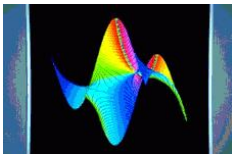


Composition of Power Series, Change of Basis and Orthogonal Polynomials

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Joint work with Alin Bostan and Éric Schost

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I Introduction

Univariate Composition

Problem (Composition of Power Series)

Input: two power series f and g at precision N , with $g(0) = 0$.

Output: $f(g)$ at precision N .

Measure of complexity: number of arithmetic operations.

Hypothesis for the talk: characteristic 0.

General algorithms:

- Naïve: $\mathcal{O}(N^3)$;
- Naïve with fast multiplication: $\mathcal{O}(NM(N))$;
- Brent & Kung (1978): $\mathcal{O}(\sqrt{N \log N} M(N))$.

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Fast Multiplication of Polynomials of degree N

- Naïve: $M(N) = \mathcal{O}(N^2)$;
- Karatsuba (1963): $M(N) = \mathcal{O}(N^{1.59})$;
- FFT: $M(N) = \mathcal{O}(N \log N)$ (Schönhage & Strassen 71).

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Nothing better known in general \rightarrow **exploit structure**

Special f 's by Newton iteration: $\mathcal{O}(M(N))$ for inverse (Sieveking 72, Kung 74), log, exp, power (Brent 75);

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$f \in \{\text{Inv}, \text{log}, \text{exp}, \text{Pow}\}$: $\mathcal{O}(M(N))$;

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$f \in \{\text{Inv}, \text{log}, \text{exp}, \text{Pow}\}$: $\mathcal{O}(M(N))$;

Special g 's by Divide-and-Conquer: $\mathcal{O}(M(N) \log N)$;

- polynomials (Brent & Kung 78);
- algebraic series (van der Hoeven 02);

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Special g 's by Divide-and-Conquer: $\mathcal{O}(M(N) \log N)$;

f a polynomial, **special g 's:**

- $x + a$: $\mathcal{O}(M(N))$ (Aho, Steiglitz, Ullman 75);
- $(ax + b)/(cx + d)$: $\mathcal{O}(M(N))$ (Pan 98);
- **NEW:** several other functions in $\mathcal{O}(M(N))$;
- **NEW:** **exp and log** in $\mathcal{O}(M(N) \log N)$.

Bivariate Evaluation

Problem (Bivariate Evaluation)

$$F(x, t) = \sum_{i,j} F_{i,j} x^i t^j = \sum_j G_j(x) t^j.$$

Input: a_0, \dots, a_N ;

Output: $\sum_j a_j G_j(x) \bmod x^{N+1}$.

Examples:

- $G_j = g^j$: univariate composition;
- G_j polynomial of degree j : change of basis from (G_j) to (x^j) .

Our Result

Good complexity for this map **and its inverse** for “nice” F .

Special Cases with Fast Change of Basis

Very fast ($\mathcal{O}(M(N))$)

- Orthogonal: Jacobi, Legendre, Chebyshev U , T , Gegenbauer, Laguerre, Hermite.
- Other: Fibonacci, Bernoulli, Euler, Mott, Spread, Bessel, . . .

Fast ($\mathcal{O}(M(N) \log N)$)

- Orthogonal: Meixner, Meixner-Pollaczek, Krawtchouk, Charlier;
- Other: Falling factorial, Bell, Actuarial, Narumi, Peters, . . .

