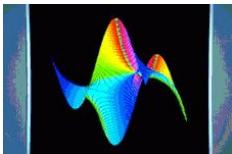


# Automatic Proofs of Special Functions or Combinatorial Identities

Bruno Salvy  
Bruno.Salvy@inria.fr

Algorithms Project, Inria



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# I Introduction

# Undecidability Issues

## Theorem (Richardson 68)

*In the class obtained from  $\mathbb{Q}(x)$ ,  $\pi$ ,  $\log 2$  by the operations  $+$ ,  $-$ ,  $\times$  and composition with  $\exp$ ,  $\sin$  and  $|\cdot|$ , testing for zero-equivalence is undecidable.*

Consequences:

- ① “Simplification” is always difficult;
- ② Automatic proofs cannot cover very general classes of identities;
- ③ Computer algebra isolates classes for which it can provide algorithms.

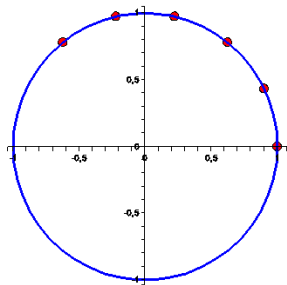
# Algebraic Numbers and Finite Dimension

$$x = \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.$$

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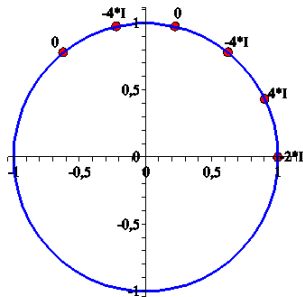


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Coordinates of  $x$



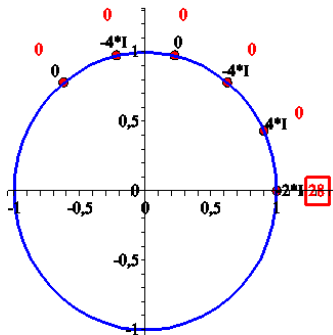
**Tools:** Euclidean division, (extended) Euclidean algorithm, linear algebra.

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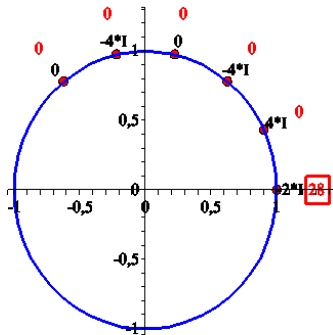
$\mathbb{Q}(\exp(i\pi/7))$  has dim 6 over  $\mathbb{Q}$

Coordinates of  $x^2$

### Definition

A number  $x \in \mathbb{C}$  is *algebraic* when its powers generate a finite-dimensional vector space over  $\mathbb{Q}$ .

**Tools:** Euclidean division, (extended) Euclidean algorithm, linear algebra.



> `x:=sin(2*Pi/7)/sin(3*Pi/7)^2-sin(Pi/7)/sin(2*Pi/7)^2+sin(3*Pi/7)/sin(Pi/7)^2;`

$$x := \frac{\sin\left(\frac{2}{7}\pi\right)}{\sin\left(\frac{3}{7}\pi\right)^2} - \frac{\sin\left(\frac{1}{7}\pi\right)}{\sin\left(\frac{2}{7}\pi\right)^2} + \frac{\sin\left(\frac{3}{7}\pi\right)}{\sin\left(\frac{1}{7}\pi\right)^2}$$

> `convert(x,exp);`

$$\frac{2I\left(e^{\frac{2}{7}i\pi} - \frac{1}{e^{\frac{2}{7}i\pi}}\right)}{\left(e^{\frac{3}{7}i\pi} - \frac{1}{e^{\frac{3}{7}i\pi}}\right)^2} - \frac{2I\left(e^{\frac{1}{7}i\pi} - \frac{1}{e^{\frac{1}{7}i\pi}}\right)}{\left(e^{\frac{2}{7}i\pi} - \frac{1}{e^{\frac{2}{7}i\pi}}\right)^2} + \frac{2I\left(e^{\frac{3}{7}i\pi} - \frac{1}{e^{\frac{3}{7}i\pi}}\right)}{\left(e^{\frac{1}{7}i\pi} - \frac{1}{e^{\frac{1}{7}i\pi}}\right)^2}$$

