Algolib can be downloaded from http://algo.inria.fr

```
> libname:="/Users/salvy/lib/maple/Algolib/11",libname:
```

Dominant singularity

Rational generating functions

Fibonacci

```
> infsing(1/(1-z-z^2),z);
```

$$\left[\left[-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right], \text{polar}, \mathit{false}\right]$$

asymptotic behaviour:

```
> equivalent(1/(1-z-z^2),z,n);
```

$$\frac{\left(e^{-n}\right)^{-\ln(2)+\ln\left(-1+\sqrt{5}\right)}}{-\frac{1}{2}+\frac{1}{2}\sqrt{5}+2\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^2}+O\left(\frac{\left(e^{-n}\right)^{-\ln(2)+\ln\left(-1+\sqrt{5}\right)}}{n}\right)$$

```
> read "conway.mpl";
```

$$\mathit{GFconway} := -\left(-1+z^2-z+z^3+12z^{78}+6z^{11}-20z^{30}-30z^{29}+z^4-20z^{73}+18z^{76} -4z^{69}+18z^{74}+31z^{71}-4z^{68}-z^{18}+3z^{19}-36z^{24}+58z^{27}+13z^{22}+8z^{12}-4z^{17} -23z^{31}+15z^{70}-6z^{23}-20z^{25}+8z^{21}-z^{13}-z^{16}+6z^{20}-6z^9-18z^{77}-5z^{14} -18z^{75}-22z^{72}-4z^{15}+45z^{55}-11z^{63}+41z^{62}+54z^{61}-56z^{60}+15z^{58}-44z^{59} -27z^{57}+62z^{66}-21z^{64}-19z^{67}-50z^{65}+34z^{28}+z^5-4z^8+35z^{32}+7z^{38}+12z^{36} -79z^{39}+107z^{43}+8z^{35}-13z^{40}+38z^{49}+16z^{41}-z^{26}+z^7-64z^{52}-15z^{56} +89z^{53}-25z^{50}-8z^{54}+126z^{48}-26z^{34}-9z^{33}+42z^{37}-39z^{47}-32z^{46}-66z^{51} -33z^{45}+14z^{42}-65z^{44}\right)\Big/\Big((z-1)\left(-1+z^2+2z^3+z^{11}-8z^{30}-6z^{29}+z^4+6z^{69} +6z^{71}-12z^{68}+3z^{18}+2z^{19}+3z^{24}+8z^{27}-z^{22}+z^{12}+10z^{17}+5z^{31}-3z^{70} -9z^{23}+7z^{25}-6z^{21}-2z^{13}+2z^{16}-6z^{20}+z^9-5z^{14}-3z^{15}+7z^{55}-5z^{62}+2z^{61} +4z^{60}-2z^{58}+12z^{59}-7z^{57}-7z^{66}-z^{64}+z^{10}+4z^{67}+7z^{65}-10z^{28}-2z^5+z^8 +12z^{32}-10z^{38}-z^{36}-z^{39}+3z^{43}-7z^{35}+6z^{40}+2z^{41}+8z^{26}-z^7-3z^{52}-12z^{56} +4z^{53}+7z^{50}+10z^{54}+8z^{48}+7z^{34}-7z^{33}+3z^{37}-14z^{47}+3z^{46}-9z^{51}-9z^{45} +10z^{42}-2z^{44}-2z^6\right)\Big)$$

```
> infsing(GFconway,z);
```

$$\Big[\Big[\mathrm{RootOf}\Big(-1+\_Z^2+2\_Z^3+\_Z^{11}-8\_Z^{30}-6\_Z^{29}+\_Z^4+6\_Z^{69}+6\_Z^{71}-12\_Z^{68} +3\_Z^{18}+2\_Z^{19}+3\_Z^{24}+8\_Z^{27}-\_Z^{22}+\_Z^{12}+10\_Z^{17}+5\_Z^{31}-3\_Z^{70} -9\_Z^{23}+7\_Z^{25}-6\_Z^{21}-2\_Z^{13}+2\_Z^{16}-6\_Z^{20}+\_Z^9-5\_Z^{14}-3\_Z^{15} +7\_Z^{55}-5\_Z^{62}+2\_Z^{61}+4\_Z^{60}-2\_Z^{58}+12\_Z^{59}-7\_Z^{57}-7\_Z^{66}-\_Z^{64} +\_Z^{10}+4\_Z^{67}+7\_Z^{65}-10\_Z^{28}-2\_Z^5+\_Z^8+12\_Z^{32}-10\_Z^{38}-\_Z^{36}-\_Z^{39}$$