 ${}_4F_3(4)$

Wilson

Racah

 ${}_3F_2(3)$ Continuous
dual HahnContinuous
Hahn

Hahn

Dual Hahn

 ${}_2F_1(2)$ Meixner
-
Pollaczek

Jacobi

Meixner

Krawtchouk

 ${}_1F_1(1)/{}_2F_0(1)$

Laguerre

Charlier

 ${}_2F_0(0)$

Hermite

Special Cases with Fast Change of Basis

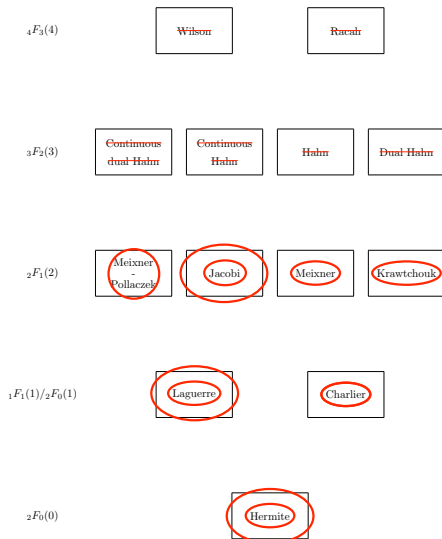
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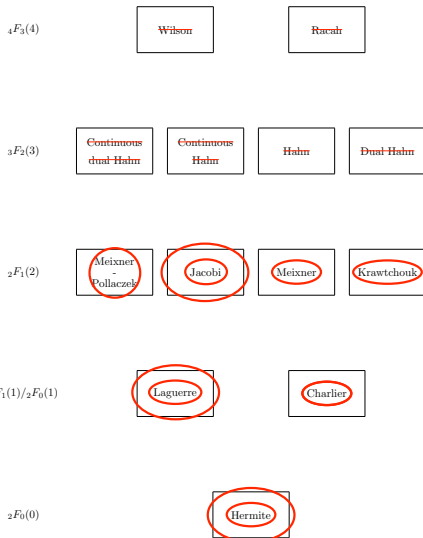
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Other algorithm giving all orthogonal polys in $\mathcal{O}(M(N) \log N)$

II Very Fast Composition ($\mathcal{O}(M(N))$ operations)

Shift is Cheap (Aho, Steiglitz, Ullman 1975)

Problem (Polynomial Shift)

Input: $P(x)$ polynomial of degree N , a a point.

Output: $P(x + a)$.

Horner: $\mathcal{O}(N^2)$ operations

Taylor:

$$P^{(N-i)}(a) = \sum_{j=0}^i P^{(N-i+j)}(0) \frac{a^j}{j!}.$$

Consequence: generating series

$$\sum_{i=0}^N \underbrace{P^{(N-i)}(a)}_{\text{coeffs of } P(x+a)} x^i + \dots = \sum_{k=0}^n \underbrace{P^{(k)}(0)}_{\text{coeffs of } P(x)} x^{N-k} \times \sum_{j \geq 0} \frac{a^j x^j}{j!}.$$

$M(N) + \mathcal{O}(N)$ operations.



Warm-up: Euler Transform for 3 Multiplications

Problem (Euler Transform)

Input: $P(x)$ polynomial of degree N (or truncated series);

Output: First N coefficients of $\frac{1}{1-x}P\left(\frac{x}{1-x}\right)$.

Algorithm: 

$$S_1(x) := P(x - 1);$$

$$S_2(x) := x^N S_1(1/x);$$

$$S_3(x) := S_2(x + 1);$$

$$S_4(x) := S_3(-x);$$

return $(1 - x)^{-N-1} S_4(x)$.

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$$S_3(x) := S_2(x + 1); \quad = (1+x)^N P\left(\frac{-x}{1+x}\right)$$

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$$S_2(x) := x^N S_1(1/x); \quad = x^N P\left(\frac{1-x}{x}\right) \quad 0 \text{ ops;}$$

$$S_3(x) := S_2(x+1); \quad = (1+x)^N P\left(\frac{-x}{1+x}\right) \quad M(N) + \mathcal{O}(N) \text{ ops;}$$

$$S_4(x) := S_3(-x); \quad = (1-x)^N P\left(\frac{x}{1-x}\right) \quad \mathcal{O}(N) \text{ ops;}$$

$$\text{return } (1-x)^{-N-1} S_4(x). \quad M(N) + \mathcal{O}(N) \text{ ops.}$$

More Systematically

Collection of operators on series...