> `R:=subs([seq(exp(I*Pi/7*j)=X^j,j=1..6)],%);`

$$R := \frac{2I\left(X^2 - \frac{1}{X^2}\right)}{\left(X^3 - \frac{1}{X^3}\right)^2} - \frac{2I\left(X - \frac{1}{X}\right)}{\left(X^2 - \frac{1}{X^2}\right)^2} + \frac{2I\left(X^3 - \frac{1}{X^3}\right)}{\left(X - \frac{1}{X}\right)^2}$$

> `minpol:=normal((X^7+1)/(X+1));`

$$\text{minpol} := X^6 - X^5 + X^4 - X^3 + X^2 - X + 1$$

> `gcdex(denom(R),minpol,X,'u','v');`

1

> `rem(numer(R)*u,minpol,X);`

$$-2I - 4IX^4 - 4IX^2 + 4IX$$

> `rem(%^2,minpol,X);`

28

## Automatic Univariate Identities

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^n$$

Cassini

$$\frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$$

Catalan num

$${}_2F_1\left(\begin{matrix} a, b \\ a + b + 1/2 \end{matrix} \middle| z\right) = {}_2F_1\left(\begin{matrix} 2a, 2b \\ a + b + 1/2 \end{matrix} \middle| \frac{1 - \sqrt{1 - z}}{2}\right)$$

Legendre

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy - u(x^2 + y^2))}{1 - 4u^2}\right)}{\sqrt{1 - 4u^2}}$$

Mehler

## Automatic Multivariate Identities

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3,$$

$$\sum_{n=0}^{+\infty} P_n(x) y^n = \frac{1}{\sqrt{1-2xy+y^2}}, \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k}^2 x^k$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\oint_0 \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} dy = \frac{n! H_n(x)}{[n/2]!},$$

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}}$$

$$\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^n \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k} (q; q)_{n-k}}.$$

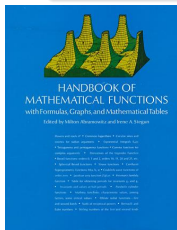
## II D-finiteness in One Variable

# D-finite Series & Sequences

## Definition

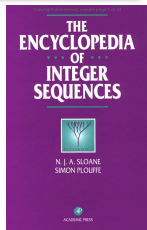
A series  $f(x) \in \mathbb{K}[[x]]$  is **D-finite** over  $\mathbb{K}$  when its derivatives generate a finite-dimensional vector space over  $\mathbb{K}(x)$ . (LDE)

A sequence  $u_n$  is **D-finite** over  $\mathbb{K}$  when its shifts  $(u_n, u_{n+1}, \dots)$  generate a finite-dimensional vector space over  $\mathbb{K}(n)$ . (LRE)



About **25%** of Sloane's encyclopedia,  
**60%** of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Tools: **right** Euclidean division; **right** (extended) Euclidean algorithm; linear algebra; equivalence via generating series.  
Implemented in **gfun** [SaZi94].

# Examples

exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech,  ${}_pF_q$  (includes Bessel  $J$ ,  $Y$ ,  $I$  and  $K$ , Airy  $Ai$  and  $Bi$  and polylogarithms), Struve, Weber and Anger fcns, the large class of algebraic functions, . . .

First Proof:  $\sin^2 + \cos^2 = 1$ 

```
> series(sin(x)^2+cos(x)^2,x,4);
```

$$1 + O(x^4)$$

Why is this a proof?

# First Proof: $\sin^2 + \cos^2 = 1$

> `series(sin(x)^2+cos(x)^2,x,4);`

$$1 + O(x^4)$$

Why is this a proof?

- ①  $\sin$  and  $\cos$  satisfy a 2nd order LDE:  $y'' + y = 0$ ;
- ② their squares (and their sum) satisfy a 3rd order LDE;
- ③ the constant 1 satisfies a 1st order LDE:  $y' = 0$ ;
- ④  $\rightarrow \sin^2 + \cos^2 - 1$  satisfies a LDE of order at most 4;
- ⑤ it is not singular at 0;
- ⑥ Cauchy's theorem concludes.

