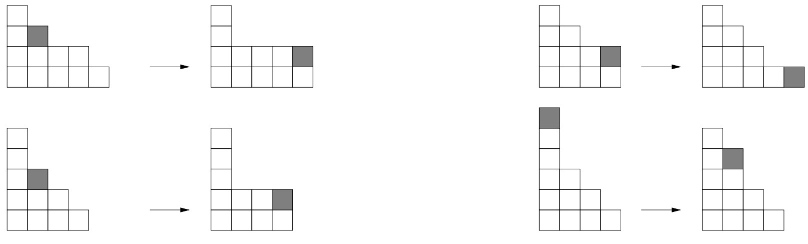

FIGURE 1. (a) Application of horizontal rule to (5,4,2,1) and (4,3,2,1,1); (b) application of vertical rule to (4,4,2,1) and (4,3,2,1,1,1).

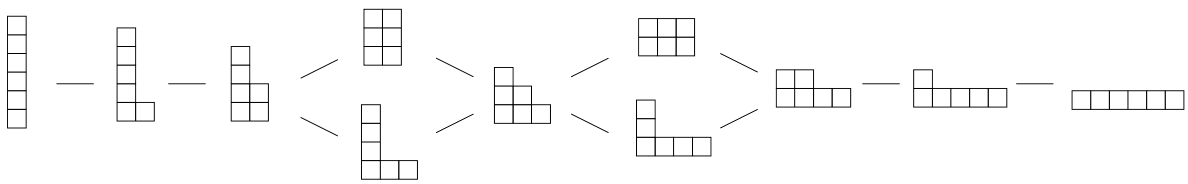

FIGURE 2. $L_B(6)$

1.2. **Generating function.** Let $p(n,k)$ denote the number of partitions of n of width k. Then:

$$F(q,x) = 1 + \sum_{n,k\geq 1} p(n,k) q^n x^k = xqF(q,x) + F(q,xq) = \prod_{i=1}^{\infty} \frac{1}{1-xq^i}.$$

1.3. **Example of bijection.** There is a bijection between partitions with odd parts and partitions with distinct parts, as is reflected by the generating functions identity

$$\prod_{i=1}^{\infty} \frac{1}{1-q^{2i+1}} = \prod_{i=1}^{\infty} \frac{1-q^{2i}}{1-q^i} = \prod_{i=1}^{\infty} \left(1+q^i\right).$$

1.4. **Order on partitions.** Let $\mu = (\mu_1, \mu_2, \dots)$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ be two partitions of n. We say that $\mu \geq \lambda$ if and only if there exists a sequence of moves of n induced by the rules to go from μ to λ. In the SPM(k) model, this order is equivalent to the dominance order $L_B(n)$ [1] on the conjugates: $\mu \geq \lambda$ if and only if $\sum_{i=1}^{j} \mu'_i \geq \sum_{i=1}^{j} \lambda'_i$ for all $j \geq 1$. Brylawski [1] showed:

Theorem 1. *Let n be an integer. The set of partitions of n with the previously defined order is a lattice, where the maximal element is $(1,1,\dots,1)$, and the minimal element is (n). Moreover, the infimum and the supremum of two partitions can be respectively defined as follows:*

1. $\inf(\mu,\lambda) = \pi$ *such that* $\pi'_j = \min\left(\sum_{i=1}^{j} \mu'_i, \sum_{i=1}^{j} \lambda'_i\right) - \sum_{i=1}^{j} \pi'_i$ *for all* $j \geq 1$.
2. $\sup(\mu,\lambda) = \alpha$ *such that* $\alpha'_j = \max\left(\sum_{i=1}^{j} \mu'_i, \sum_{i=1}^{j} \lambda'_i\right) - \sum_{i=1}^{j} \alpha'_i$ *for all* $j \geq 1$.

In Figure 2, the maximal element $(1,1,1,1,1,1)$ is on the left.

Length of a maximal chain. The length of a maximal chain is greater than $2n-3$ [1], and smaller than $2\binom{l+1}{3} + lj + 1$ [3], where l and j are defined by $n = j + l(l+1)/2$ and $0 \leq j \leq l$. For $n = 6$, the two bounds are equal to 9, which shows that they both can be attained. The corresponding maximal chain is displayed in Figure 2.

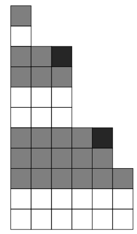

FIGURE 3. Decomposition of a sand pile in IPM(1).