$$A_a : g \mapsto a + g$$

$$M_\lambda : g \mapsto \lambda g$$

$$\text{Inv} : g \mapsto 1/g$$

$$P_k : g \mapsto g^k$$

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...and associativity rules

$$f(A_a(g)) = f|_{x \mapsto x+a} \circ g$$

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Theorem

For any well-defined g composition of $\{A_a, M_\lambda, \text{Inv}, P_k, R_k\}$ and any polynomial f , $f \circ g \bmod x^N$ computed in $\mathcal{O}(M(N))$ ops.

Proof. Fast right and left compositions.

Polynomials & Rational Functions

Proposition (Degree 2)

All polynomials and rational functions of degree 2, and their (compositional) inverses.

- ① $ax + b$: easy;
- ② $ax^2 + bx + c = (b - a^2/4) + (x + a/2)^2$;
- ③ $(ax + b)/(cx + d) = a/c + (b - ad/c)/(cx + d)$ (Pan, 98);
- ④ $x/(x^2 - d) = \frac{1}{2\sqrt{d}} \sqrt{\left(\frac{2d}{d-x^2} - 1\right)^2 - 1}$;
- ⑤ $(x^2 - d)/x = \textcircled{4}^{-1}$;
- ⑥ $(x^2 + ax - d)/x = a + \textcircled{5}$;
- ⑦ $(x^2 + ax + b)/(x + c) = \textcircled{6} \Big|_{x \mapsto x+c, d \mapsto ac-b}$;
- ⑧ $(x + c)/(x^2 + ax + b) = \textcircled{7}^{-1}$;
- ⑨ $(x^2 + dx + e)/(x^2 + ax + b) = 1 + (d - a) \cdot \textcircled{8} \Big|_{c \mapsto (e-b)/(d-a)}$;
- ⑩ $(x^2 + ax + e)/(x^2 + ax + b) = 1 + (e - b) \cdot \textcircled{2} \Big|_{a \mapsto 1, b \mapsto a, c \mapsto b}$.

Sequence Transformations

Problem (Binomial Convolution)

Input: sequence (a_n) , $n = 0, \dots, N$;

Output: sequence (b_n) , $n = 0, \dots, N$, where

$$b_n = \sum_{k \geq 0} \binom{n + pk}{m + qk} a_k.$$

Generating series

$$\sum_n b_n z^n = \frac{z^m}{(1-z)^{m+1}} A\left(\frac{z^{p-q}}{(1-z)^p}\right).$$

Algorithm in $\mathcal{O}(M(N))$ for

- $p = 1, q = 0$ (Euler);
- $p = 2$ and $q = 0$ or $q = 1$.

III Fast Composition ($\mathcal{O}(M(n)\log n)$ operations)

Result

Theorem

For any well-defined g composition of $\{A_a, M_\lambda, \text{Inv}, P_k, R_k, \text{polynomial}, \text{algebraic function}, \text{exp}, \text{log}\}$ and any polynomial f , $f \circ g \bmod x^N$ computed in $\mathcal{O}(M(N) \log N)$ ops.

Proof.

- Fast left compositions
(Newton, Brent & Kung);
- Fast right compositions:
 - polynomials: Brent & Kung (78);
 - algebraic functions: van der Hoeven (02);
 - exp, log: next screens.

Exponential

Problem (Right Composition with Exp)

Input: (a_0, \dots, a_N)

Output: coefficients of $\sum_j a_j e^{jx} \bmod x^{N+1}$.

Laplace: $\mathcal{L}(x^i) := i!x^i \Rightarrow \sum_{j=0}^N a_j e^{jx} = \mathcal{L}^{-1} \underbrace{\sum_{j=0}^N \frac{a_j}{1 - jx}}_{:= S_0^N}$.



