Gfun — 15 Years Later

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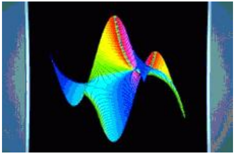
Algorithms Project, Inria



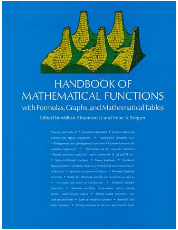
Joint Mathematics Meetings
January 5–8, 2007
New Orleans

I Introduction

Framework: D-finite Series & Sequences

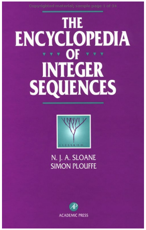
Definition

A *series* $f(x) \in \mathbb{K}[[x]]$ is D-finite over $\mathbb{K}$ when its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$. (LDE)
A *sequence* u_n is D-finite over $\mathbb{K}$ when its shifts $(u_n, u_{n+1}, \dots)$ generate a finite-dimensional vector space over $\mathbb{K}(n)$. (LRE)



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Gfun [SaZi94]: Maple package to guess, manipulate and prove D-finite identities.
References: Stanley vol. 2, $A = B$ [PeWiZe96]

II Guessing Identities

Mehler's Identity on Hermite Polynomials (1866)

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = ?$$

Answer: compute the first 10 polynomials and guess!

```
> L:=[seq(orthopoly[H](n,x)*orthopoly[H](n,y),n=0..9)];
```

$$L := [1, 4xy, (-2+4x^2)(-2+4y^2), (8x^3-12x)(8y^3-12y), (12+16x^4-48x^2)(12+16y^4-48y^2), (32x^5-160x^3+120x)(32y^5-160y^3+120y), (-120+64x^6-480x^4+720x^2)(-120+64y^6-480y^4+720y^2), (128x^7-1344x^5+3360x^3-1680x)(128y^7-1344y^5+3360y^3-1680y), (1680+256x^8-3584x^6+13440x^4-13440x^2)(1680+256y^8-3584y^6+13440y^4-13440y^2), (512x^9-9216x^7+48384x^5-80640x^3+30240x)(512y^9-9216y^7+48384y^5-80640y^3+30240y)]$$

```
> deq:=gfun[listtodiffeq](L,F(u),['egf']);
```

$$deq := \left[\left\{(1-8u^2+16u^4)\left(\frac{d}{du}F(u)\right) + (-4xy+8ux^2-4u+8uy^2-16u^2xy+16u^3)F(u), F(0)=1\right\}, egf\right]$$

```
> dsolve(deq[1],F(u)) assuming 0<u,u<1/2;
```

$$F(u) = \frac{e^{\frac{-4xyu+x^2+y^2}{(2u+1)(2u-1)}}\sqrt{\frac{1}{(-2u+1)(2u+1)}}}{e^{-x^2-y^2}}$$

```
>
```

Guess = Good Approximant

Definition (Padé-Hermite Approximant)

The vector of polynomials $(P_1, \dots, P_k)$ with $\deg P_i \leq d_i$ is a Padé-Hermite approximant of type $(d_1, \dots, d_k)$ for a vector of power series $(f_1, \dots, f_k)$ when

$$P_1 f_1 + \cdots + P_k f_k = O(x^{d_1+\cdots+d_k+k-1}).$$

Special cases: (given one series y)

- $k = 2$, $f_1 = -1$, $f_2 = y$: Padé-approximant;
- $f_i = y^{i-1}$, $i = 1, \dots, k$: algebraic approximants;
- $f_i = y^{(i-1)}$, $i = 1, \dots, k$: differential approximants.

Guess = Good Approximant

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Algorithms and complexity ($D = d_1 + \dots + d_k$):

- Linear algebra: $O(D^\omega)$ ops ($\omega \le 3$ complexity of matrix product);
- minimal basis of approximants in $O(k^\omega D^{1+\epsilon})$ ops [BeLa94];
- genset in $O(k^\omega (D/k)^{1+\epsilon})$ ops [Storjohann06].