Second proof (same idea):  $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

> for n to 5 do

`fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;`

## Third Proof: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

- 1 Definition of Hermite polynomials (D-finite over  $\mathbb{Q}(x)$ ):  
recurrence of order 2
- 2 Product by linear algebra:  $H_{n+k}(x)H_{n+k}(y)/(n+k)!$ ,  $k \in \mathbb{N}$   
generated over  $\mathbb{Q}(x, n)$  by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

→ recurrence of order at most 4;

- 3 Translation into a differential equation



## I. Definition

$$> R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\} :$$

$$> R_2 := \text{subs}(H=H_2, x=y, R_1);$$

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

## II. Product

$$> R_3 := \text{gfum} :- \text{poltorec}(H(n) \cdot H_2(n) \cdot v(n), [R_1, R_2, \{v(n+1) \cdot (n+1) = v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n));$$

$$R_3 := \left\{ c(0)=1, c(1)=4xy, c(2)=8x^2y^2 + 2 - 4y^2 - 4x^2, c(3) = \frac{32}{3}x^3y^3 + 24xy - 16xy^3 - 16x^3y, (16n \right.$$

$$\left. + 16)c(n) - 16xyc(n+1) + (-8n - 20 + 8y^2 + 8x^2)c(n+2) - 4xc(n+3)y + (n+4)c(n+4) \right\}$$

## III. Differential Equation

$$> \text{gfum} :- \text{rectodiffeq}(R_3, c(n), f(u));$$

$$\left\{ (16u^3 - 16u^2yx - 4u + 8uy^2 + 8ux^2 - 4xy)f(u) + (16u^4 - 8u^2 + 1) \left( \frac{d}{du} f(u) \right), f(0)=1 \right\}$$

$$> \text{dsolve}(\%, f(u));$$

$$f(u) = \frac{\text{Ie} \left( \frac{-4xyu + y^2 + x^2}{(2u-1)(2u+1)} \right)}{e^{(-y^2-x^2)} \sqrt{2u+1} \sqrt{2u-1}}$$

# Euclidean Division & Finite Dimension

Theorem (XIXth century)

*D*-finite series and sequences over  $\mathbb{K}$  form  $\mathbb{K}$ -algebras.

Proof.

Linear algebra □

Corollary

*D*-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform ( $ogf \leftrightarrow egf$ ).

## Euclidean Division &amp; Finite Dimension

Theorem (Tannery 1874)

*D*-finite series composed with algebraic power series are *D*-finite.

Proof.

$$\begin{aligned}
 P(x, y) = 0 \text{ and } AP + BP_y = 1 &\Rightarrow y' = -\frac{P_x}{P_y} = -BP_x \text{ mod } P \\
 &\Rightarrow y^{(k)} \in \underbrace{\bigoplus_{i < \deg_y P} \mathbb{K}(x)y^i}_{\text{finite dim}}.
 \end{aligned}$$

$(f \circ y)^{(p)}$  linear combination of  $(f^{(j)} \circ y)y^k$ .



Also,  $\exp \int y$ .

# Example: Airy Ai at Infinity

$$\begin{aligned}
 \text{Ai}(z) &= \frac{\sqrt{z}e^{-\xi}}{2\pi} \int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]} dv, \quad \xi = \frac{2}{3}z^{3/2}, u = \sqrt{1 + \frac{v^2}{3}} \\
 &\sim \frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi} \sum_{n=0}^{\infty} (-1)^n \xi^{-n} \frac{\Gamma(3n + \frac{1}{2})}{54^n n! \Gamma(n + \frac{1}{2})}.
 \end{aligned}$$

# Example: Airy Ai at Infinity

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## Computation:

- ① **algebraic** change of variables  $t^2 = (u-1)(4u^2+4u+1)$ ;

$$\rightarrow \int_{-\infty}^{\infty} e^{-\xi t^2} f(t) dt, \quad f(t) = \frac{dv}{dt},$$

- ② recurrence satisfied by the coefficients of  $f$  (**generating series**);
- ③ termwise integration (**Hadamard product**).