$$+ 3\,\_Z^{43} - 7\,\_Z^{35} + 6\,\_Z^{40} + 2\,\_Z^{41} + 8\,\_Z^{26} - \_Z^{7} - 3\,\_Z^{52} - 12\,\_Z^{56} + 4\,\_Z^{53}$$
$$+ 7\,\_Z^{50} + 10\,\_Z^{54} + 8\,\_Z^{48} + 7\,\_Z^{34} - 7\,\_Z^{33} + 3\,\_Z^{37} - 14\,\_Z^{47} + 3\,\_Z^{46} - 9\,\_Z^{51}$$
$$- 9\,\_Z^{45} + 10\,\_Z^{42} - 2\,\_Z^{44} - 2\,\_Z^{6}, 0.7671198507\big)\big], \mathrm{polar}, \mathit{false}\big]$$

It's the root of this polynomial that is approximately 0.767. We call it α later.

```
> alias(alpha=%[1][1]):
```

asymptotic behaviour of the coefficients:

```
> equivalent(GFconway,z,n);
```

$$-\Big(\big(-1 - 64\,\alpha^{52} - 4\,\alpha^{15} - 26\,\alpha^{34} + 38\,\alpha^{49} + 16\,\alpha^{41} - \alpha^{26} - 66\,\alpha^{51} - 33\,\alpha^{45} + 41\,\alpha^{62}$$
$$+ 12\,\alpha^{36} + 58\,\alpha^{27} + 42\,\alpha^{37} - 39\,\alpha^{47} - 32\,\alpha^{46} + \alpha^{7} - \alpha^{16} + 13\,\alpha^{22} - \alpha^{13} - 9\,\alpha^{33}$$
$$+ 31\,\alpha^{71} - 4\,\alpha^{68} - 50\,\alpha^{65} - \alpha^{18} + \alpha^{5} - 4\,\alpha^{8} + 35\,\alpha^{32} + 54\,\alpha^{61} - 56\,\alpha^{60} + 6\,\alpha^{20}$$
$$- 4\,\alpha^{17} - 23\,\alpha^{31} - 13\,\alpha^{40} - 65\,\alpha^{44} + 62\,\alpha^{66} - 22\,\alpha^{72} + 7\,\alpha^{38} + 34\,\alpha^{28} - 25\,\alpha^{50}$$
$$- 6\,\alpha^{23} - 20\,\alpha^{25} + 8\,\alpha^{21} + 8\,\alpha^{12} - 21\,\alpha^{64} - 19\,\alpha^{67} + 15\,\alpha^{70} + 15\,\alpha^{58} + 3\,\alpha^{19} - 36\,\alpha^{24}$$
$$- 15\,\alpha^{56} + 89\,\alpha^{53} - 44\,\alpha^{59} - 27\,\alpha^{57} - 8\,\alpha^{54} + 126\,\alpha^{48} + 45\,\alpha^{55} + 14\,\alpha^{42} - 79\,\alpha^{39}$$
$$+ 18\,\alpha^{74} + \alpha^{2} + \alpha^{3} + 12\,\alpha^{78} - 20\,\alpha^{73} - \alpha - 5\,\alpha^{14} - 18\,\alpha^{75} - 11\,\alpha^{63} - 6\,\alpha^{9} - 18\,\alpha^{77}$$
$$+ 18\,\alpha^{76} - 30\,\alpha^{29} + \alpha^{4} - 20\,\alpha^{30} + 6\,\alpha^{11} + 107\,\alpha^{43} + 8\,\alpha^{35} - 4\,\alpha^{69}\big)\,\big(4\,\alpha^{52} + 2\,\alpha^{15}$$
