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A Criterion for Non-Complete Integrability of Hamiltonian Systems

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Summary by Philippe Dumas and Bruno Salvy

Abstract

Finding polynomial solutions of linear differential equations is a building block implemented in several algorithms of computer algebra systems. In particular, this is a necessary sub-step when looking for rational, algebraic or Liouvillian solutions of linear differential equations. When there are no parameters, several algorithms are available, but the general case with parameters is undecidable. However, special families can be handled by *ad hoc* methods. Such methods were developed by Boucher who applied them to the nice example of integrability of the 3-body problem. The key idea there is to rely on a recent result of Morales-Ruiz and Ramis who relate complete integrability and differential Galois group. It turns out that special properties of this group can be related to computable properties of an appropriate linear differential equation, which leads Boucher to a "simple" sufficient condition for non-complete integrability.

1. Polynomial Solutions of Linear Differential Equations

The classical method to find polynomial solutions of linear differential equations over $\mathbb{K}(x)$, where $\mathbb{K}$ is a field, starts by determining a bound on the degree of potential solutions. This is a bound on the integer solutions of the *indicial equation* at infinity.

Once a bound on the degree has been found, one uses an indeterminate coefficients method. The linear system on these coefficients has a band-matrix structure which can be exploited to accelerate the computation [1]. This linear system is rectangular, with more equations than unknown coefficients, thus existence of solution is related to the vanishing of a determinant.

When parameters occur in the equation ($\mathbb{K}$ is a field of rational functions), there are two difficulties: the size of the matrix may depend on the parameters and even when it does not, the determinant which must vanish is a polynomial in the parameters. Using Matijasevich's result on the undecidability of Hilbert's 10th problem (*Is there a finite process which determines if a polynomial equation is solvable in integers?*), it is possible to show that this problem itself is undecidable. More precisely, Jacques-Arthur Weil observes that the equation

$$y'(x) - \left(\frac{a_1}{x-1} + \cdots + \frac{a_m}{x-m} + P(a_1, \ldots, a_m)\right) y(x) = 0$$

has rational solutions if and only if $P(a_1, \ldots, a_m) = 0$ has integral solutions.

There are still cases where all polynomial solutions can be found: this happens when either the size of the matrix is bounded independently of the parameters and the vanishing of the required determinant can be determined or when the structure of the matrix is sufficiently regular to make the decision possible. Examples of both cases are given in [4].

2. Complete Integrability

2.1. Hamiltonian Mechanics. In the Hamiltonian approach to classical mechanics, the state of a system is characterized by $2n$ variables, q_i (positions) and p_i (momenta), $i = 1, \dots, n$, living in an open subset U of $\mathbb{R}^{2n}$ (the phase space). More generally, the phase space of a system is the cotangent fibre bundle $T^\star M$ of an n-dimensional real manifold M. The formulae we give below are expressions in a chart of useful quantities. The state variables satisfy

$$\dot{p}_i = \frac{\partial H}{\partial q_i}, \quad \dot{q}_i = -\frac{\partial H}{\partial p_i}, \tag{1}$$

where a dot denotes a derivative with respect to time and $H(p, q, t)$ is the *Hamiltonian.* Physically, the Hamiltonian often represents the energy of the system. The system (1) governs the evolution of the system (in the phase space U). Solutions $\gamma(t)$ of (1) are the trajectories of the system.

In a more abstract setting, $\mathbb{R}^{2n}$ is endowed with a non-degenerate 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

known as Liouville's symplectic 2-form. Since ω is non-degenerate, it induces an isomorphism between $\mathbb{R}^{2n}$ and its dual under which $-dH$ is the image of a vector field X_H. In this language, the Hamiltonian system (1) reduces to

$$\dot{\gamma} = X_H(\gamma).$$

First integrals are functions $F(p, q)$ that are constant along the solutions $\gamma(t)$. A necessary and sufficient condition is

$$\{F, H\} := \sum_i \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} = 0,$$

where $\{F, H\}$ is known as the *Poisson bracket* of F and H. In particular, the Hamiltonian itself is a first integral.

Two first integrals are *in involution* if their Poisson bracket vanishes. A Hamiltonian system is *completely integrable* when it possesses a set of n first integrals in involution that are independent (i.e., their Jacobian matrix is regular in the open set U).

Informally, a completely integrable system can be "solved" in terms of its first integrals. Indeed, given a first integral, a process known as *symplectic reduction* makes it possible to reduce the number of degrees of freedom by 1, i.e., the dimension by 2 [2, p. 91].