2. IPM(k) Model

A more realistic generalization of SPM(k) model limits the lengths of the possible horizontal shifts of a grain.

2.1. **Definition.** In [4], the sand piles in IPM(k) are characterized in the following way:

Proposition 1. *A sand pile in* IPM(k) *is a partition* $\pi = (\pi_1, \dots, \pi_l)$ *of* n *such that*

- *for* $1 \le i \le l$, $0 \le \pi_i - \pi_{i+1} \le k+1$;
- *for any* $i < j$ *with* $\pi_i - \pi_{i+1} = k+1$ *and* $\pi_j - \pi_{j+1} = k+1$, *there exists* z *with* $i < z < j$ *such that* $\pi_z - \pi_{z+1} < k$.

2.2. **Generating functions.**

2.2.1. *Area and height.*

Theorem 2. *The generating function* $S_k(q,x)$ *of* IPM(k) *sand piles, with* q *and* x *respectively counting area and height, satisfies*

$$S_k(q,x) = 1 + \sum_{\pi \in \mathrm{IPM}(k)} x^{l(\pi)} q^{|\pi|} = 1 + \sum_{i=1}^{k} \frac{xq^i}{1-xq^i} S_k(q, xq^i) + xq^{k+1} S_k(q, xq^k).$$

Proof. A sand pile in IPM(k) is either the empty partition, or a partition in IPM(k) where one duplicates i times the highest column and adds to it at least one part i ($1 \le i \le k$), or a partition in IPM(k) where one duplicates k times the highest column and adds to it one part of length $k+1$. This decomposition yields the last expression for $S_k(q,x)$ in the statement of the theorem, after noting that $S_k(q, xq^r)$ is the generating function obtained by duplicating r times the highest column in each sand pile. □

Note the particular cases:

$$S_1(q,x) = 1 + \sum_{n \ge 1} x^n q^{n(n+1)/2} \prod_{i=1}^{n} \left(q + \frac{1}{1-xq^i} \right); \qquad S_\infty(q,x) = \prod_{i=1}^{n} \left(q + \frac{1}{1-xq^i} \right).$$

2.2.2. *Area and width.*

Theorem 3. *The generating function* $\mathcal{S}_k(q,y)$ *of* IPM(k) *sand piles, with* q *and* y *respectively counting area and width, satisfies:*

$$\mathcal{S}_k(q,y) = \left(\frac{1-(yq)^{k+1}}{1-yq} + y^k q^{k-1} \right) \mathcal{S}_k(q,yq) + y^k q^{k-1} \left(\mathcal{S}_k(q,yq) - \mathcal{S}_k(q,yq^2) \right).$$

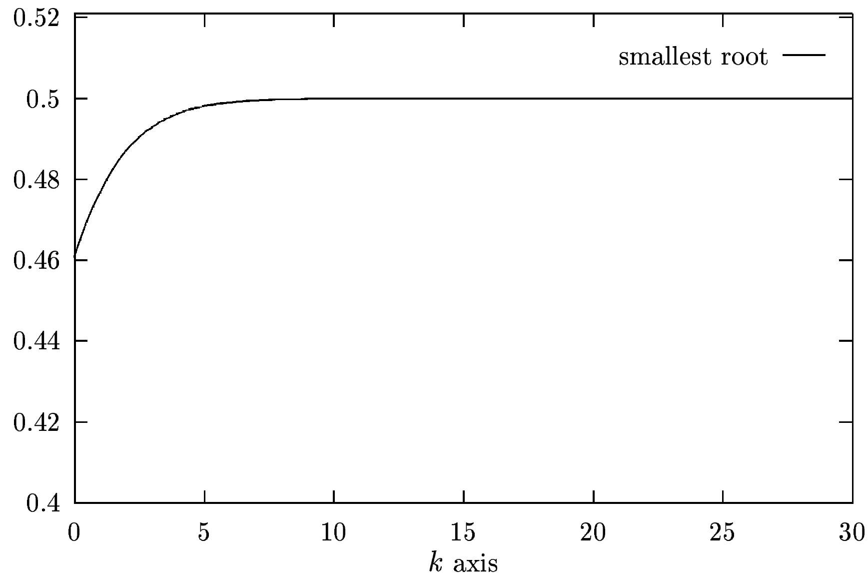

FIGURE 4. Evolution of the smallest root of the polynomial $1-2x-3x^{k+2}+x^{k+3}+x^{k+1}$.