Transposed Multipoint Evaluation (Bostan, Lecerf, Schost 03)

- 1 Divide and conquer: numerator and denominator of S_0^N from those of $S_0^{N/2}$ and $S_{N/2+1}^N$;
- 2 Compute the series expansion of the quotient
- 3 Multiply the i th coeff by $1/i!$, $i = 0, \dots, N$

Exponential

Problem (Right Composition with Exp)

Input: (a_0, \dots, a_N)

Output: coefficients of $\sum_j a_j e^{jx} \bmod x^{N+1}$.

Laplace: $\mathcal{L}(x^i) := i!x^i \Rightarrow \sum_{j=0}^N a_j e^{jx} = \mathcal{L}^{-1} \underbrace{\sum_{j=0}^N \frac{a_j}{1 - jx}}_{:= S_0^N}$.



Transposed Multipoint Evaluation (Bostan, Lecerf, Schost 03)

- 1 Divide and conquer: numerator and denominator of S_0^N from those of $S_0^{N/2}$ and $S_{N/2+1}^N$; $\frac{3}{2} M(N) \log N + \mathcal{O}(M(N))$ ops
- 2 Compute the series expansion of the quotient $\mathcal{O}(M(N))$ ops
- 3 Multiply the i th coeff by $1/i!$, $i = 0, \dots, N$ $\mathcal{O}(N)$ ops

Matrices Acting on the Vector of Coefficients

Free (0 op)

- Rec: permutation $i \mapsto N - i$;

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- $\text{Diag}(u_i)$, e.g., $M_\lambda(u_i = \lambda^i)$; $\mathcal{L}(u_i = i!)$; $\mathcal{L}^{-1}(u_i = 1/i!)$;

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Very fast ($\mathcal{O}(M(N))$):

- Multiplication by a fixed P : $\text{Mul}(P)$;
- Shift A_a : $\mathcal{L}^{-1} \cdot \text{Rec} \cdot \text{Mul}(e^{ax}) \cdot \text{Rec} \cdot \mathcal{L}$;

Check

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- Multipoint evaluation: $V(a_0, \dots, a_N)$ (Vandermonde);
- Interpolation: $V(a_0, \dots, a_N)^{-1}$;
- Exp: $\mathcal{L}^{-1} \cdot V(0, \dots, N)^t$;

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Check

Slow ($\mathcal{O}(\sqrt{N} \log N M(N))$)

- Composition with g : $(g_{i,k})$, with $g^k = \sum g_{i,k} x^i \bmod x^N$.

$\log(1+x)$ and Transposed Interpolation

$$P(x) \xrightarrow{A_{-1}} \sum_{j=0}^N a_j x^j \xrightarrow{\mathcal{L}} \sum_{j=0}^N \frac{b_j}{1-jx} \xrightarrow{\text{MEval}^t} \sum_{j=0}^N c_j x^j \xrightarrow{\mathcal{L}^{-1}} P(e^x - 1) + O(x^{N+1}).$$

Problem: Transposed Interpolation

Input: $C(x)$ polynomial of degree N

Output: (b_0, \dots, b_N) such that $C(x) = \sum_{j=0}^N \frac{b_j}{1-jx} + O(x^{N+1})$.

log(1 + x) and Transposed Interpolation

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Let $P(x) := \prod_{j=0}^N (X - j)$, then

$$\frac{1}{x} C\left(\frac{1}{x}\right) = \sum_{j=0}^N \frac{b_j}{x-j} = \frac{Q(x)}{P(x)} + O(x^{-N-2}) \Rightarrow b_j = \frac{Q(j)}{P'(j)}.$$

Algorithm (Bostan, Lecerf, Schost 03)

- 1 Expand P by divide-and-conquer;
- 2 Compute $C \times \text{Rec}(P) = \text{Rec}(Q) + O(x^{N+1})$;
- 3 Evaluate Q and P' at $0, \dots, N$;
- 4 Return $(Q(0)/P'(0), \dots, Q(N)/P'(N))$.