## I. Algebraic change of variables

$$eq_u := u^2 - \left(1 + \frac{v^2}{3}\right);$$

$$eq_t := t^2 - (u-1) \cdot (4 \cdot u^2 + 4 \cdot u + 1);$$

$$res := \text{resultant}(eq_u, eq_t, u);$$

$$t^4 + 2t^2 - 3v^2 - \frac{8}{3}v^4 - \frac{16}{27}v^6$$

$$gfun := \text{algeqtodiffeq}\left(res, v(t), \left\{v(0)=0, D(v)(0)=\text{sqrt}\left(\frac{2}{3}\right)\right\}\right);$$

$$\left\{-4v(t) + 9t \left(\frac{d}{dt} v(t)\right) + (9t^2 + 18) \left(\frac{d^2}{dt^2} v(t)\right), v(0)=0, (D(v))(0) = \frac{1}{3}\sqrt{6}\right\}$$

## II. Recurrence satisfied by the coefficients of f

$$gfun := \text{poltodiffeq}(\text{diff}(v(t), t), [\%], [v(t)], f(t));$$

$$\left\{5f(t) + 27t \left(\frac{d}{dt} f(t)\right) + (9t^2 + 18) \left(\frac{d^2}{dt^2} f(t)\right), f(0) = \frac{1}{3}\sqrt{6}, (D(f))(0) = 0\right\}$$

$$R_f := gfun := \text{diffeqtorec}(\%, f(t), c(n));$$

$$\left\{(5 + 18n + 9n^2)c(n) + (18n^2 + 54n + 36)c(n+2), c(0) = \frac{1}{3}\sqrt{6}, c(1) = 0\right\}$$

### III. Hadamard product

assume ( $\xi > 0$ );  $s := \text{Int}(\exp(-\xi * t^2) * t^n, t = -\infty .. \infty)$ ;  $s = \text{student}[\text{intparts}](s, \exp(-\xi * t^2))$ ;

$$\int_{-\infty}^{\infty} e^{(-\xi - t^2)} t^n dt = - \int_{-\infty}^{\infty} \frac{2 \xi \sim t e^{(-\xi - t^2)} t^{(n+1)}}{n+1} dt$$

$$R_i := \left\{ c(n) = \frac{2 \cdot \xi}{(n+1)} \cdot c(n+2), c(0) = \text{value}(\text{eval}(s, n=0)), c(1) = \text{value}(\text{eval}(s, n=1)) \right\};$$

$$\left\{ c(n) = \frac{2 \xi \sim c(n+2)}{n+1}, c(0) = \frac{\sqrt{\pi}}{\sqrt{\xi \sim}}, c(1) = 0 \right\}$$

> **FinalRec:=gfun:-`rec\*rec` (R[i],R[f],c(n));**

$$\text{FinalRec} := \left\{ (5 + 18n + 9n^2) c(n) + (36 \xi \sim n + 72 \xi \sim) c(n+2), c(1) = 0, c(0) = \frac{1}{3} \frac{\sqrt{\pi} \sqrt{6}}{\sqrt{\xi \sim}} \right\}$$

Sol := rsolve(FinalRec, c(n));

$$\left\{ \begin{array}{ll} \frac{1}{3} \frac{(-1)^{\left(\frac{1}{2}n\right)} {}_2\left(-1 - \frac{1}{2}n\right) \Gamma\left(\frac{1}{2}n + \frac{5}{6}\right) \Gamma\left(\frac{1}{2}n + \frac{1}{6}\right) \xi \sim^{\left(-\frac{1}{2}n\right)} \sqrt{6}}{\sqrt{\pi} \Gamma\left(\frac{1}{2}n + 1\right) \sqrt{\xi \sim}} & n::\text{even} \\ 0 & n::\text{odd} \end{array} \right.$$

# Other Uses of the LDE as a Data-Structure

ESF: <http://algo.inria.fr/esf> [MeSa03]

## III Definite Hypergeometric Summation

# Creative Telescoping

$$I_n := \sum_{k=0}^n \underbrace{\binom{n}{k}}_{u_{n,k}} = 2^n.$$

**IF** one knows Pascal's triangle:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k},$$

then summing over  $k$  gives

$$I_{n+1} = I_n + I_n = 2I_n.$$

The initial condition  $I_0 = 1$  concludes the proof.

# Creative Telescoping

$$I_n := \sum_{k=0}^n \underbrace{\binom{n}{k}}_{u_{n,k}} = 2^n.$$

Zeilberger's idea: look for rational  $r(n, k)$  and  $\lambda(n)$  such that

$$v_{n,k+1} - v_{n,k} = u_{n+1,k} + \lambda(n)u_{n,k}, \quad \text{with } v_{n,k} := r(n, k)u_{n,k}$$

When summing over  $k$ , the left-hand side **telescopes**.