2.2. Many-Body Problem. In the many-body problem, n particles obeying Newton's law are governed by the following Hamiltonian:

$$H(p, q) = \frac{1}{2}\sum_i \frac{\|p_i\|^2}{m_i} - \sum_{i \neq j} \frac{m_i m_j}{\|q_j - q_i\|}.$$

Note that here each p_i and q_i has coordinates in $\mathbb{R}^3$, thus the phase space has dimension $6n$.

Apart from the Hamiltonian itself, known first integrals for this system are the momentum of the centre of mass and the angular momentum $\sum q_i \wedge p_i$. Thus, the number of degrees of freedom can be reduced from $3n$ to $3n - 6$ (or from $2n$ to $2n - 4$ in the planar case).

For the 3-body problem, Poincaré proved that there are no other *complex analytic* first integrals. Bruns proved a similar result for *complex algebraic* first integrals.

2.3. **Theorem of Morales-Ruiz and Ramis.** We now present a simple version of a result of Morales-Ruiz and Ramis in [10, 11, 12, 13] (see also [3]) on non-complete integrability in terms of *meromorphic* first integrals. The Hamiltonian is analytic over an open set of $\mathbb{C}^{2n}$ and t (time) is a *complex* variable. Given a non-stationary trajectory $\Gamma(t)$, following an idea of Poincaré, one considers the *linear* differential equation that must satisfy a "small" variation η, such that $\Gamma(t)+\eta(t)$ is solution of the Hamiltonian system. This equation

$$\dot{\eta} = X'_H(\gamma) \cdot \eta \tag{2}$$

is called the *variational equation* along Γ. A theorem of Morales-Ruiz and Ramis relates complete integrability and Galois group of this equation. (For an introduction to differential Galois theory, see [14] or the summary of Ulmer's talk in this seminar in 1994.) However, since the Galois group is often very difficult to compute, it is useful to consider a differential equation of lower order. This is achieved by the following result.

Theorem 1 (Morales-Ruiz and Ramis)**.** *If the system possesses n meromorphic first integrals in the neighbourhood of Γ, independent and in involution, then the connected component of identity in the differential Galois group of the normal variational equation along Γ is abelian.*

Similar earlier results of Ziglin based on the monodromy group and of Churchill, Singer *et alii* based on the Galois group did not extend to the case where the variational equation has an irregular singular point. In this theorem, the *normal* variational equation is an equation obtained from the variational equation through symplectic reduction. Indeed, $dH\big(\Gamma(t)\big) \cdot \eta$ is a first integral of the variational equation, as can be seen by a first-order expansion.

3. Boucher's Criterion and its Application

It is not necessary to compute the Galois group of a linear differential equation in order to detect that it is not abelian. Thanks to a sufficient criterion [5, 6], Boucher has proved that the planar 3-body problem is not completely integrable in terms of meromorphic first integrals. Unfortunately, the formulae involved in this derivation are much too large to be reproduced here. Thus we content ourselves with a sketch of the steps and a description of the tools used in the calculations.

3.1. Criterion.

Theorem 2. *Assume that the linear differential operator L can be factored as KM, with $M = \operatorname{lcm}(L_1, \dots, L_m)$ where the L_i, $i = 1, \dots, m$, are irreducible (and* lcm *denotes the least common left multiple). Assume moreover that $M(y) = 0$ has a formal solution with a logarithm. Then the connected component of the differential Galois group of $L(y) = 0$ is not abelian.*

Given a linear differential equation, this theorem reduces the task to factoring and finding formal solutions. Factoring can be done by an algorithm of van Hoeij [19, 20], and formal solutions can be computed at any singularity, including infinity [15, 20].

3.2. **Application to the 3-Body Problem.** Tsygvintsev and Boucher have proved independently that the planar 3-body problem is not completely integrable in terms of meromorphic first integrals. Their approaches [5, 17] follow the same initial steps till the normal variational equation. Then [17] uses Ziglin's result. We now outline Boucher's approach.

Reduced Hamiltonian. Using the first integrals obtained in Section 2.2, the problem is reduced to a Hamiltonian with three degrees of freedom, given in [17]. The parameters in this equation are the three masses m_1, m_2, m_3 and the value c of the angular momentum (which reduces to a scalar in this dimension). By homogeneity, we can freely assume $m_3 = 1$. (Note that these transformations make the resulting expressions asymmetric with respect to the bodies.)