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Algorithm (Bostan, Lecerf, Schost 03) with linear rec for P'

- 1 Expand P by divide-and-conquer; $\frac{1}{2} M(N) \log N + O(N)$
- 2 Compute $C \times \text{Rec}(P) = \text{Rec}(Q) + O(x^{N+1})$; $M(N) + O(N)$
- 3 Evaluate Q and P' at $0, \dots, N$; $M(N) \log N + O(N)$
- 4 Return $(Q(0)/P'(0), \dots, Q(N)/P'(N))$. $O(N)$.

IV Change of Basis

Composition Sequences

Definition (Composition Sequence)

A map $F = o_1 \circ \cdots \circ o_m$ with o_i 's in $\{A_a, M_\lambda, P_k, R_k, \text{Inv}, \text{exp}, \text{log}\}$.
It *defines* $F(x)$ if all intermediates $o_i \circ \cdots \circ o_m(x)$ are power series.

Summary so far

P a polynomial, F a composition sequence defining $F(x)$, then

- $F(x) \bmod x^N$;
- $P(F(x)) \bmod x^N$

can be computed in a number of operations

$$T_F(N) = \begin{cases} \mathcal{O}(M(N)) & \text{if no } o_i \text{ is exp or log,} \\ \mathcal{O}(M(N) \log N) & \text{otherwise.} \end{cases}$$

Main Theorem

Bivariate Evaluation with $F(x, t) = \sum F_{ij}x^i t^j = \sum G_j(x)t^j$

Input: (a_0, \dots, a_N) ;

Output: $\sum_{i=0}^N b_i x^i = \sum a_j G_j(x) \bmod x^{N+1}$.

Main Theorem of (Bostan, S., Schost, 08)

Generating function $F(x, t) = u(x) f(g(x) \times h(t)) v(t)$ with

- g and h given by composition sequences;
- u, v, f given by their coefficients $\bmod x^{N+1}$;
- well-defined: $g(0)h(0) = 0$;

then $(a_i) \mapsto (b_i)$ in $\mathcal{O}(T_g(N) + T_h(N) + M(N))$ ops

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- g and h given by composition sequences;
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- well-defined: $g(0)h(0) = 0$;
- $g'(0)h'(0)u(0)v(0) \neq 0$, all coefficients of $f \neq 0$,

then $(a_i) \mapsto (b_i)$ in $\mathcal{O}(T_g(N) + T_h(N) + M(N))$ ops

then $(b_i) \mapsto (a_i)$ in $\mathcal{O}(T_g(N) + T_h(N) + M(N))$ ops.

Examples in $\mathcal{O}(M(N))$ Sheffer Sequences: $F = \exp(xh(t))v(t)$

Name	Notation	h	v
Laguerre	L_n^α	$t(1-t)^{-1}$	$(1-t)^{-1-\alpha}$
Hermite	H_n	$2t$	$\exp(-t^2)$
Euler	E_n^α	t	$2^\alpha(1+e^t)^{-\alpha}$
Bernoulli	B_n^α	t	$t^\alpha(1+e^t)^{-\alpha}$
Mott	M_n	$(\sqrt{1-t^2}-1)/t$	1
Bessel	p_n	$1-\sqrt{1-2t}$	1

Examples in $\mathcal{O}(M(N))$

Non-Sheffer:

- Fibonacci

$$\sum_{n \geq 0} F_n(x) t^n = \frac{1}{1-t^2} \frac{1}{1-\frac{xt}{1-t^2}};$$

- Chebyshev

$$-\frac{1}{2} + \sum_{n \geq 0} T_n(x) t^n = \frac{1-t^2}{2(1+t^2)} \frac{1}{1-\frac{2xt}{1+t^2}};$$