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When summing over  $k$ , the left-hand side **telescopes**.

- 1 Dividing out by  $u_{n,k}$  gives a recurrence for  $r$ :

$$\frac{n-k}{k+1}r(n, k+1) - r(n, k) = \frac{n+1}{n+1-k} + \lambda(n).$$

- 2 discussion on poles and degree numerator  $\rightarrow r = \frac{a(n)k+b(n)}{n-k}$ ;
- 3 normalize and extract coefficients of  $k \rightarrow$  **linear** system
- 4 solution:  $a = -1, b = 0, \lambda = -2$ .

The rational function  $r(n, k) = k/(k-n)$  is the **certificate**.

# Zeilberger's Algorithm

Input: a **hypergeometric** term  $u_{n,k}$ , i.e.,  $u_{n+1,k}/u_{n,k}$  and  $u_{n,k+1}/u_{n,k}$  rational functions;

Output: a linear recurrence satisfied by  $\sum_k u_{n,k}$  and a certificate.  
For  $m = 1, 2, 3, \dots$

- 1 Set up the recurrence for  $v_{n,k} = r(n, k)u_{n,k}$


$$v_{n,k+1} - v_{n,k} = u_{n+m,k} + \lambda_1(n)u_{n+m-1,k} + \dots + \lambda_{m-1}(n)u_{n,k}$$

with unknown  $r$  and  $\lambda_i$ ;

- 2 Discuss denominator of  $r$  (Gosper's or Abramov's algorithm);
- 3 Look for numerator  $\rightarrow$  linear system in its coefficients and the  $\lambda_i$ 's;
- 4 If a solution is found, break.

# Example: $\zeta(3)$ is Irrational [Apéry78]

$$a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \quad b_n = a_n \sum_{k=1}^n \frac{1}{k^3} + \sum_{k=1}^n \sum_{m=1}^k \frac{(-1)^{m+1} \binom{n}{k}^2 \binom{n+k}{k}^2}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

- ①  $b_n/a_n \rightarrow \zeta(3)$ ,  $n \rightarrow \infty$ ;  $d_n^3 b_n \in \mathbb{Z}$ , where  $d_n = \text{lcm}(1, \dots, n)$ ;
- ② By **creative telescoping**, both  $a_n$  and  $b_n$  satisfy 

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}, \quad n \geq 1;$$

*"Neither Cohen nor I had been able to prove [this] in the intervening two months."* [Van der Poorten]

- ③  $0 < \zeta(3) - \frac{b_n}{a_n} = \sum_{k \geq n+1} \frac{b_k}{a_k} - \frac{b_{k-1}}{a_{k-1}}: b_k a_{k-1} - b_{k-1} a_k = \frac{6}{k^3}$ ;
- ④  $\lambda a_n + \mu b_n \approx \alpha_{\pm}^n$ , with  $\alpha_{\pm}^2 = 34\alpha_{\pm} - 1$ ;
- ⑤ Conclusion:  $0 < \underbrace{a_n d_n^3}_{\in \mathbb{N}} \zeta(3) - \underbrace{d_n^3 b_n}_{\in \mathbb{N}} \approx \alpha_-^n e^{3n} \rightarrow 0$ .

Algolib can be downloaded from <http://algo.inria.fr/libraries>.

> `libname := "/Users/salvy/lib/maple/Algolib", libname :`

> `a := binomial(n, k)^2 · binomial(n + k, k)^2;`

`a := binomial(n, k)^2 binomial(n + k, k)^2`

> `Mgfun[creative_telescoping](a, n :: shift, k :: shift);`

$$\left[ \begin{aligned} & (-n^3 - 3n^2 - 3n - 1) f(n, k) + (34n^3 + 153n^2 + 231n + 117) f(n + 1, k) + (-n^3 - 6n^2 - 12n - 8) f(n \\ & + 2, k), - \frac{4k^4 (4n^2 + 12n + 8 + 3k - 2k^2) (2n + 3) f(n, k)}{4 + 12n - 12k - 4nk^3 + 13n^2 + 13k^2 + k^4 - 26nk + n^4 + 6n^3 - 6k^3 + 6n^2k^2 - 18n^2k + 18nk^2 - 4n^3k} \end{aligned} \right]$$