$$- 7\,\alpha^{34} + 7\,\alpha^{49} + 10\,\alpha^{41} + 8\,\alpha^{26} - 3\,\alpha^{51} + 3\,\alpha^{45} + 3\,\alpha^{36} - 10\,\alpha^{27} - 10\,\alpha^{37} + 8\,\alpha^{47}$$
$$- 14\,\alpha^{46} + \alpha^{7} + 10\,\alpha^{16} - 9\,\alpha^{22} - 5\,\alpha^{13} + 7\,\alpha^{33} + 6\,\alpha^{68} - 7\,\alpha^{65} + 2\,\alpha^{18} - 2\,\alpha^{5} + \alpha^{8}$$
$$- 7\,\alpha^{32} - 5\,\alpha^{61} + 2\,\alpha^{60} - 6\,\alpha^{20} + 3\,\alpha^{17} + 12\,\alpha^{31} + 2\,\alpha^{40} - 9\,\alpha^{44} + 4\,\alpha^{66} - \alpha^{38}$$
$$- 6\,\alpha^{28} - 9\,\alpha^{50} + 3\,\alpha^{23} + 8\,\alpha^{25} - \alpha^{21} - 2\,\alpha^{12} + 7\,\alpha^{64} - 12\,\alpha^{67} + 6\,\alpha^{70} + 12\,\alpha^{58}$$
$$- 6\,\alpha^{19} + 7\,\alpha^{24} - 7\,\alpha^{56} + 10\,\alpha^{53} + 4\,\alpha^{59} - 2\,\alpha^{57} + 7\,\alpha^{54} - 12\,\alpha^{55} + 3\,\alpha^{42} + 6\,\alpha^{39}$$
$$+ 2\,\alpha^{2} + \alpha^{3} + \alpha - 3\,\alpha^{14} - \alpha^{63} + \alpha^{9} - 8\,\alpha^{29} - 2\,\alpha^{4} + 5\,\alpha^{30} + \alpha^{11} - 2\,\alpha^{43} - \alpha^{35}$$
$$- 3\,\alpha^{69} - \alpha^{6} + \alpha^{10}\big)^{-\ln(e^{-n})}\Big) \Big/ \Big(\big((\alpha - 1)\,\big(156\,\alpha^{52} + 45\,\alpha^{15} - 238\,\alpha^{34} - 82\,\alpha^{41}$$
$$- 208\,\alpha^{26} + 459\,\alpha^{51} + 405\,\alpha^{45} + 310\,\alpha^{62} + 36\,\alpha^{36} - 216\,\alpha^{27} - 111\,\alpha^{37} + 658\,\alpha^{47}$$
$$- 138\,\alpha^{46} + 7\,\alpha^{7} - 32\,\alpha^{16} + 22\,\alpha^{22} + 26\,\alpha^{13} + 231\,\alpha^{33} - 426\,\alpha^{71} + 816\,\alpha^{68} - 455\,\alpha^{65}$$
$$- 54\,\alpha^{18} + 10\,\alpha^{5} - 8\,\alpha^{8} - 384\,\alpha^{32} - 122\,\alpha^{61} - 240\,\alpha^{60} + 120\,\alpha^{20} - 170\,\alpha^{17} - 155\,\alpha^{31}$$
$$- 240\,\alpha^{40} + 88\,\alpha^{44} + 462\,\alpha^{66} + 380\,\alpha^{38} + 280\,\alpha^{28} - 350\,\alpha^{50} + 207\,\alpha^{23} - 175\,\alpha^{25}$$
$$+ 126\,\alpha^{21} - 12\,\alpha^{12} + 64\,\alpha^{64} - 268\,\alpha^{67} + 210\,\alpha^{70} + 116\,\alpha^{58} - 38\,\alpha^{19} - 72\,\alpha^{24}$$