In particular:

$$\mathcal{S}_1(q,y) = 1 + \sum_{n\geq 1} y^n q^{n(n-1)/2} \prod_{i=1}^{n} \frac{1+q-q^{i-1}}{1-q^i}.$$

2.2.3. *Height and width.* Let $p_k(h,w)$ be the number of sand piles in IPM(k) of height h and width w and $P_{k,h}(y)$ the generating function $\sum_{w\geq 0} p_k(h,w)y^w$.

Theorem 4. *The generating function $P_{k,h}(y)$ follows the recurrence:*

$$P_{k,0}(y) = 1; \qquad P_{k,1}(y) = y\frac{1-y^{k+1}}{1-y}; \qquad P_{k,2}(y) = y\frac{1-y^{k+1}}{1-y}\frac{1-y^{k+2}}{1-y} - y^{2(k+1)};$$

$$P_{k,h}(y) = \left(\frac{1-y^{k+1}}{1-y} + y^k\right) P_{k,h-1}(y) - y^k P_{k,h-2}(y) \qquad \text{for } h \geq 3.$$

Now, let $\mathcal{P}_k(x,y)$ be the width (variable x) and the height (variable y) generating function $\sum_{h\geq 0} P_{k,h}(y)x^h$. From the previous recurrence one gets:

Theorem 5. *The generating function $\mathcal{P}_k(x,y)$ is given by:*

$$\mathcal{P}_k(x,y) = \frac{1-x(y^k+1-y^{k+1})+x^2y^k(1-y)}{1-x\left(\frac{1-y^{k+1}}{1-y}+y^k(1-x)\right)}.$$

Let $p_k(n)$ be the number of sand piles in IPM(k) of half perimeter (width + height) n, and $\mathcal{P}_k(x) = \sum_{n\geq 0} p_k(n)x^n$ be its generating function. As $\mathcal{P}_k(x) = \mathcal{P}_k(x,x)$, its expression is:

$$\mathcal{P}_k(x) = \frac{(1-x)^2(1-x^{k+1}+x^{k+2})}{1-2x-3x^{k+2}+x^{k+3}+x^{k+1}}.$$

When k grows, the quantity $p_k(n)$, asymptotically equal for large n to c_k/ρ_k^n with $c_k \in \mathbb{R}$ and ρ_k the smallest root of the denominator $1-2x-3x^{k+2}+x^{k+3}+x^{k+1}$, gets closer to 2^n, the number of partitions of semi-perimeter n.

2.3. **Asymptotics.** Define

$$p_k = [q^n] \prod_{i \geq 1} \frac{1 - q^{ki}}{1 - q^i}, \qquad B_k = \sqrt{\frac{k-1}{6k}}, \qquad \text{and} \qquad C_k = \frac{1}{2}\left(\frac{k-1}{6k^3}\right)^{1/4}.$$

Then $p_k(n) = C_k n^{-3/4} \exp\left(B_k n^{1/2}\right) O\left(1 + n^{-1/4}\right)$. If $I_k(n)$ is the number of partitions of n in IPM(k), then $p_{k+1}(n) \leq I_k(n) \leq p_{k+2}(n)$.

3. The Model $L(\theta)$

The model $L(\theta)$ generalizes the SPM(k) by restricting its vertical rule, instead of the horizontal rule as IPM(k). Namely, the difference between the two consecutive columns involved must be greater then θ.

3.1. **Definition.** In [4], the sand piles in $L(\theta)$ are characterized in the following way:

Proposition 2. *A sand pile in $L(\theta)$ is a partition $\pi = (\pi_1, \ldots, \pi_l)$ of n such that*
- *for $1 \leq \pi_1$, $\pi'_i - \pi'_{i+1} \geq \theta - 1$;*
- *for any $i < j$ with $\pi'_i - \pi'_{i+1} = \theta - 1$ and $\pi'_j - \pi'_{j+1} = \theta - 1$, there exists z with $i < z < j$ such that $\pi_z - \pi_{z+1} > \theta$.*

Let $\mathcal{L}_\theta(q, x) = 1 + \sum_{\pi \in L(\theta)} x^{l(\pi)} q^{|pi|}$ the generating function of sand piles in $L(\theta)$ according to their height and area.