In order to apply Theorem 1, we need a particular solution of the system. This is provided by the celebrated *Lagrange solutions*. In these solutions, the three particles have orbits on similar conics with a common focus located at their centre of mass (see [8, p. 400]). Since any particular solution can be chosen, Tsygvintsev and Boucher concentrate on the parabolic orbit (for angular momentum $c \neq 0$).

Variational Equation. The variational equation (2) is a linear system of order $n = 6$. The normal variational equation is obtained via a linear change of symplectic basis as follows. We observe that X_H itself is a solution of the variational equation. It will be the first vector e_1 of the new basis. Next, we compute a basis $(e_1 = X, e_2, \ldots, e_n, e_{n+2}, \ldots, e_{2n})$ of the kernel of $dH\big(\Gamma(t)\big)$ satisfying $\omega(e_i, e_{n+i}) = 1$ for $1 < i \leq n$ and $\omega(e_i, e_j) = 0$ otherwise. Finally, we compute a vector $e_{n+1} = Y$ such that $\omega(e_i, Y) = 0$ for $i \neq 1$ and $\omega(X, Y) = 1$. In the new basis $(e_1, \ldots, e_{2n})$, the first column of the matrix of the variational equation is 0, since X_H is a solution. Now, for any vector field η, $\omega(X, \eta) = -dH\big(\Gamma(t)\big) \cdot \eta$, therefore for any solution η of the variational equation, the value of this first integral is the coordinate of η on the vector Y in the new basis. The normal variational equation is obtained by setting this coordinate to 0 and considering the induced matrix A on the subspace with basis $(e_2, \ldots, e_n, e_{n+2}, \ldots, e_{2n})$.

Cyclic Vector. The criterion of Theorem 1 applies to equations rather than systems. A classical method to convert a system of order m into an equation $L(u) = 0$ is to start from a random vector u and find a linear dependency between the $m+1$ vectors $u, u', \ldots, u^{(m)}$ where the derivatives are computed using the matrix A. Unfortunately, this process generically introduces spurious singularities that are roots of the determinant of the change of basis $(u, u', \ldots, u^{(m-1)})$. Boucher therefore selects cyclic vectors in such a way that no new singularity occurs and this requires distinguishing two cases depending on the value of the mass m_1.

Right Factors. In the simplest case of Boucher's criterion, the operator L has an irreducible right factor M whose formal solutions exhibit logarithms. This requires M to have order at least 2. Factors of order k are found by constructing an auxiliary equation $L^{\wedge k}$ of order $\binom{m}{k}$ whose solutions are Wronskians of k independent solutions of L [7]. (Note that this can be computed directly from L.) Indeed, a monic right factor of order k has for coefficient of order $k-1$ the logarithmic derivative w'/w of some particular Wronskian of its solutions. Finding right factors then amounts to looking for so-called *exponential solutions* of $L^{\wedge k}$ (i.e., those with logarithmic derivative that is rational). From a basis of such solutions, corresponding to linear combinations of Wronskians, Plücker's relations help select those that are indeed Wronskians [16]. From there, the complete factor can be reconstructed. Exponential solutions are found by looking at formal solutions at all singularities of the equation [19]. This requires a discussion in the parametric case. If a factor is found, the next step is to check whether this factor is irreducible, or to find conditions on the parameters that make it irreducible. This is done again by searching for factors of the factor. It turns out that in this application, in all cases an irreducible right factor of order 2 is found.

Logarithms. Logarithms in formal solutions occur when the indicial equation at a singularity has roots that differ by an integer. A necessary and sufficient condition has been given by Frobenius [9, p. 404–406]. Again, in all generality nothing can be said when parameters are present but Boucher manages to show that logarithms are present in all cases for this application.

4. **Conclusion**

This application is a very good showcase for many of the algorithms that have been developed in computer algebra for linear differential equations: formal solutions, factorization, polynomial solutions, ...

What Boucher has shown is that, even in the presence of parameters, these algorithms can be exploited to provide useful information by concentrating on those points where specific quantities such as the indicial equation or its solutions do not depend "too much" on the parameters.

A recent trend in computer algebra is to revisit all these algorithms that have been designed for equations and extend them to deal with systems, without using the cyclic vector. It would be a natural step to try and adapt Boucher's criterion so that the symplectic structure is not lost. (Work on this has been started by Boucher and Weil.)

Remark. A new result of Tsygvintsev [18] shows the stronger result that there is no additional meromorphic first integral. Also, Theorem 2 has been extended to the case when L is a product of irreducible factors one of which has a solution with logarithms.

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