$$
\begin{aligned}
&+ 672\,\alpha^{56} - 212\,\alpha^{53} - 708\,\alpha^{59} + 399\,\alpha^{57} - 540\,\alpha^{54} - 384\,\alpha^{48} - 385\,\alpha^{55} - 420\,\alpha^{42} \\
&+ 39\,\alpha^{39} - 2\,\alpha^{2} - 6\,\alpha^{3} + 70\,\alpha^{14} - 9\,\alpha^{9} + 174\,\alpha^{29} - 4\,\alpha^{4} + 240\,\alpha^{30} - 11\,\alpha^{11} - 129\,\alpha^{43} \\
&+ 245\,\alpha^{35} - 414\,\alpha^{69} + 12\,\alpha^{6} - 10\,\alpha^{10}\big)\big) + \mathrm{O}\Big(\frac{1}{n}\big(4\,\alpha^{52} + 2\,\alpha^{15} - 7\,\alpha^{34} \\
&+ 7\,\alpha^{49} + 10\,\alpha^{41} + 8\,\alpha^{26} - 3\,\alpha^{51} + 3\,\alpha^{45} + 3\,\alpha^{36} - 10\,\alpha^{27} - 10\,\alpha^{37} + 8\,\alpha^{47} - 14\,\alpha^{46} \\
&+ \alpha^{7} + 10\,\alpha^{16} - 9\,\alpha^{22} - 5\,\alpha^{13} + 7\,\alpha^{33} + 6\,\alpha^{68} - 7\,\alpha^{65} + 2\,\alpha^{18} - 2\,\alpha^{5} + \alpha^{8} - 7\,\alpha^{32} \\
&- 5\,\alpha^{61} + 2\,\alpha^{60} - 6\,\alpha^{20} + 3\,\alpha^{17} + 12\,\alpha^{31} + 2\,\alpha^{40} - 9\,\alpha^{44} + 4\,\alpha^{66} - \alpha^{38} - 6\,\alpha^{28} \\
&- 9\,\alpha^{50} + 3\,\alpha^{23} + 8\,\alpha^{25} - \alpha^{21} - 2\,\alpha^{12} + 7\,\alpha^{64} - 12\,\alpha^{67} + 6\,\alpha^{70} + 12\,\alpha^{58} - 6\,\alpha^{19} \\
&+ 7\,\alpha^{24} - 7\,\alpha^{56} + 10\,\alpha^{53} + 4\,\alpha^{59} - 2\,\alpha^{57} + 7\,\alpha^{54} - 12\,\alpha^{55} + 3\,\alpha^{42} + 6\,\alpha^{39} + 2\,\alpha^{2} + \alpha^{3} \\
&+ \alpha - 3\,\alpha^{14} - \alpha^{63} + \alpha^{9} - 8\,\alpha^{29} - 2\,\alpha^{4} + 5\,\alpha^{30} + \alpha^{11} - 2\,\alpha^{43} - \alpha^{35} - 3\,\alpha^{69} - \alpha^{6} \\
&+ \alpha^{10}\big)^{n}\Big)
\end{aligned}
$$

Numerical value:

```
> evalf(%);
```

$$2.042160079\ 1.303577270^{-1.\ln(e^{-1.n})} + \mathrm{O}\left(\frac{1.303577270^{n}}{n}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$2.042160079\ e^{0.2651122315\,n} + \mathrm{O}\left(\frac{e^{0.2651122315\,n}}{n}\right)$$

Meromorphic functions

Derangements

```
> derangements:={S=Set(Cycle(Z,card>1))};
```

$$derangements := \{S = Set(Cycle(Z, 1 < card))\}$$

```
> combstruct[gfsolve](derangements,labelled,z);
```

$$\left\{Z(z) = z, S(z) = e^{\ln\left(\frac{1}{1-z}\right) - z}\right\}$$

```
> der:=simplify(subs(%,S(z)));
```

$$der := -\frac{e^{-z}}{z-1}$$

```
> infsing(der,z);
```

$$[[1], \mathrm{polar}, false]$$

asymptotic number:

```
> equivalent(der,z,n);
```

$$e^{-1} + \mathrm{O}\left(\frac{1}{n}\right)$$

Surjections

```
> surjections:={S=Sequence(Set(Z,card>0))};
```

$$surjections := \{S = Sequence(Set(Z, 0 < card))\}$$

```
> combstruct[gfsolve](surjections,labelled,z);
```