- Jacobi with $(\alpha + \beta + 1) \neq 0$

$$\sum_{n \geq 0} \frac{(\alpha + \beta + 1)_n}{(\beta + 1)_n} P_n^{(\alpha, \beta)}(x) t^n =$$

$$(1+t)^{-\alpha-\beta-1} {}_2F_1 \left(\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+1}{2} \\ \beta+1 \end{matrix} \middle| \frac{2t(1+x)}{(1+t)^2} \right).$$

Other Sheffer Sequences, but in $\mathcal{O}(M(N) \log N)$

Name	Notation	h	v
Falling factorial	$(x)_n$	$\log(1+t)$	1
Bell	ϕ_n	$\exp(t) - 1$	1
Bernoulli, 2nd kind	b_n	$\log(1+t)$	$t / \log(1+t)$
Poisson-Charlier	$c_n(x; a)$	$\log(1+t/a)$	$\exp(-t)$
Actuarial	$a_n^{(\beta)}$	$1 - \exp(t)$	$\exp(\beta t)$
Narumi	$N_n^{(a)}$	$\log(1+t)$	$t^a \log(1+t)^{-a}$
Peters	$P_n^{(\lambda, \mu)}$	$\log(1+t)$	$(1 + (1+t)^\lambda)^{-\mu}$
Meixner-Pollaczek	$P_n^{(\lambda)}(x; \phi)$	$i \log\left(\frac{1-te^{i\phi}}{1-te^{-i\phi}}\right)$	$(1 + t^2 - 2t \cos \phi)^{-\lambda}$
Meixner	$m_n(x; \beta, c)$	$\log\left(\frac{1-t/c}{1-t}\right)$	$(1-t)^{-\beta}$
Krawtchouk	$K_n(x; p, N)$	$\log\left(\frac{p-(1-p)t}{p(1+t)}\right)$	$(1+t)^N$

Falling factorial known (Gerhard 00);

Last 3 known in direct sense (P_i) $\mapsto (x^i)$

(Driscoll, Healy, Rockmore 97; Pott, Steidl, Tasche 98).

Algorithm

Bivariate Eval, $f(g(x)h(t)) = \sum F_{ij}x^i t^j = \sum G_j(x)t^j$

Input: (a_0, \dots, a_N) ; composition seq. for g and h ; series for f

Output: $\sum_{i=0}^N b_i x^i = \sum a_j G_j(x) \bmod x^{N+1}$.

$$[x^i t^j] f(g(x)h(t)) = F_{ij} = \sum_k f_k g_{ik} h_{jk}, \quad \text{where } \begin{cases} g^k &= \sum_i g_{ik} x^i, \\ h^k &= \sum_j h_{jk} x^j. \end{cases}$$

In matrix form,

$$(b) = (F_{ij})(a) = (g_{ik}) \text{Diag}(f)(h_{jk})^t(a).$$

We know how to multiply by (g_{ik}) (fast right composition) if we have fast multiplication by $(h_{jk})^t$, we're done.

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Effective by code transposition (Bostan, Lecerf, Schost 03).

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Basic Transpositions

Transpose

Free ($\mathcal{O}(N)$)

- Rec: permutation $i \mapsto N - i$;

Super fast ($\mathcal{O}(N)$):

- $\text{Diag}(u)$, e.g., $M_\lambda, \mathcal{L}, \mathcal{L}^{-1}$;

Very fast ($\mathcal{O}(M(N))$):

- $\text{Mul}(P)$;
- $A_a = \mathcal{L}^{-1} \text{Rec Mul}(e^{ax}) \text{Rec } \mathcal{L}$;

Fast ($\mathcal{O}(M(N) \log N)$):

- Multipoint evaluation: $V(a_0, \dots, a_N)$ (Vandermonde);
- Interpolation: $V(a_0, \dots, a_N)^{-1}$;
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- $\log(1 + x)$: $A_1 \cdot (V(0, \dots, N)^{-1})^t \cdot \mathcal{L}$.

