Pólya Urns
THE Analytic Approach

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Philippe Flajolet and Analytic Combinatorics
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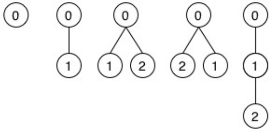
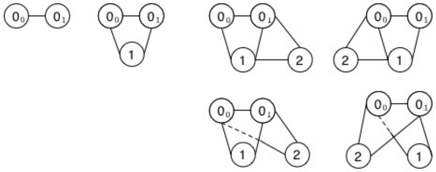
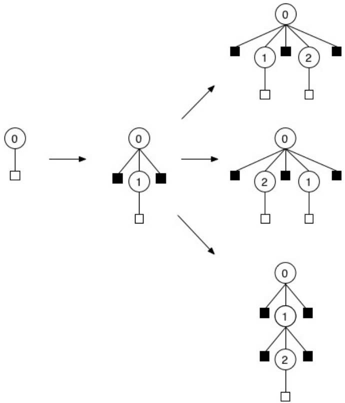
INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Additive balanced urns of some k-trees







$$\begin{pmatrix} k-1 & 2 \\ k & 1 \end{pmatrix}$$

$$\frac{\frac{d}{dz}H(x,z)}{H(x,z)^k \left(H(x,z)-x+1\right)^2} = 1$$

Singularity analysis,
Probabilistic results:

- Gaussian limit law + rate of convergence
- local limit law
- large deviations

[M., Flajolet, July 2010]

Additive balanced urns with preferential growth

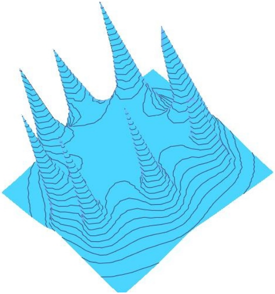
$$\begin{pmatrix} 2\alpha & \beta \\ \alpha & \alpha+\beta \end{pmatrix}$$

Ex: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad$ $\mathtt{x} \to \mathtt{x\,x\,y}$, $\quad \mathtt{y} \to \mathtt{x\,y\,y}$

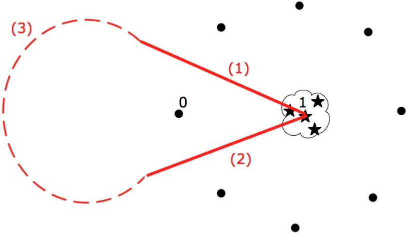
$$\left(z - \frac{x^{-\alpha}-1}{\alpha+\beta} - \frac{1}{2\alpha+\beta}\right) H(x,z)^{2\alpha+\beta} + \frac{x^{-\alpha}-1}{\alpha+\beta} H(x,z)^{\alpha} + \frac{1}{2\alpha+\beta} = 0$$

$$h_n(x) = \frac{\sigma^{n+1}}{2i\pi} \oint a_x(w) g_x(w)^{n+1} \mathrm{d}w$$

Multiple coalescing saddle-points

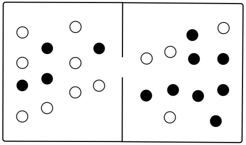


[M., LATIN2012]



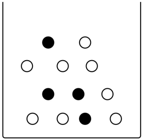
$$p_n(x) = \frac{h_n(x)}{h_n(1)} \sim \exp\left(\frac{\alpha\sigma}{\alpha+\beta}\sqrt{n}\tilde{x} - \frac{\alpha^3\sigma}{2(\alpha+\beta)^2}\tilde{x}^2\right)$$

Urns with multiple draws: the Bernoulli-Laplace process



Chamber A
a white
b black

Chamber B
b white
a black



One urn
a white, *b* black

$$\begin{array}{c} \\ WW \\ BW \\ BB \end{array} \begin{array}{cc} W & B \\ \end{array} \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$a + b = N$$

$$x^a y^b \xrightarrow{\mathcal{G}} a^2\, x^{-1} y^1\, x^a y^b + b^2\, x^1 y^{-1}\, x^a y^b + 2ab\, x^0 y^0\, x^a y^b$$

$$\Theta_u := u\partial_u \ (\textit{pick \& replace}\text{ a ball})$$

$$\mathcal{G} := x^{-1} y^1\, \Theta_x^2 + x^1 y^{-1}\, \Theta_y^2 + 2\, x^0 y^0 \Theta_x \Theta_y$$

$$H_n(x, y) := \mathcal{G}^n \circ x^{a_0} y^{b_0}$$

$$\mathbb{P}\{(a_0, b_0) \rightsquigarrow (a, b)\} = \frac{1}{N^{2n}} \left[x^a y^b\right] H_n(x, y)$$

[Flajolet, July 2010]

Unbalanced urns: the Knuth's strings

2.1. **Cutting loops.** The following problem by Daniel Shine was published on pages 144–145 of the *Journal of Recreational Mathematics*, Volume 33:

> **2680**. One thousand loops of string, each of unit length, are placed in a box. One piece of string is removed at random. It is cut at a random location and put back into the box. This select-and-cut process is repeated 1000 times. All pieces, whether loops or not, are equally likely to be selected. After the 1000 repetitions, what is the average length of the pieces of string in the box?

$$\mathcal{K} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$(a_0, b_0) = (1000, 0)$$

$$\psi_n(x,y) := \sum_{a,b} \mathbb{P}\left[A_n = a, B_n = b\right] \frac{x^a y^b}{a+b}$$

$$(\Theta_x + \Theta_y)\, \psi_{n+1}(x,y) = \left(x^\alpha y^\beta\, \Theta_x + x^\gamma y^\delta\, \Theta_y\right) \psi_n(x,y)$$

$\Psi(x,y,z) = \sum_n \psi_n(x,y) z^n$ satisfies the PDE

$$\boxed{\left((1 - z\, x^\alpha y^\beta)\Theta_x + (1 - z\, x^\gamma y^\delta)\Theta_y\right) \circ \Psi = x^{a_0} y^{b_0}}$$

[M., Flajolet, January 2011]

Balanced urns with random entries: isomorphism theorem strikes again

$$\begin{pmatrix} 1-\mathcal{B} & \mathcal{B} \\ \mathcal{B} & 1-\mathcal{B} \end{pmatrix}, \quad \text{with } \mathcal{B} \sim \mathrm{Ber}(p)$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with proba p $\qquad$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with proba $1-p$

Again, $H(x,y,z) = X(x,y,z)^{a_0}\ Y(x,y,z)^{b_0}$, with

$$\begin{cases} \dot{X} &= p\,X\,Y + (1-p)\,X^2 \\ \dot{Y} &= p\,X\,Y + (1-p)\,Y^2 \end{cases}$$

Probability to have a black balls and b white balls after n draws:

$$p_{n,a,b} = \frac{[x^a y^b z^n] H(x,y,z)}{[z^n] H(1,1,z)}.$$

True for any balanced urn $\left(\begin{smallmatrix} \mathcal{A} & \sigma-\mathcal{A} \\ \sigma-\mathcal{B} & \mathcal{B} \end{smallmatrix}\right)$, with σ constant, and $\mathcal{A}, \mathcal{B}$ random variables on a finite state space $\{-1, 0, 1, \ldots, \sigma\}$.

[M., Mahmoud, since July 2011]