$$\left\{Z(z) = z, S(z) = -\frac{1}{-2+e^z}\right\}$$

```
> surj:=subs(%,S(z));
```

$$surj := -\frac{1}{-2+e^z}$$

```
> infsing(surj,z);
```

$$[[\ln(2)], \text{polar}, false]$$

asymptotic number

```
> equivalent(surj,z,n);
```

$$\frac{1}{2}\,\frac{(e^{-n})^{\ln(\ln(2))}}{\ln(2)} + O\left(\frac{(e^{-n})^{\ln(\ln(2))}}{n}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\ln(2)^{-1-n} + O\left(\frac{\ln(2)^{-n}}{n}\right)$$

Bernoulli numbers

```
> infsing(z/(exp(z)-1),z);
```

$$[[-2\,I\,\pi, 2\,I\,\pi], \text{polar}, false]$$

Iterative constructions

Binary trees of cycles of cycles

```
> btcc:={S=Union(CC,Prod(S,S)),CC=Cycle(Cycle(Z))};
```

$$btcc := \{S = Union(CC, Prod(S,S)), CC = Cycle(Cycle(Z))\}$$

```
> combstruct[gfsolve](btcc,labelled,z);
```

$$\left\{Z(z) = z, S(z) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\ln\left(-\frac{1}{-1+\ln\left(-\frac{1}{z-1}\right)}\right)}, CC(z) = \ln\left(-\frac{1}{-1+\ln\left(-\frac{1}{z-1}\right)}\right)\right\}$$

```
> btcc:=subs(%,S(z));
```

$$btcc := \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\ln\left(-\frac{1}{-1+\ln\left(-\frac{1}{z-1}\right)}\right)}$$

```
> infsing(btcc,z);
```

$$\left[\left[\frac{\mathrm{e}^{\mathrm{e}^{\frac{1}{4}}-1}{\mathrm{e}^{\frac{1}{4}}}-1}{\mathrm{e}^{\frac{\mathrm{e}^{\frac{1}{4}}-1}{\mathrm{e}^{\frac{1}{4}}}}}\right], algebraic, false\right]$$

Singularity analysis

Binary trees:

```
> equivalent((1-sqrt(1-4*z))/2/z,z,n,5);
```

$$\frac{\left(\frac{1}{n}\right)^{3/2}\left(\mathrm{e}^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}-\frac{9}{8}\frac{\left(\frac{1}{n}\right)^{5/2}\left(\mathrm{e}^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+\frac{145}{128}\frac{\left(\frac{1}{n}\right)^{7/2}\left(\mathrm{e}^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}-\frac{1155}{1024}\frac{\left(\frac{1}{n}\right)^{9/2}\left(\mathrm{e}^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+\frac{36939}{32768}\frac{\left(\frac{1}{n}\right)^{11/2}\left(\mathrm{e}^{-n}\right)^{-2\ln(2)}}{\sqrt{\pi}}+\mathrm{O}\left(\left(\frac{1}{n}\right)^{13/2}\left(\mathrm{e}^{-n}\right)^{-2\ln(2)}\right)$$

Cayley trees:

```
> Cayley:={T=Prod(Z,Set(T))};
```

$$Cayley := \{T = Prod(Z, Set(T))\}$$

```
> combstruct[gfsolve](Cayley,labelled,z);
```

$$\{Z(z) = z, T(z) = -\mathrm{LambertW}(-z)\}$$

```
> equivalent(subs(%,T(z)),z,n);
```

$$\frac{1}{2}\frac{\sqrt{2}\sqrt{\mathrm{e}}\sqrt{\mathrm{e}^{-1}}\left(\frac{1}{n}\right)^{3/2}\mathrm{e}^{n}}{\sqrt{\pi}}+\mathrm{O}\left(\frac{\mathrm{e}^{n}}{n^{2}}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\frac{\sqrt{2}\,\mathrm{e}^{n}}{\sqrt{\pi}\,n^{3/2}}+\mathrm{O}\left(\frac{\mathrm{e}^{n}}{n^{2}}\right)$$