Neither Cohen nor I had been able to prove [this] in the intervening two months. [Van der Poorten]

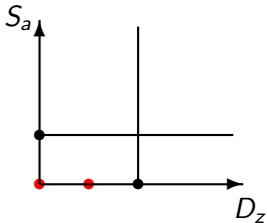
## IV D-finiteness in Several Variables

# Example: Contiguity of Hypergeometric Series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n (b)_n}{(c)_n n!}}_{u_{a,n}} z^n, \quad (x)_n := x(x+1) \cdots (x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1, b; c; z) = \frac{z}{a} F' + F.$$



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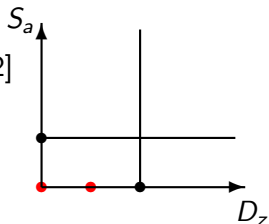
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$\dim=2 \Rightarrow S_a^2 F, S_a F, F$  linearly dependent [Gauss1812]

Also:

- $S_a^{-1}$  in terms of  $\text{Id}, D_z$ ;
- relation between any three polynomials in  $S_a, S_b, S_c$ ;
- generalizes to any  ${}_pF_q$  and multivariate case [Takayama89].



# Ore Polynomials & Ore Algebras

- **Skew polynomial ring:**  $\mathbb{A}[\partial; \sigma, \delta]$ ,  $\mathbb{A}$  integral domain and commutation  $\partial P = \sigma(P)\partial + \delta(P)$ ,  $P \in \mathbb{A}$   
 (ex.  $\partial_x P(x) = P(x)\partial_x + P'(x)$ ,  $S_n P(n) = P(n+1)S_n$ ).  
 Technical conditions on  $\sigma, \delta$  to make product associative.
- **Ore algebra:**  $\mathbb{A}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$ ,  $\sigma, \delta$  s. t.  $\partial_i \partial_j = \partial_j \partial_i$ .  
**Aim** [ChSa98]: manipulate (solutions of) systems of mixed linear ( $q$ -)differential or ( $q$ -)difference operators.

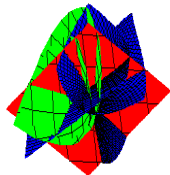


- **Main property:** the leading term of a product is (up to a cst) the product of leading terms.

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- **Main property:** the leading term of a product is (up to a cst) the product of leading terms.
- **Consequences:**
  - 1 Univariate: Right Euclidean division and extended Euclidean algorithm [Ore 33];
  - 2 Multivariate: Buchberger's algorithm for Gröbner bases works in Ore algebras [Kredel93].

# 0-dimensionality & D-finiteness



Polynomial algebra

0-dimensional ideal



quotient is a **finite dimensional** vector space

Ore algebra

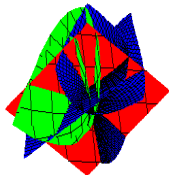
**D-finite** left ideal



**Exs:** Orthogonal polynomials, hypergeometric series, their  $q$ -analogues,...

system+ini. cond.=data structure

# 0-dimensionality & D-finiteness



Polynomial algebra

0-dimensional ideal



quotient is a **finite dimensional** vector space



polynomial expressions  
are algebraic

Ore algebra

**D-finite** left ideal



polynomials and  $\partial$ 's  
are D-finite

Tools: linear algebra, Gröbner bases. Implemented in **Mgfun** [Chyzak98]

Exs: Orthogonal polynomials, hypergeometric series, their  $q$ -analogues,...

system+ini. cond.=data structure

# Example: Binomial Coefficients and Pascal's Triangle

- Algebra:  $A = \mathbb{Q}(n, k)[S_n; S_n, 0][S_k; S_k, 0]$
- $\binom{n}{k}$  is annihilated by  $S_n - \frac{n+1}{n+1-k}$  and  $S_k - \frac{n-k}{k+1}$
- They generate a left ideal  $\mathcal{I}$ .
- The quotient has dimension 1 ( $\equiv$  hypergeometric).
- Pascal's triangle is  $P = S_n S_k - S_k - 1 \in \mathcal{I}$ .
- Creative telescoping is obtained by left division by  $S_k - 1$ :

$$P = (S_k - 1) \underbrace{(S_n - 1)}_{\text{certificate}} + \underbrace{S_n - 2}_{\text{result}}$$

## General Creative Telescoping [Zeilberger 90]

$$F_n = \sum_k u_{n,k} = ?$$

**IF** one knows  $A(n, S_n)$  and  $B(n, k, S_n, S_k)$  such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum “telescopes”, leading to  $A(n, S_n) \cdot F_n = 0$ .