III Computing Identities

Proof of Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x, n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\longrightarrow$ recurrence of order at most 4;
3. Translation into a differential equation

I. Definition

> $R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\}$:

> $R_2 := subs(H=H_2, x=y, R_1)$;

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

II. Product

> $R_3 :=$ gfun :- poltorec$\left(H(n)\cdot H_2(n)\cdot v(n), \left[R_1, R_2, \{v(n+1)\cdot(n+1) = v(n), v(1)=1\}\right], \left[H(n), H_2(n), v(n)\right], c(n)\right)$;

$$R_3 := \left\{c(0)=1, c(1)=4xy, c(2)=8x^2y^2+2-4y^2-4x^2, c(3)=\frac{32}{3}x^3y^3+24xy-16xy^3-16x^3y, (16n+16)c(n)-16xyc(n+1)+\left(-8n-20+8y^2+8x^2\right)c(n+2)-4xc(n+3)y+(n+4)c(n+4)\right\}$$

III. Differential Equation

> gfun :- rectodiffeq$\left(R_3, c(n), f(u)\right)$;

$$\left\{\left(16u^3-16u^2yx-4u+8uy^2+8ux^2-4xy\right)f(u)+\left(16u^4-8u^2+1\right)\left(\frac{d}{du}f(u)\right), f(0)=1\right\}$$

> $dsolve(\%, f(u))$;

$$f(u) = \frac{I\, e^{\left(\frac{-4xyu+y^2+x^2}{(2u-1)(2u+1)}\right)}}{e^{\left(-y^2-x^2\right)}\sqrt{2u+1}\sqrt{2u-1}}$$

>

Closure Properties

Theorem (XIXth century)

- *D-finite series and sequences over* $\mathbb{K}$ *form* $\mathbb{K}$*-algebras;*
- y *algebraic,* f *is D-finite* $\Longrightarrow y, f \circ y, \exp \int y$ *D-finite.*

Proof.

Linear algebra in finite dimension. □

Corollary

D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform (ogf↔egf).

All implemented in `Gfun`.

IV Multivariate Functions & Sequences

Definition (D-finiteness)

A series $F(x, y) \in \mathbb{K}[[x, y]]$ is D-finite when its derivatives generate a finite-dimensional space over $\mathbb{K}(x, y)$.

More generally, 0-dimensional ideals of operators in suitable Ore algebras [ChSa98].

New Operation: Creative telescoping (Zeilberger)

If F satisfies an equation $A(x, \partial_x)F = \partial_y B(x, y, \partial_x, \partial_y)F$, then

$$A(x, \partial_x) \int_\Omega F(x, y)\, dy = 0, \qquad \text{for “good” } \Omega.$$

B is called the *certificate* of this identity.

Applications: definite summation and integration, computation of generating functions, extraction of coefficients,. . . .

Mgfun: Multivariate Gfun [Chyzak98]

Example: an identity on Bessel Functions [GlMo94]

$$\int_0^\infty x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = \frac{1}{2\pi a^2} \ln \frac{1}{1-a^4}.$$

```
> libname:="/Users/salvy/lib/maple/Algolib",libname:
> f:=x*BesselJ(1,a*x)*BesselI(1,a*x)*BesselY(0,x)*BesselK(0,x);
```

$$f := x\,\mathrm{BesselJ}(1, a\,x)\,\mathrm{BesselI}(1, a\,x)\,\mathrm{BesselY}(0, x)\,\mathrm{BesselK}(0, x)$$

```
> sys:=Mgfun:-dfinite_expr_to_sys(f,y(x::diff,a::diff));
```