Basic Transpositions

Free ($\mathcal{O}(1)$)

- Rec: permutation $i \mapsto N - i$;

Transpose

Rec

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$\text{Diag}(u)$

Very fast ($\mathcal{O}(M(N))$):

- $\text{Mul}(P)$:
- $A_a = \mathcal{L}^{-1} \text{RecMul}(e^{ax}) \text{Rec}\mathcal{L}$;

$$\text{Mul}(P)^t : A \mapsto A \times \text{Rec}(P) \text{ div } x^N$$

$$\mathcal{L} \text{RecMul}(e^{ax})^t \text{Rec}\mathcal{L}^{-1}$$

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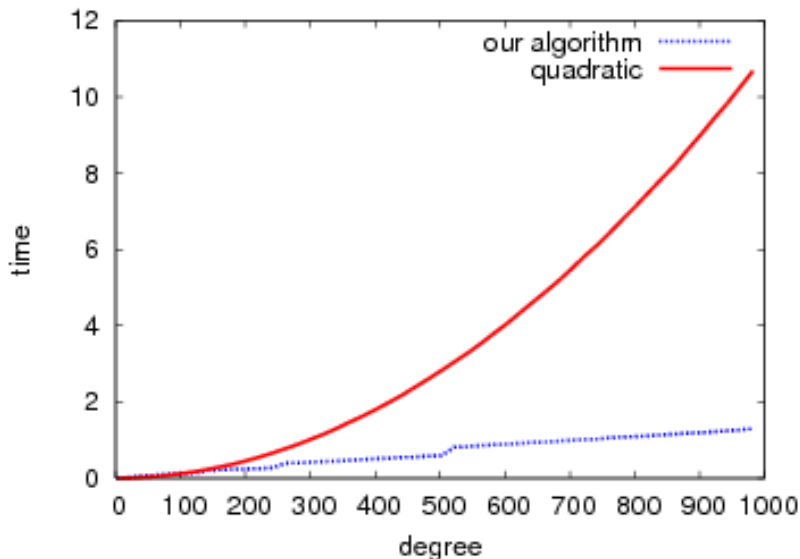
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- $\log(1+x)$: $A_1 \cdot (V(0, \dots, N)^{-1})^t \cdot \mathcal{L}$. $\mathcal{L} \cdot \text{Interpolation} \cdot A_1^t$

End of Proof

Example: Conversion to Jacobi



V General Orthogonal Polynomials

Basic Facts on Orthogonal Polynomials

- ① A non-decreasing function μ on (a, b) .
- ② A scalar product $\langle f, g \rangle = \ell(fg) = \int_a^b fg \, d\mu$ s.t. $\forall n, \ell(x^n) < \infty$.
- ③ Gram-Schmidt for (x^n) produces (P_n) s.t. $\deg P_n = n$ and if $\deg q < n$, then $\langle P_n, q \rangle = 0$.
- ④ Therefore $\langle xP_n | P_i \rangle = \langle P_n | xP_i \rangle = 0$ for $i < n - 1$, and

$$P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x), \quad a_n c_n \neq 0.$$

(R)

Starting from $P_{-1} = 0, P_0 = 1$, this defines the (P_n) .

Direct Conversion in $\mathcal{O}(M(N) \log N)$ ops

$$\begin{pmatrix} P_{n+1} \\ P_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_n x + b & c_n \\ 1 & 0 \end{pmatrix}}_{:= M^{(n-1,n)}} \begin{pmatrix} P_n \\ P_{n-1} \end{pmatrix} = M^{(n-1,n)} \dots M^{(0,1)} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix}.$$

$$\text{Aim: } \sum_{i=0}^{2n+1} \alpha_i P_i = \sum_{i=0}^n (\alpha_{2i} \quad \alpha_{2i+1}) M^{(0,2i)} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix} =: S^{(0,2n)} \begin{pmatrix} P_1 \\ P_0 \end{pmatrix}.$$

Algorithm Direct Conversion

Input: $(\alpha_{2m}, \dots, \alpha_{2p+1})$

Output: $M^{(2m,2p)}$ and $S^{(2m,2p)}$

- 1 $q := \lceil \frac{m+p}{2} \rceil$;
- 2 Compute recursively $M^{(2m,2q)}, S^{(2m,2q)}, M^{(2q+2,2p)}, S^{(2q+2,2p)}$;
- 3 Return $M^{(2q+2,2p)} M^{(2m,2q)}$ and $S^{(2m,2q)} + S^{(2q+2,2p)} M^{(2m,2q)}$.