## General Creative Telescoping [Zeilberger 90]

$$I(x) = \int_{\Omega} u(x, y) dy = ?$$

**IF** one knows  $A(x, \partial_x)$  and  $B(x, y, \partial_x, \partial_y)$  such that

$$(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x, y) = 0,$$

then the integral “telescopes”, leading to  $A(x, \partial_x) \cdot I(x) = 0$ .

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*Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.*

Richard P. Feynman 1985

Creative telescoping = “differentiation” under integral + “integration” by parts

## General Creative Telescoping [Zeilberger 90]

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Creative telescoping = “differentiation” under integral + “integration” by parts

- General case: Find annihilators of

$$I(x_1, \dots, x_{n-1}) = \partial_n^{-1} \Big|_{\Omega} f(x_1, \dots, x_n)$$

knowing generators of  $\text{Ann}_f$  in

$$\mathbb{O}_n = \mathbb{K}(x_1, \dots, x_n)[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n];$$

- Crucial step: compute  $(\mathbb{O}_n \text{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$ .

## Applications of Creative Telescoping

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad [\text{Strehl92}]$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2} \quad [\text{GIMo94}]$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{[n/2]!} \quad [\text{Doetsch30}]$$

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}} \quad [\text{Andrews74}]$$

$$\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^n \frac{(-1)^k q^{7/2k^2+1/2k}}{(q; q)_{n+k} (q; q)_{n-k}} \quad [\text{Paule85}]$$

## (Partial) Algorithms for Creative Telescoping

Aim:  $\mathcal{I} = (\mathbb{O}_n \text{Ann}_f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$

- By Gröbner bases, eliminate  $x_n$  and set  $\partial_n$  to 0 [ChSa98]  
 $\rightarrow (\mathbb{O}_n \text{Ann}_f \cap \mathbb{O}_{n-1}[\partial_n] + \partial_n \mathbb{O}_{n-1}) \cap \mathbb{O}_{n-1} \subset \mathcal{I}$
- Differential case: algorithms from  $\mathcal{D}$ -module theory [SaStTa00, Tsai00], Gröbner bases with negative weights.
- Shift case,  $n = 2$ ,  $\dim 1$  (= hypergeometric): [Zeilberger91]  
 For increasing  $k$ , search for  $a_i$  and  $B$

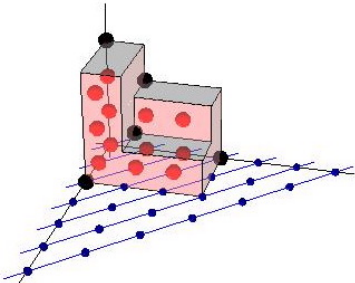
$$\mathbb{O}_{n-1} \ni \sum_{i=0}^k a_i \partial_{n-1}^i f = \partial_n B f$$

Termination [Abramov03].

- Arbitrary  $n$  and  $\mathbb{O}_n$ : [Chyzak00]

$$\mathbb{O}_{n-1} \ni \sum_{\lambda} a_{\lambda} \partial^{\lambda} = \partial_n B \text{ mod } \text{Ann}_f$$

$B$  is given by rational solutions of a linear system in  $\sigma_n, \partial_n$ .



# Open Problems

## Efficiency

- Faster Gröbner bases;
- Other elimination techniques (adapt geometric resolution [GiHe93,GiLeSa01] to Ore algebras);
- Structured Padé-Hermite approximants.




## Understand **non-minimality**

- Remove **apparent** singularities by Ore closure, a generalization of Weyl closure [Tsai00], and of [AbBavH05] ([ChDuLeMaMiSa05] in progress);
- Exploit symmetry (extend [Paule94]).

## Easy-to-use Implementations

- Improve `gfun` and `Mgfun`. Make the ESF interactive.

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