Lemma 1. *$\mathcal{L}_\theta(q, x)$ satisfies the q-equation:*

$$\mathcal{L}_\theta(q, x) = \frac{1 - (xq)^{\theta - 1}}{1 - xq} + \left(\frac{(xq)^{\theta - 1}}{1 - xq} + x^{\theta - 1} q^\theta\right) \mathcal{L}_\theta(q, xq).$$

Theorem 6. *$\mathcal{L}_\theta(q, x)$ is given by:*

$$\mathcal{L}_\theta(q, x) = \sum_{n \geq 0} x^{\theta n} q^{\theta n(n+1)/2} \frac{1 - (xq^{n+1})^\theta}{1 - xq^{n+1}} \prod_{i=1}^{n} \left(q + \frac{1}{1 - xq^i}\right).$$

Bounds can be obtained for all θ for the number $l_{n,\theta}$ of partitions in $L(n, \theta)$.

4. Frobenius Model

Another generalization consists in allowing the grains to move both to the left and to the right. In [2], Corteel defines such a model, called the *Frobenius sand pile*, in the following way:

Definition 1. Let l be an integer. A *Frobenius sand pile* is a pair consisting of a pivot indice $p(a) \leq l$ and a sequence of integers $(a_1, a_2, \ldots, a_l)$ such that

$$a_1 \leq a_2 \leq \cdots \leq a_{p(a)} \geq a_{p(a)+1} \geq \cdots \geq a_l.$$

4.1. **Order on Frobenius sand piles.**

Definition 2. Let $a = \big(p(a), (a_1, a_2, \ldots, a_l)\big)$ and $b = \big(p(b), (b_1, b_2, \ldots, b_l)\big)$ be two Frobenius sand piles. Then $a \geq_F b$ if and only if, for all $i, j \geq 0$,

$$\sum_{l=p(a)-i}^{p(a)+j} a_l \geq \sum_{l=p(b)-i}^{p(b)+j} b_l.$$

Proposition 3. *Let $L_F(n)$ be the set of Frobenius partitions ordered by $\geq_F$. Then $L_B(n)$ is a suborder of $L_F(n)$.*

Length of a maximal chain. For $n \geq 3$, the length of a maximal chain is greater than $2n-4$, and smaller than $2\binom{l+1}{3} + lj + 1$, where l and j are defined by $n = j + l(l+1)/2$ and $0 \leq j \leq l$.

Definition 3. Let $a = \big(p(a), (a_1, a_2, \ldots, a_l)\big)$ be a sand pile. $a_<$, $a_>$, $a_\leq$, and $a_\geq$ are defined by

$$a_< = \big(a_{p(a)-1}, a_{p(a)-1}, \ldots, a_1\big)\,; \qquad a_> = \big(a_{p(a)+1}, a_{p(a)+2}, \ldots, a_l\big)\,;$$
$$a_\leq = \big(a_{p(a)}, a_{p(a)-1}, \ldots, a_1\big)\,; \qquad a_\geq = \big(a_{p(a)}, a_{p(a)+1}, \ldots, a_l\big)\,.$$

If we constrain horizontal shifts to be smaller than k, we can create an increasing sequence of orders IFPM(k) with the relations of order $\geq_k$. The Frobenius sand piles of IFPM(k) are characterized by:

Proposition 4. *Let $a = \big(p(a), (a_1, a_2, \ldots, a_l)\big)$ be a sand pile. This sand pile belongs to* IFPM(k) *if and only if both of $a_<$ and $a_>$, and at least one of $a_\leq$ and $a_\geq$ belong to* IPM(k).

4.2. **Generating functions.** The only available generating function is the series of F-partitions given by

$$1 + \sum_{k \geq 1} q^k \prod_{i=1}^{k} \frac{1}{(1-q^i)^2}.$$

For IFPM(k), we must so far satisfy ourselves with the bound $F_k(n) \leq \big|\mathrm{IFPM}(n,k)\big| \leq F_{k+1}(n)$ for

$$F_k(n) = [q^n]\left(1 + \sum_{j \geq 1} q^j \prod_{i=1}^{j} \frac{1 - q^{(k+1)i}}{(1-q^i)^2}\right).$$

5. Conclusion and Open Questions

We have studied different sand pile models related to integer partitions, and in particular we have computed generating functions and asymptotic bounds. A question of interest would consist in getting exact asymptotics instead of asymptotic bounds only. One could start with the area generating function in the SPM case, given by

$$\sum_{n \geq 0} x^n q^{n(n+1)/2} \prod_{i=1}^{n} \left(q + \frac{1}{1-q^i}\right).$$

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