Binary trees of cycles of cycles

```
> equivalent(btcc,z,n):
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\frac{\left(\mathrm{e}^{\left(\mathrm{e}^{\frac{1}{4}}-1\right)\mathrm{e}^{-\frac{1}{4}}}-1\right)^{\frac{1}{2}-n}\mathrm{e}^{\frac{1}{8}\left(\mathrm{e}^{\frac{1}{4}}+8n\,\mathrm{e}^{\frac{1}{4}}-8n\right)\mathrm{e}^{-\frac{1}{4}}}}{\sqrt{\pi}\,n^{3/2}}$$

$$+\mathrm{O}\left(\frac{e^{-\left(-e^{\frac{1}{4}}+1+\ln\left(e^{\left(e^{\frac{1}{4}}-1\right)e^{-\frac{1}{4}}}-1\right)e^{\frac{1}{4}}\right)e^{-\frac{1}{4}}n}}{n^{5/2}}\right)$$

Saddle-point method

Sets

```
> equivalent(exp(z),z,n);
```

$$\frac{1}{2}\,\frac{\sqrt{2}\,\sqrt{\frac{1}{n}}\,e^{n}\,n^{-n}}{\sqrt{\pi}}+\mathrm{O}\left(\left(\frac{1}{n}\right)^{3/2}e^{n}\,n^{-n}\right)$$

Involutions

```
> equivalent(exp(z+z^2/2),z,n);
```

$$\frac{1}{2}\,\frac{e^{-\frac{1}{4}}\,\sqrt{\frac{1}{n}}\,e^{\frac{1}{\sqrt{\frac{1}{n}}}}\,\sqrt{n^{-n}}}{\sqrt{\pi}\,\sqrt{e^{-n}}}+\mathrm{O}\left(\frac{e^{\frac{\cos\left(\left(\frac{1}{4}-\frac{1}{4\,\mathrm{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}}\,\sqrt{n^{-n}}}{n\,\sqrt{e^{-n}}}\right)$$

```
> map(simplify,%) assuming n::posint;
```

$$\frac{1}{2}\,\frac{e^{-\frac{1}{4}+\sqrt{n}+\frac{1}{2}n}\,n^{-\frac{1}{2}-\frac{1}{2}n}}{\sqrt{\pi}}+\mathrm{O}\left(e^{\sqrt{n}+\frac{1}{2}n}\,n^{-1-\frac{1}{2}n}\right)$$

An example with a singularity at finite distance

```
> equivalent(exp(z/(1-z)),z,n);
```

$$\frac{1}{8}\,\frac{\sqrt{2}\,e^{-\frac{1}{2}}\,4^{3/4}\left(\frac{1}{n}\right)^{3/4}e^{\frac{2}{\sqrt{\frac{1}{n}}}}}{\sqrt{\pi}}+\mathrm{O}\left(\left(\frac{1}{n}\right)^{5/4}e^{\frac{2\cos\left(\left(\frac{1}{4}-\frac{1}{4\,\mathrm{signum}(n)}\right)\pi\right)}{\sqrt{\frac{1}{|n|}}}}\right)$$

Set partitions

```
> equivalent(exp(exp(z)-1),z,n);
```

The saddle point is , LambertW$(n+1)$

Saddle point's expansion:

$$\ln(n)-\ln(\ln(n))+\frac{\ln(\ln(n))}{\ln(n)}+\mathrm{O}\left(\frac{\ln(\ln(n))^{2}}{\ln(n)^{2}}\right)$$

$$\frac{1}{2}\,\frac{\sqrt{2}\,e^{-1}\sqrt{e^{-\_saddlepoint}}\,e^{e^{\_saddlepoint}}}{\_saddlepoint^{n}\sqrt{\pi}\,\_saddlepoint}+\mathrm{O}\left(\frac{\sqrt{e^{-\_saddlepoint}}\,e^{e^{\_saddlepoint}}}{\_saddlepoint^{2}\,\_saddlepoint^{n}}\right)$$

```
>
```