Note: This gives fast multiplication by $\mathbf{P} := ([x^i] P_j)$.

Inverse Conversion in $\mathcal{O}(M(N) \log N)$ operations

$$P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x), \quad a_n c_n \neq 0. \quad (R)$$

Starting from $P_{-1} = 0$, $P_0 = 1$, this defines the (P_n) .

Theorem (Favard (35); Shohat (36))

Conversely, given $(a_n), (b_n), (c_n)$ and (R) , there exists ℓ such that

- $\ell(P_j P_i) = 0, j < i; d_i := \ell(P_i^2) = (-1)^i c_2 \cdots c_{i+1} / a_{i+1} \neq 0;$

- $L(x) := \sum_{i \geq 0} \ell(x^i) x^i = \frac{Q_{n-1}}{\text{Rec}(P_n)} + \mathcal{O}(x^{2n}),$ where

$$Q_n = (a_{n+1} + b_{n+1}x)Q_{n-1} + c_{n+1}x^2 Q_{n-2}, \quad Q_{-1} = 0, \quad Q_0 = 1.$$

Inverse Conversion in $\mathcal{O}(M(N) \log N)$ operations

Theorem (Favard (35); Shohat (36))

Conversely, given $(a_n), (b_n), (c_n)$ and (R) , there exists ℓ such that

- $\ell(P_j P_i) = 0, j < i; d_i := \ell(P_i^2) = (-1)^i c_2 \cdots c_{i+1} / a_{i+1} \neq 0;$

- $L(x) := \sum_{i \geq 0} \ell(x^i) x^i = \frac{Q_{n-1}}{\text{Rec}(P_n)} + \mathcal{O}(x^{2n}),$ where

$$Q_n = (a_{n+1} + b_{n+1}x)Q_{n-1} + c_{n+1}x^2Q_{n-2}, Q_{-1} = 0, Q_0 = 1.$$

$$\mathbf{P}^t \cdot (\ell(x^{i+j})) \cdot \mathbf{P} = \text{Diag}(d_i) \Rightarrow \mathbf{P}^{-1} = \text{Diag}(d_i)^{-1} \cdot \mathbf{P}^t \cdot (\ell(x^{i+j})).$$

Algorithm $(a) \mapsto \mathbf{P}^{-1}(a)$ (Bostan, S., Schost 08b)

- 1 Compute $(d_i);$
- 2 Compute $L(x) \bmod x^{2n}$ using Direct Conversion;
- 3 $(a) \mapsto (\ell(x^{i+j})) \cdot (a)$ is a transposed multiplication by $L(x);$
- 4 Algorithm $(a) \mapsto \mathbf{P}^t(a)$ obtained by transposing the code for Direct Conversion.

VI Conclusion

Conclusion, Further Work, Work in Progress

- **Summary:** transposition helps conversions to and from polynomials and power series with generating function $u(x)v(t)f(g(x)h(t))$.
- **General orthogonal polynomials** in $\mathcal{O}(M(N) \log N)$ by a fast effective Shohat-Favard theorem and again transposition;
- **k -term recurrences** with $k > 3$?
- **D-finite series** decompose on orthogonal polynomials in $\mathcal{O}(N)$ ops: A. Benoit, extending (Pazkowski, Lewanowicz, Rebillard, . . .);
- **Numerically stable algorithms?**
- $\mathcal{O}(M(N))$: How far can one go?