$$sys := \Big\{ (-4a^4x^4+4x^4+3)\,y(x,a) + 12x^2a\left(\frac{\partial^3}{\partial x^2\,\partial a}y(x,a)\right) - 4a^3x\left(\frac{\partial^4}{\partial x\,\partial a^3}y(x,a)\right) - 3x\left(\frac{\partial}{\partial x}y(x,a)\right) + 26a\left(\frac{\partial}{\partial a}y(x,a)\right) - 26ax\left(\frac{\partial^2}{\partial x\,\partial a}y(x,a)\right) + 40a^2\left(\frac{\partial^2}{\partial a^2}y(x,a)\right) + 6a^2x^2\left(\frac{\partial^4}{\partial x^2\,\partial a^2}y(x,a)\right) + x^2\left(\frac{\partial^2}{\partial x^2}y(x,a)\right) - 24a^2x\left(\frac{\partial^3}{\partial x\,\partial a^2}y(x,a)\right) + 8a^3\left(\frac{\partial^3}{\partial a^3}y(x,a)\right) - 4x^3a\left(\frac{\partial^4}{\partial x^3\,\partial a}y(x,a)\right) + x^4\left(\frac{\partial^4}{\partial x^4}y(x,a)\right),\ 4x^4a^3y(x,a) + a^3\left(\frac{\partial^4}{\partial a^4}y(x,a)\right) + 3\left(\frac{\partial}{\partial a}y(x,a)\right) - 3a\left(\frac{\partial^2}{\partial a^2}y(x,a)\right) + 4a^2\left(\frac{\partial^3}{\partial a^3}y(x,a)\right)\Big\}$$

```
> deq:=Mgfun:-int_of_sys(sys,x=0..infinity,_takayama_algo);
```

$$deq := \Big\{ 32a^3y(a) + (16a^6-4a^2)\left(\frac{d^3}{da^3}y(a)\right) + (-a^3+a^7)\left(\frac{d^4}{da^4}y(a)\right) + (73a^5+3a)\left(\frac{d^2}{da^2}y(a)\right) + (103a^4-3)\left(\frac{d}{da}y(a)\right)\Big\}$$

```
> normal(eval(deq,y(a)=1/2/Pi/a^2*ln(1/(1-a^4))));
```

$$\{0\}$$

```
> sol:=subs(dsolve(deq,y(a)),y(a)) assuming a>0,a<1;
```

$$sol := \frac{\_C1}{a^2} + \frac{\_C2\,\ln\big((-1+a)(a+1)(a^2+1)\big)}{a^2} + \frac{\_C3\,\big(\ln(a+1)+\ln(-1+a)-\ln(a^2+1)\big)}{a^2} + \frac{1}{a^2}\Big(\_C4\,\big(2\,\mathrm{dilog}(a+1) + 2\ln(a)\ln(a+1) - 2\,\mathrm{dilog}(a) + 2\ln(a)\ln(I(-I+a)) + 2\ln(a)\ln(-I(I+a)) + 2\,\mathrm{dilog}(I(-I+a)) + 2\,\mathrm{dilog}(-I(I+a)) - \ln\big((-1+a)(a+1)(a^2+1)\big)\big)\Big)$$

V Guess as Proof

Guess + Bound = Proof = Algorithm

$$> \texttt{series}(\texttt{sin}(\texttt{x})^2 + \texttt{cos}(\texttt{x})^2, \texttt{x}, 4);$$

$$1 + O(x^4)$$

Why is this a proof?

Guess + Bound = Proof = Algorithm

$> \texttt{series}(\texttt{sin}(\texttt{x})^2 + \texttt{cos}(\texttt{x})^2, \texttt{x}, 4);$

$$1 + O(x^4)$$

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. it is not singular at 0, Cauchy's theorem concludes.

Same for Creative Telescoping

Zeilberger *et alii* [WiZe92,Yen93,Yen97,MoZe05,ApZe06]:
Bounds for classes of

- hypergeometric sums;
- hyperexponential integrals;
- q-analogues,. . .

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Prop. Linear Recurrences for Algebraic Series (BoChLeSaSc07)

If $P(x, y)$ has degree at most d then the coefficients of its series solutions obey a recurrence of order at most $(d+1)^2$.

(First observed experimentally, then proved rigorously).
Proof by bounds for creative telescoping on

$$\frac{1}{2\pi i}\oint \frac{yP_y}{P}\,dy.$$

VI Conclusion

In the Next Versions of Gfun

- Minimal recurrences and differential equations by removing apparent singularities [ChDuLeMaMiSa07];
- Better/faster guessing;
- Initial conditions at singular points (see also http://algo.inria.fr/esf);
- Fast evaluation of D-finite sequences.

Do not hesitate to ask for more, or provide code!